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Stability of Markov processes nonhomogeneous in time

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Abstract. We study the asymptotic behaviour of discrete time processes which are products of time dependent transformations defined on a complete metric space. Our sufficient condition is applied to products of Markov operators corresponding to stochastically perturbed dynamical systems and fractals.

0. Introduction. In some analytical models we need to study the asymptotic behaviour of sequences of the form

$$x_n = T_n \circ \ldots \circ T_1 x_0, \tag{0.1}$$

where $T_i : X \to X$ are given transformations from a metric space X into itself and $x_0 \in X$ is a starting point. The behaviour of the sequence may be quite complicated even in the case when all the transformations T_i are contractions. As the simplest example consider constant transformation $T_i(x) = a_i$ for $x \in X$. Then, of course, $x_n = a_n$ and the fact that all T_i have Lipschitz constant equal to zero is irrelevant.

A. Lasota proposed to study the behaviour of (x_n) under the assumption

$$\sum_{n=1}^{\infty} \sup_{x\in X} \varrho(T_n(x), T_{n+1}(x)) < \infty. \tag{0.2}$$

We show that in the case when all T_i are contractive some more restrictive condition (see (1.2)) is sufficient for the convergence of (x_n). In the specific case when all T_i are contractive with the same constant smaller than 1, our condition reduces to (0.2).

The plan of the paper is as follows. In Section 1 we formulate theorems on asymptotic properties of sequences of the form (0.1) and give some remarks. The proof of the main result is given in Section 2. Section 3 contains basic notions and facts concerning Markov operators acting on measures. Finally,

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in Section 4 we apply our theorem to stochastically perturbed systems and iterated function systems (related to fractals).

1. The convergence theorem. Let (E,d) be an arbitrary metric space. We call a mapping $T : E \to E$ *nonexpansive* with respect to the metric d if it satisfies

$$d(T(u),T(v)) \le d(u,v) \quad \text{for } u,v \in E,$$

and λ-*contractive* with respect to the metric d if $\lambda \in [0,1)$ and

$$d(T(u),T(v)) \le \lambda d(u,v) \quad \text{for } u,v \in E.$$

As usual, by T^n we denote the nth iterate of T. The set of all positive integers is denoted by $\mathbb{N}$.

Our goal is to study a family $T(n,m)$ ($n \ge m,\ n,m \in \mathbb{N}$) of transformations from E into itself. We call a family $\{T(n,m)\}$ a *process* if $T(m,m) = \mathrm{Id}$ (the identity transformation) and

$$T(n,m) = T(n,k)T(k,m) \quad \text{for } n \ge k \ge m.$$

Observe that in view of the above condition, a family $\{T(n,m)\}$ is a process if and only if there is a sequence $(T_n)_{n\in\mathbb{N}}$ of transformations such that

$$T(n,m) = T_{n-1} \circ \ldots \circ T_m \quad \text{for } n > m,\ n,m \in \mathbb{N}.$$

When $T(n,m)$ is generated by one transformation $T : E \to E$, then

$$T(n,m) = T^{n-m}, \quad n \ge m\ (T^0 = \mathrm{Id}).$$

We call a process $\{T(n,m)\}$ *asymptotically stable* if there exists a unique element $u_* \in E$ such that

$$\lim_{n\to\infty} d(T(n,m)v, u_*) = 0 \quad \text{for all } v \in E \text{ and } m \in \mathbb{N}. \tag{1.1}$$

Now, we are in a position to state our main result.

THEOREM 1. *Let (E,d) be a metric space and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of arbitrary transformations from E into itself. Assume that there exists an increasing sequence (n_k) of positive integers and a sequence (λ_k) of non-negative real numbers such that for each $k \in \mathbb{N}$ the transformation T_{n_k} is λ_k-contractive and*

$$\lim_{k\to\infty} \frac{1}{1-\lambda_k} \sum_{i=n_k}^{\infty} \sup_{u\in E} d(T_i(u), T_{i+1}(u)) = 0. \tag{1.2}$$

Then for every $m \in \mathbb{N}$ and $u \in E$ we have:

(a) *The sequence $(T(n,m)(u))_{n\ge m}$ is Cauchy.*

(b) $\lim_{n\to\infty} d(T(n,m)(u), T(n,m)(v)) = 0$ *for all $v \in E$.*

If (E,d) is in addition complete then the process $\{T(n,m)\}$ is asymptotically stable.

