ANNALES
POLONICI MATHEMATICI
LXXI.1 (1999)

On bifurcation intervals for nonlinear eigenvalue problems

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Abstract. We give a sufficient condition for $[\mu - M, \mu + M] \times \{0\}$ to be a bifurcation interval of the equation $u = L(\lambda u + F(u))$, where L is a linear symmetric operator in a Hilbert space, $\mu \in r(L)$ is of odd multiplicity, and F is a nonlinear operator. This abstract result provides an elementary proof of the existence of bifurcation intervals for some eigenvalue problems with nondifferentiable nonlinearities. All the results obtained may be easily transferred to the case of bifurcation from infinity.

1. The abstract result. Following Berestycki [1], by a *bifurcation interval* we understand an interval which contains at least one bifurcation point. The purpose of this paper is to provide a sufficient condition for bifurcation of equations of the form

$$u = L(\lambda u + F(u)). \tag{1}$$

Let $(E, \| \, \|_E)$ be a real Banach space imbedded in a Hilbert space $(H, \| \, \|_H)$. We assume that L is a linear symmetric operator in H, $\operatorname{Range} L \subset E$, and $L : E \to E$ is compact. We denote by $r(L)$ the set of characteristic values of L. The nonlinear operator $F : E \to H$ satisfies the following conditions:

$$L \circ F : E \to E \text{ is compact and } \exists_{M>0} \, \forall_{u \in E} \ \|F(u)\|_H \le M \|u\|_H. \tag{2}$$

By a *solution* of (1) is meant a pair $(\lambda, u) \in \mathbb{R} \times E$ satisfying (1). In particular, (1) has the line of trivial solutions.

Now we show that (1) possesses no nontrivial solutions when $\lambda \notin [\mu - M, \mu + M]$ for $\mu \in r(L)$.

Theorem 1. *If* (λ, u) *is a nontrivial solution of* (1), *then*

$$\operatorname{dist}(\lambda, r(L)) \le M. \tag{3}$$

1991 *Mathematics Subject Classification*: Primary 35P30.

Key words and phrases: bifurcation interval, symmetric operator, Sturm–Liouville problem, Dirichlet problem, Leray–Schauder degree, characteristic values.

P r o o f. The pair (λ, u) satisfies (1). For $\lambda \notin r(L)$, $I - \lambda L$ is invertible, so that (1) is equivalent to

$$u = (I - \lambda L)^{-1} L F(u).$$

Hence

$$\|u\|_H \le \|(I - \lambda L)^{-1} L\| \cdot \|F(u)\|_H \le \|(I - \lambda L)^{-1} L\| \cdot M \cdot \|u\|_H,$$

so

$$1 \le \|(I - \lambda L)^{-1} L\| \cdot M.$$

From the spectral mapping theorem for symmetric operators ([3], p. 273) we have $\|(I - \lambda L)^{-1} L\|^{-1} = \mathrm{dist}(\lambda, r(L))$, which completes the proof.

We can now formulate the main theorem.

THEOREM 2. *If* $\mu \in r(L)$ *is of odd multiplicity and* $\mathrm{dist}(\mu, r(L) - \{\mu\}) > 2M$, *then* $[\mu - M, \mu + M] \times \{0\}$ *is a bifurcation interval for* (1).

P r o o f. It is sufficient to show that for any $\delta > 0$ there exists a solution (λ, u) of (1) with $\|u\|_E = \delta$ and $\lambda \in [\mu - M, \mu + M]$. Fix δ and set $B = \{u \in E : \|u\|_E < \delta\}$. By assumption, there exists $\varepsilon > 0$ such that $\mathrm{dist}(\mu, r(L) - \{\mu\}) = 2M + \varepsilon$. Set $\underline{\lambda} = \mu - M - \varepsilon/2$ and $\overline{\lambda} = \mu + M + \varepsilon/2$. It is clear that

$$\mathrm{dist}(\underline{\lambda}, r(L)) = \mathrm{dist}(\overline{\lambda}, r(L)) = M + \varepsilon/2. \tag{4}$$

From (3) it follows that it suffices to prove the existence of a solution $(\lambda, u) \in [\underline{\lambda}, \overline{\lambda}] \times \partial B$.