The proof will be given in the next section. Now we discuss some problems related to condition (1.2), which is a key assumption in Theorem 1.

Remark 1. First observe that if the sequence $(\lambda_k)_{k\in\mathbb{N}}$ tends to a constant $\lambda < 1$ or is bounded by a constant $\lambda < 1$ then condition (1.2) is equivalent to

$$\lim_{k\to\infty}\sum_{i=n_k}^{\infty}\sup_{u\in E} d(T_i(u), T_{i+1}(u)) = 0. \tag{1.3}$$

Remark 2. It is worth pointing out that even in the case of a compact metric space assumption (1.2) of Theorem 1 cannot be replaced by condition (1.3) without additional assumptions concerning the transformations T_n. Consider the following example. Take $E = [0,1]$. Let T_n be the identity transformation for odd positive integers n, whereas for even n set $T_n(u) = (1-1/n^2)u$, $u \in E$. Take $n_k = 2k$, $k \in \mathbb{N}$. Then T_{2k} is λ_k-contractive with $\lambda_k = 1-1/(4k^2)$. Note that $\sup_{u\in E} d(T_i(u), T_{i+1}(u)) \le 1/i^2$ for every $i \in \mathbb{N}$, hence (1.3) holds. It is easy to calculate that $T(n,1)(u) = T(n,2)u = a_k u$ for $2k \le n < 2k+2$, where

$$a_k = \prod_{i=1}^{k}\left(1 - \frac{1}{4i^2}\right) \quad \text{for } k \in \mathbb{N}.$$

Since the sequence (a_k) tends to $2/\pi$ as $k \to \infty$, we have

$$\lim_{n\to\infty} T(n,1)(u) = \frac{2}{\pi}u$$

and the limit depends on u, so the process is not asymptotically stable.

The following theorem shows that the assumptions of Theorem 1 can be modified in a way that will be useful later.

Theorem 2. *Let (E,d) be a complete metric space and, for every $n \in \mathbb{N}$, the mapping $T_n : E \to E$ be a nonexpansive transformation with respect to the metric d. Assume that there is a subset $E_0 \subset E$ and a metric $d_0 : E_0 \times E_0 \to \mathbb{R}_+$ such that*

(i) *E_0 is dense in E with respect to the metric d and invariant under every T_n, i.e. $T_n(E_0) \subset E_0$ for $n \in \mathbb{N}$;*

(ii) *d_0 is stronger than d, i.e.*

$$d(u,v) \le d_0(u,v) \quad \textit{for } u, v \in E_0.$$

Assume, moreover, that there exists an increasing sequence (n_k) of positive integers and a sequence (λ_k) of nonnegative real numbers so that

(iii) $\lim_{k\to\infty} \frac{1}{1-\lambda_k} \sum_{i=n_k}^{\infty} \sup_{u\in E_0} d_0(T_i(u), T_{i+1}(u)) = 0$;

(iv) *for each* $k \in \mathbb{N}$ *the transformation* T_{n_k} *restricted to* E_0 *is* λ_k*-contractive with respect to the metric* d_0.

Under the above assumptions the process $\{T(n,m)\}$ *is asymptotically stable and the unique element* $u_* \in E$ *described by condition* (1.1) *is such that the sequence* $(T_n(u_*))$ *tends to* u_*.

Proof. By conditions (iii), (iv) and T_n-invariance of E_0 we can use Theorem 1 for (E_0, d_0). From Theorem 1(b) and assumption (ii) it follows that for each $m \in \mathbb{N}$ we have

$$\lim_{n\to\infty} d(T(n,m)(u), T(n,m)(v)) = 0 \quad \text{for all } u, v \in E_0. \tag{1.4}$$

Since E_0 is dense in (E,d) and each T_n is nonexpansive with respect to d, (1.4) remains true for $u, v \in E$. The properties (a) and (ii) imply that for every $m \in \mathbb{N}$ and $u \in E_0$ the sequence $(T(n,m)(u))$ is also Cauchy with respect to the metric d, thus it is convergent in (E,d).

By what we have just shown, for each $m \in \mathbb{N}$ there exists exactly one point, say u_m, such that

$$\lim_{n\to\infty} d(T(n,m)v, u_m) = 0 \quad \text{for all } v \in E. \tag{1.5}$$

Fix an integer $m \ge 2$ and $u \in E$. Substituting $v = T(m,1)(u)$ into (1.5) we get

$$\lim_{n\to\infty} d(T(n,m)T(m,1)(u), u_m) = 0.$$

On the other hand, the sequence $(T(n,1)(u))$ tends to u_1. Since for each n sufficiently large $T(n,m)T(m,1)u = T(n,1)u$ and this sequence has exactly one limit point, u_m must be u_1. Moreover, by nonexpansiveness of T_n,

$$d(T_{n+1}(u_1), u_1) \le d(u_1, T(n+1,1)(u)) + d(T(n,1)(u_1), u_1) \quad \text{for } n \in \mathbb{N}.$$

From (1.5) it now follows that the sequence $(T_n(u_1))$ tends to u_1. ∎

Now consider a special case when every transformation is independent of n, i.e. $T_n = T$. Then obviously condition (iii) is satisfied and we have the following corollary, which was stated by A. Lasota [6].

COROLLARY 1. *Assume that a mapping* $T : E \to E$ *defined on a complete metric space is nonexpansive. Suppose there is a subset* $E_0 \subset E$ *and a metric* $d_0 : E_0 \times E_0 \to \mathbb{R}_+$ *such that*

(i′) E_0 *is dense in* E *with respect to the metric* d *and* T*-invariant*;

(ii′) d_0 *is stronger than* d;

(iii′) *the transformation* T *restricted to* E_0 *is* λ*-contractive with respect to the metric* d_0, *where* $\lambda < 1$ *is a constant.*

Then T has a unique fixed point u_* in E and

$$\lim_{n\to\infty} d(T^n(u), u_*) = 0 \quad \textit{for all } u \in E.$$

2. Proof of Theorem 1. We precede the proof of Theorem 1 with the following lemmas.

LEMMA 1. *Let (E, d) be a metric space. Assume that a sequence $(z_n)_{n\in\mathbb{N}}$ in E has the following property*:

(I) *For every $\varepsilon > 0$ there exists a Cauchy sequence $(v_n)_{n\in\mathbb{N}}$ in E such that*

$$\limsup_{n\to\infty} d(v_n, z_n) \le \varepsilon.$$

Then the sequence (z_n) is Cauchy in (E, d).

The proof of the above lemma is a straightforward consequence of condition (I).

LEMMA 2. *Let (E, d) be a metric space and T_n, $n \in \mathbb{N}$, be arbitrary transformations from E into itself. If there exists a $k \in \mathbb{N}$ and a nonnegative real number a_k so that*

$$d(T_k(u), T_k(v)) \le a_k d(u, v) \quad \textit{for all } u, v \in E, \tag{2.1}$$

then for every $z \in E$ and $n > k$, $n \in \mathbb{N}$,

$$\begin{aligned} d(T(n+1, k+1)(z), T_k^{n-k}(z)) \\ \le \sum_{i=k}^{n-1} \varepsilon_i + a_k d(T(n, k+1)(z), T_k^{n-k-1}(z)), \end{aligned} \tag{2.2}$$

where

$$\varepsilon_i = \sup_{u\in E} d(T_i(u), T_{i+1}(u)) \quad \textit{for } i \in \mathbb{N}. \tag{2.3}$$

Proof. Let $z \in E$. For each fixed $n > k$ define $y_n = T(n+1, k+1)(z)$ and $x_n = T_k^{n-k}(z)$. Observe that, according to the recurrent formulas $y_n = T_n(y_{n-1})$ and $x_n = T_k(x_{n-1})$, we have

$$d(y_n, x_n) \le \sum_{i=k}^{n-1} d(T_i(y_{n-1}), T_{i+1}(y_{n-1})) + d(T_k(y_{n-1}), T_k(x_{n-1})).$$

From this and assumption (2.1) it follows that

$$d(y_n, x_n) \le \sum_{i=k}^{n-1} \varepsilon_i + a_k d(y_{n-1}, x_{n-1}),$$

where ε_i are given by (2.3). The last inequality is equivalent to (2.2). ∎

Proof of Theorem 1. Fix a positive integer m. We begin by showing that for every $\varepsilon > 0$ there exists $k = k(\varepsilon, m) \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} d(T(n,m)(u), v_n(u)) \le \varepsilon \quad \text{for all } u \in E, \tag{2.4}$$

where $v_n(u) = T_{n_k}^{n-1-n_k}(T(n_k+1,m)(u))$ for $n > n_k$.