We argue by contradiction, so suppose that $u \ne L(\lambda u + F(u))$ for all $u \in \partial B$ and $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. Since $L \circ (\lambda I + F)$ is compact on $\overline{B}$, the Leray–Schauder degree $d(\Phi(\lambda), B, 0)$ of $\Phi(\lambda) = I - L \circ (\lambda I + F)$ with respect to B and the point 0 is well defined for all $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. By the homotopy invariance of the degree,

$$d(\Phi(\lambda), B, 0) = \mathrm{const} \qquad \text{for } \lambda \in [\underline{\lambda}, \overline{\lambda}].$$

In particular we have

$$d(\Phi(\underline{\lambda}), B, 0) = d(\Phi(\overline{\lambda}), B, 0). \tag{5}$$

Consider now the first term in (5). Notice that $u \ne L(\underline{\lambda} u + tF(u))$ for $u \in \partial B$ and $t \in [0, 1]$. If not, proceeding as in the proof of Theorem 1, we obtain $\mathrm{dist}(\underline{\lambda}, r(L)) \le M$, which contradicts (4).

So, using the homotopy invariance again we obtain

$$d(\Phi(\underline{\lambda}), B, 0) = d(I - \underline{\lambda} L, B, 0) = i(\underline{\lambda}).$$

The same argument can be used for $\overline{\lambda}$. Hence $i(\underline{\lambda}) = i(\overline{\lambda})$. Since μ is the only characteristic value of L in $[\underline{\lambda}, \overline{\lambda}]$ and μ is of odd multiplicity k_μ, it follows that $i(\underline{\lambda}) \cdot (-1)^{k_\mu} = i(\overline{\lambda}) \ne 0$, which is impossible. The theorem is proved.

Assuming additionally that

$$\exists_{m\in\mathbb{R}} \quad (L\circ F)'(0) = mL \tag{6}$$

we obtain

THEOREM 3. *Under the assumptions of Theorem* 2, *if moreover* (6) *holds*, *then the bifurcation interval* $[\mu - M, \mu + M]\times\{0\}$ *degenerates to the point* $(\mu - m, 0)$.

P r o o f. Let $(\lambda, 0) \in [\mu - M, \mu + M]\times\{0\}$ be a bifurcation point for (1). This means that there exists a sequence $\mathbb{R}\times E \ni (\lambda_n, u_n) \to (\lambda, 0)$ satisfying (1). Dividing (1) by $\|u_n\|_E$ and setting $u_n/\|u_n\|_E = w_n$ yields

$$w_n = L\biggl(\lambda_n w_n + \frac{F(u_n)}{\|u_n\|_E}\biggr),$$

which is equivalent to

$$w_n = (\lambda_n + m)Lw_n + \frac{L(F(u_n)) - mLu_n}{\|u_n\|_E}. \tag{7}$$

The second term on the right tends to 0 as $n\to\infty$. Since L is compact, a subsequence of Lw_n converges. Hence the left-hand side of (7) has a convergent subsequence $w_{n_k} \to w$ with $\|w\|_E = 1$ and $w = (\lambda + m)Lw$. Consequently, $\lambda + m \in r(L)$.

Since $|\lambda - \mu| \le M$ and $|m| \le M$ we obtain $|\lambda + m - \mu| \le 2M$, which is only possible when $\lambda = \mu - m$. The proof is complete.