Given $\varepsilon > 0$, by assumption (1.2) we can choose k_0 so that

$$\frac{1}{1-\lambda_k}\sum_{i=n_k}^{\infty} \varepsilon_i < \varepsilon \quad \text{for } k \ge k_0, \tag{2.5}$$

where

$$\varepsilon_i = \sup_{u\in E} d(T_i(u), T_{i+1}(u)) \quad \text{for } i \in \mathbb{N}.$$

Let k be an integer such that $n_k > \max\{m, n_{k_0}\}$ and let $u \in E$. Applying Lemma 2 to the transformation T_{n_k} we infer that inequality (2.2) is valid for every $n > n_k$ and $z \in E$. In particular, for $z = T(n_k+1,m)(u)$ and $n > n_k$ we obtain

$$\begin{aligned} &d(T(n+1,n_k+1)(T(n_k+1,m)(u)), T_{n_k}^{n-k}(T(n_k+1,m)(u))) \\ &\quad\le \sum_{i=n_k}^{n-1}\varepsilon_i + \lambda_k d(T(n,n_k+1)(T(n_k+1,m)(u)), T_{n_k}^{n-1-k}(T(n_k+1,m)(u))). \end{aligned}$$

This estimate and (2.5) imply that

$$\begin{aligned} d(T(n+1,m)(u), v_{n+1}(u)) &\le (1-\lambda_k)\varepsilon + \lambda_k d(T(n,m)(u), v_n(u)), \\ \text{where} \quad v_n(u) &= T_{n_k}^{n-1-n_k}(T(n_k+1,m)(u)) \text{ for } n > n_k. \end{aligned}$$

It follows that the numerical sequence $(d(T(n,m)(u), v_n(u)))_{n>n_k}$ is bounded and that

$$\begin{aligned} &\limsup_{n\to\infty} d(T(n+1,m)(u), v_{n+1}(u)) \\ &\qquad\qquad \le (1-\lambda_k)\varepsilon + \lambda_k \limsup_{n\to\infty} d(T(n,m)(u), v_n(u)). \end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} d(T(n,m)(u), v_n(u)) \le \varepsilon,$$

which completes the proof of (2.4).

Since for each $k \in \mathbb{N}$ the transformation T_{n_k} is λ_k-contractive, the sequence $(T_{n_k}^{n-n_k}(z))_{n\ge n_k}$ is Cauchy for $z \in E$. From this and (2.4) it follows that for every $u \in E$ the sequence $(T(n,m)(u))_{n\ge m}$ satisfies condition (I) of Lemma 1, so the proof of (a) is complete.

To prove (b) fix $\varepsilon > 0$ and choose k such that (2.4) holds. Let $u, v \in E$. Clearly,

$$\begin{aligned} d(T(n,m)(u), T(n,m)(v)) \le{} & d(T(n,m)(u), T_{n_k}^{n-1-n_k}(T(n_k+1,m)(u))) \\ & + d(T(n,m)(v), T_{n_k}^{n-1-n_k}(T(n_k+1,m)(v))) \\ & + \lambda_k^{n-1-n_k} d(T(n_k+1,m)(u), T(n_k+1,m)(v)) \end{aligned}$$

for all $n > n_k$. By assumption, $\lambda_k < 1$, therefore the last term on the right-hand side converges to zero as $n \to \infty$. Hence and from (2.4) we obtain

$$\limsup_{n\to\infty} d(T(n,m)(u), T(n,m)(v)) < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of (b).

The second part of the theorem is obvious. ∎

3. Markov operators. Let (X, ϱ) be a Polish space, i.e. a separable complete metric space. We denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. The space of all finite Borel measures (nonnegative, σ-additive) on X will be denoted by $\mathcal{M}$. The subspace of $\mathcal{M}$ which contains only normalized measures (i.e. $\mu(X) = 1$, $\mu \in \mathcal{M}$) will be denoted by $\mathcal{M}_1$ and its elements will be called *distributions*. Furthermore,

$$\mathcal{M}_{\text{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\}$$

denotes the space of finite signed measures.

As usual, we denote by $B(X)$ the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ and by $C(X)$ its subspace containing all continuous functions. Both spaces are considered with the norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

For $f \in B(X)$ and $\mu \in \mathcal{M}_{\text{sig}}$ we write

$$\langle f, \mu\rangle = \int_X f(x)\, \mu(dx).$$

The space $\mathcal{M}_{\text{sig}}$ is a normed vector space with the *Fortet–Mourier norm* ([3], [9])

$$\|\mu\|_{\mathcal{F}} = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{F}\} \quad \text{for } \mu \in \mathcal{M}_{\text{sig}},$$

where

$$\mathcal{F} = \{f \in C(X) : \|f\| \le 1 \text{ and } |f(x) - f(y)| \le \varrho(x,y) \text{ for } x, y \in X\}.$$

In general, $(\mathcal{M}_{\text{sig}}, \|\cdot\|_{\mathcal{F}})$ is not a complete space. However, it is known that the set $\mathcal{M}_1$ with the distance $\|\mu_1 - \mu_2\|_{\mathcal{F}}$ is a complete metric space ([9]) and the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{F}} = 0 \quad \text{for } \mu_n, \mu \in \mathcal{M}_1$$

is equivalent to weak convergence of distributions defined by

$$\lim_{n\to\infty}\langle f,\mu_n\rangle = \langle f,\mu\rangle \quad \text{for all } f\in C(X).$$

In $\mathcal{M}_1$ we introduce another distance, the *Hutchinson metric* ([5], [6]):

$$\|\mu_1-\mu_2\|_{\mathcal{H}} = \sup\{|\langle f,\mu_1-\mu_2\rangle| : f\in\mathcal{H}\} \quad \text{for } \mu_1,\mu_2\in\mathcal{M}_1,$$

where

$$\mathcal{H} = \{f\in C(X) : |f(x)-f(y)|\le \varrho(x,y) \text{ for } x,y\in X\};$$

$\|\mu_1-\mu_2\|_{\mathcal{H}}$ is always defined but for some $\mu_1,\mu_2\in\mathcal{M}_1$ it may be infinite. Note that, because of the inclusion $\mathcal{F}\subset\mathcal{H}$, we always have

$$\|\mu_1-\mu_2\|_{\mathcal{F}} \le \|\mu_1-\mu_2\|_{\mathcal{H}} \quad \text{for } \mu_1,\mu_2\in\mathcal{M}_1.$$

A linear mapping $P:\mathcal{M}_{\mathrm{sig}}\to\mathcal{M}_{\mathrm{sig}}$ is called a *Markov operator* if $P(\mathcal{M}_1)\subset\mathcal{M}_1$ (see [6, 7, 9]). Now we will show how to construct a Markov operator.

Let a linear operator $U: B(X)\to B(X)$ be given. Assume that U satisfies the following conditions:

(U1) $Uf\ge 0$ for $f\in B(X)$, $f\ge 0$;

(U2) $U1_X = 1_X$;

(U3) if a nonincreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $B(X)$ is pointwise convergent to 0 then

$$\lim_{n\to\infty} Uf_n(x) = 0 \quad \text{for } x\in X;$$

(U4) $Uf\in C(X)$ for $f\in C(X)$.

Define an operator $P:\mathcal{M}_{\mathrm{sig}}\to\mathcal{M}_{\mathrm{sig}}$ by

$$P\mu(A) = \langle U1_A,\mu\rangle \quad \text{for } A\in\mathcal{B}_X,\ \mu\in\mathcal{M}_{\mathrm{sig}}. \tag{3.1}$$

It can be easily shown (see [6]) that P is the unique Markov operator satisfying

$$\langle Uf,\mu\rangle = \langle f,P\mu\rangle \quad \text{for } f\in B(X),\ \mu\in\mathcal{M}_{\mathrm{sig}}, \tag{3.2}$$

so U is the dual operator to P. In particular, substituting $\mu=\delta_x$ into (3.2) we obtain

$$Uf(x) = \langle f,P\delta_x\rangle \quad \text{for } x\in X,\ f\in B(X),$$

where $\delta_x\in\mathcal{M}_1$ is the point (Dirac) unit measure supported at x.

We call P a *Feller operator* if its dual operator U satisfies condition (U4).

Finally, for convenience, we present some facts concerning Markov operators which we need in the sequel (see [6]).

PROPOSITION 1. *Let $P:\mathcal{M}_{\mathrm{sig}}\to\mathcal{M}_{\mathrm{sig}}$ be a Feller operator and let its dual operator U satisfy*

$$|Uf(x)-Uf(\overline{x})|\le\lambda\varrho(x,\overline{x}) \quad \textit{for } x,\overline{x}\in X \textit{ and } f\in\mathcal{H},$$

where $\lambda \leq 1$ *is a nonnegative constant. Then* P *is nonexpansive with respect to the Fortet–Mourier norm and*

$$\|P\mu_1 - P\mu_2\|_{\mathcal{H}} \leq \lambda\|\mu_1 - \mu_2\|_{\mathcal{H}} \quad \textit{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{3.3}$$

If, *moreover*, *there is a measure* $\nu \in \mathcal{M}_1$ *such that*

$$\|P\nu - \nu\|_{\mathcal{H}} < \infty, \tag{3.4}$$

then $\mathcal{M}_0 = \{\mu \in \mathcal{M}_1 : \|\mu - \nu\|_{\mathcal{H}} < \infty\}$ *is a dense and* P*-invariant subset of the metric space* $(\mathcal{M}_1, \|\cdot\|_{\mathcal{F}})$, *and it is a metric space when equipped with the Hutchinson metric.*

4. Dynamical systems. Throughout this section (X, ϱ) is a Polish space and $(I, \mathcal{A})$ is a measurable space. We consider dynamical systems in a general form (for the homogeneous cases see [7–8, 10]). Let $(\Omega, \Sigma, \mathrm{prob})$ be a probability space and let $\eta_n : \Omega \to I$, $n \in \mathbb{N}$, be a sequence of independent random elements (measurable transformations) having the same distribution, i.e. the measure

$$\psi(A) = \mathrm{prob}(\eta_n \in A) \quad \text{for } A \in \mathcal{A}$$

is the same for all n. Assume that for each $n \in \mathbb{N}$ a measurable transformation $S_n : X \times I \to X$ is given.

Consider a sequence $\xi_n : \Omega \to X$ of random elements defined by the recurrent formula

$$\xi_n = S_n(\xi_{n-1}, \eta_n) \quad \text{for } n \in \mathbb{N}, \tag{4.1}$$

where the initial value $\xi_0 : \Omega \to X$ is a random element independent of the sequence (η_n).

We make the following assumptions:

(A1) For each n there exists a measurable function $L_n : I \to \mathbb{R}_+$ such that

$$\varrho(S_n(x, y), S_n(\overline{x}, y)) \leq L_n(y)\varrho(x, \overline{x}) \quad \text{for } x, \overline{x} \in X,\ y \in I \tag{4.2}$$

and

$$a_n = \int_I L_n(y)\, \psi(dy) \leq 1. \tag{4.3}$$

(A2) There exists a point $x_0 \in X$ such that

$$b_n = \int_I \varrho(x_0, S_n(x_0, y))\, \psi(dy) < \infty \quad \text{for } n \in \mathbb{N}.$$

(A3) There exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers so that

$a_{n_k} < 1$ for $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \frac{1}{1-a_{n_k}} \sum_{i=n_k}^{\infty} \sup_{x\in X} \int_I \varrho(S_i(x,y), S_{i+1}(x,y))\, \psi(dy) = 0.$$

The sequence given by (4.1) is a Markov process for which the one-step transition function may depend on n. We now give a rule on how the distributions of ξ_n evolve in time by means of Markov operators. For each integer n define an operator U_n acting on $B(X)$ by setting

$$U_n f(x) = \int_I f(S_n(x,y))\, \psi(dy) \quad \text{for } x \in X,\ f \in B(X). \tag{4.4}$$

Of course, $U_n : B(X) \to B(X)$ is a linear operator satisfying (U1)–(U3). Moreover, from (4.2) it follows that for every $y \in I$ the transformation $S_n(\cdot, y) : X \to X$ is continuous, therefore $U_n f \in C(X)$ for $f \in C(X)$. Hence, according to (3.1), the Markov operator P_n is of the form

$$P_n\mu(A) = \int_X \Big\{ \int_I 1_A(S_n(x,y))\, \psi(dy) \Big\} \mu(dx) \quad \text{for } A \in \mathcal{B}_X,\ \mu \in \mathcal{M}_{\text{sig}}.$$

We are interested in the asymptotic behaviour of the distributions

$$\mu_n(A) = \operatorname{prob}(\xi_n \in A) \quad \text{for } A \in \mathcal{B}_X,\ n = 0, 1, 2, \ldots,$$

where (ξ_n) is defined by (4.1). Using the form of P_n it is easy to check (see [7]) that

$$\mu_n = P_n \mu_{n-1} \quad \text{for } n \in \mathbb{N}.$$

Consequently, $\mu_n = P(n+1, 1)\mu_0$, $n \in \mathbb{N}$.

Now, using Theorem 2 we can prove the main result of this section, which is a nonhomogeneous (in time) version of a result due to A. Lasota and M. C. Mackey [7] (p. 423).

THEOREM 3. *Assume that the sequence* (S_n) *satisfies* (A1)–(A3). *Then there exists a unique measure* $\mu_* \in \mathcal{M}_1$ *such that* $\lim_{n\to\infty} \|P_n\mu_* - \mu_*\|_{\mathcal{F}} = 0$ *and*

$$\lim_{n\to\infty} \|P(n,m)\mu - \mu_*\|_{\mathcal{F}} = 0 \quad \textit{for all } \mu \in \mathcal{M}_1,\ m \in \mathbb{N}. \tag{4.5}$$

Proof. We show that the Markov operators $P_n : \mathcal{M}_1 \to \mathcal{M}_1$, $n \in \mathbb{N}$, satisfy the requirements of Theorem 2. Fix n. It is easy to calculate that, in view of (4.4) and (A1),

$$|U_n f(x) - U_n f(\overline{x})| \le a_n \varrho(x, \overline{x}) \quad \text{for } x, \overline{x} \in X \text{ and } f \in \mathcal{H},$$

where, according to (4.3), $a_n \le 1$. Now, we are going to verify that

$$\|P_n \delta_{x_0} - \delta_{x_0}\|_{\mathcal{H}} \le b_n,$$

where x_0 and b_n are described in (A2). Indeed, if $f \in \mathcal{H}$ then $|\langle f, P_n\delta_{x_0} - \delta_{x_0}\rangle| = |U_n f(x_0) - f(x_0)|$. Since $\psi(I) = 1$, we have $f(x_0) = \int_I f(x_0)\, \psi(dy)$, and consequently,

$$|\langle f, P_n\delta_{x_0} - \delta_{x_0}\rangle| \le \int_I \varrho(S_n(x_0, y), x_0)\,\psi(dy).$$

The right-hand side does not depend on f, hence the desired estimate follows. Thus, by Proposition 1 the Markov operator P_n is nonexpansive with respect to the Fortet–Mourier metric and the metric space $(\mathcal{M}_0, \|\cdot\|_{\mathcal{H}})$ satisfies condition (i) of Theorem 2, where

$$\mathcal{M}_0 = \{\mu \in \mathcal{M}_1 : \|\mu - \delta_{x_0}\|_{\mathcal{H}} < \infty\}.$$

Moreover, by (3.3) we have $\|P_n\mu_1 - P_n\mu_2\|_{\mathcal{H}} \le a_n\|\mu_1 - \mu_2\|_{\mathcal{H}}$ for all n, and $a_{n_k} < 1$ for all $k \in \mathbb{N}$ by (A3), therefore condition (iv) is satisfied as well.

It remains to verify (iii). Observe that for $f \in \mathcal{H}$ and $\mu \in \mathcal{M}_0$ we have $|\langle f, P_n\mu - P_{n+1}\mu\rangle| = |\langle U_n f - U_{n+1} f, \mu\rangle| \le \|U_n f - U_{n+1} f\|$ for all $n \in \mathbb{N}$. The last term can be estimated as follows:

$$\begin{aligned}\|U_n f - U_{n+1} f\| &\le \sup_{x\in X}\int_I |f(S_n(x,y)) - f(S_{n+1}(x,y))|\,\psi(dy)\\ &\le \sup_{x\in X}\int_I \varrho(S_n(x,y), S_{n+1}(x,y))\,\psi(dy).\end{aligned}$$

The right-hand side does not depend on $f \in \mathcal{H}$ and $\mu \in \mathcal{M}_0$, thus

$$\sup_{\mu\in\mathcal{M}_0} \|P_n\mu - P_{n+1}\mu\|_{\mathcal{H}} \le \sup_{x\in X}\int_I \varrho(S_n(x,y), S_{n+1}(x,y))\,\psi(dy),$$

which, according to (A3), proves condition (iii). Consequently, making use of Theorem 2 completes the proof. ■

Now, we give some examples of applications of Theorem 3. First, we consider iterated function systems [1–2, 6–8, 9, 10]. In our case transformations vary in each step.

Example 1. Let N be a positive integer and for each $n \in \mathbb{N}$ let $S_i^n : X \to X$, $i = 1, \ldots, N$, be a sequence of transformations such that

$$\varrho(S_i^n(x), S_i^n(\overline{x})) \le L_i^n\,\varrho(x, \overline{x}) \quad \text{for } x, \overline{x} \in X.$$

Moreover, let p_i, $i = 1, \ldots, N$, be a sequence of positive numbers such that $p_1 + \ldots + p_N = 1$. We define a random sequence (ξ_n) in the following way. If an initial point x_0 is given, we select a transformation S_i^1 with probability p_i and define $x_1 = S_i^1(x_0)$. Having defined the points $x_1, \ldots, x_n$ we select a transformation S_i^{n+1} with probability p_i and define $x_{n+1} = S_i^{n+1}(x_n)$. This scheme can be described in terms of the following dynamical system. Let $I = \{1, \ldots, N\}$ and let $\eta_n : \Omega \to I$, $n \in \mathbb{N}$, be a sequence of independent random variables with $\mathrm{prob}(\eta_n = i) = p_i$. Set $S_n(x, i) = S_i^n(x)$ for $x \in X$, $i \in I$, $n \in \mathbb{N}$.

If we assume that $a_n = \sum_{i=1}^{N} p_i L_i^n \le 1$ for $n \in \mathbb{N}$, $\liminf_{n\to\infty} a_n < 1$, and the series $\sum_{n=1}^{\infty} \sup_{x\in X} \varrho(S_i^n(x), S_i^{n+1}(x))$ is convergent for each $i \in I$, then all the assumptions of Theorem 3 are satisfied. Thus, the process $\{P(n,m)\}$ generated by the Markov operators

$$P_n\mu(A) = \sum_{i=1}^{N} p_i \mu((S_i^n)^{-1}(A)) \quad \text{for } A \in \mathcal{B}_X,\ \mu \in \mathcal{M}_1,\ n \in \mathbb{N}$$

is asymptotically stable.

The next example concerns dynamical systems with multiplicative perturbations [4, 11].

Example 2. Let $(X, \|\cdot\|)$ be a separable Banach space or a closed cone in such a space and $I = [0,\infty)$. For each $n \in \mathbb{N}$ consider the map $S_n : X \times I \to X$ of the form

$$S_n(x,y) = yT_n(x) \quad \text{for } x \in X,\ y \in I,$$

where $T_n : X \to X$ satisfies $\|T_n(x) - T_n(\overline{x})\| \le c_n \|x - \overline{x}\|$ for $x, \overline{x} \in X$ with a nonnegative constant c_n. Assume that the first moment of the random variables $\eta_n : \Omega \to I$ is finite, i.e.

$$\int_I y\,\psi(dy) = K < \infty.$$

If $c_n K \le 1$ for $n \in \mathbb{N}$, $\liminf_{n\to\infty} c_n < 1/K$ and $\sum_{n=1}^{\infty} \sup_{x\in X} \|T_n(x) - T_{n+1}(x)\|$ is convergent, then all the assumptions of Theorem 3 are satisfied. Thus, the process $\{P(n,m)\}$ generated by the Markov operators

$$P_n\mu(A) = \int_X \Big\{ \int_I 1_A(yT_n(x))\,\psi(dy) \Big\} \mu(dx) \quad \text{for } A \in \mathcal{B}_X,\ \mu \in \mathcal{M}_1,\ n \in \mathbb{N}$$

is asymptotically stable.

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