2. Applications to differential equations. Perhaps the nicest applications of Theorem 2 are to nonlinear Sturm–Liouville boundary value problems with nondifferentiable nonlinearities. Consider

$$\mathcal{L}u = -(pu')' + qu = \lambda u + f(\cdot, u, u') \quad \text{ in } (0,\pi) \tag{8}$$

together with the separated boundary conditions

$$a_1u(0) - \alpha_2u'(0) = 0, \qquad \beta_1u(\pi) + \beta_2u'(\pi) = 0, \tag{8.1}$$

where $\alpha_i, \beta_i \ge 0$ and $(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2) \ne 0$. As usual we assume $p\in C^1[0,\pi]$, $q \in C[0,\pi]$ and $p > 0$, $q \ge 0$ on $[0,\pi]$. Suppose f is continuous on $[0,\pi]\times\mathbb{R}^2$ and satisfies

$$\exists_{M>0}\ \forall_{(x,\xi,\eta)\in[0,\pi]\times\mathbb{R}^2} \quad |f(x,\xi,\eta)| \le M|\xi|.$$

For $f = 0$ the boundary value problem (8), (8.1) becomes a linear Sturm–Liouville problem, which has an increasing sequence of simple eigenvalues $0 < \mu_1 < \mu_2 < \ldots$ with $\lim_{k\to\infty}(\mu_k - \mu_{k-1}) = \infty$. This means in particular that the intervals $[\mu_k - M, \mu_k + M]$ are disjoint for k large enough.

The boundary value problem (8), (8.1) is equivalent to the integral equation

$$u(\cdot)=\int_0^\pi g(\cdot,y)[\lambda u(y)+f(y,u(y),u'(y))]\,dy=L(\lambda u+F(u)),$$

where

$$Lu=\int_0^\pi g(\cdot,y)u(y)\,dy,\qquad F(u)=f(\cdot,u,u'),$$

and g is the Green function for $\mathcal{L}$ with (8.1).

Taking $H=L^2(0,\pi)$ and $E=\{u\in C^1[0,\pi]:u$ satisfies (8.1)$\}$ it is easily seen that all the requirements on L and F are satisfied. Thus we have the following theorem.

THEOREM 4. *For every* $k>k_0=\min\{\overline{k}\in\mathbb{N}:\mu_k-\mu_{k-1}>2M$ *for* $k>\overline{k}\}$, $[\mu_k-M,\mu_k+M]\times\{0\}$ *is a bifurcation interval for* (8), (8.1).

REMARK 1. This result is well known and may be found in many papers ([1], [2], [7]) presenting different viewpoints and different methods.

The same theorem is formulated and proved in [5] for compositions of Sturm–Liouville operators.

REMARK 2. If we assume additionally that

$$\exists_{m\in\mathbb{R}}\ \forall_{x\in[0,\pi]}\qquad \lim_{(\xi,\eta)\to(0,0)}\frac{f(x,\xi,\eta)}{\xi}=m,$$

then, according to Theorem 3, the bifurcation interval $[\mu_k-M,\mu_k+M]\times\{0\}$ degenerates to the point $(\mu_k-m,0)$.

We conclude this section with some applications to nonlinear elliptic partial differential equations.

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a smooth boundary. Consider the Dirichlet boundary value problem

$$\text{(9)}\qquad \begin{cases}\mathcal{N}u=-\displaystyle\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+qu=\lambda u+f(\cdot,u,Du) & \text{in } \Omega,\\ u|_{\partial\Omega}=0,\end{cases}$$

where $Du=(\partial u/\partial x_1,\ldots,\partial u/\partial x_n)$.

We assume that $\mathcal{N}$ is uniformly elliptic in $\overline{\Omega}$, $a_{ij}=a_{ji}>0$, $a_{ij}\in C^1(\overline{\Omega})$ and $0\le q\in C(\overline{\Omega})$.

Suppose f is continuous on $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$ and satisfies

$$\exists_{M>0}\ \forall_{(x,\xi,\eta)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n}\qquad |f(x,\xi,\eta)|\le M|\xi|.$$

For $p > 1$, let $W_0^{m,p}(\Omega)$ denote the closure of $C_0^m(\overline{\Omega}) = \{u \in C^m(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ with respect to the norm

$$|u|_{m,p} = \Big(\sum_{|\alpha|\le m}\int_\Omega |D^\alpha u(x)|^p dx\Big)^{1/p},$$

and let $|\cdot|_{0,p} = |\cdot|_p$. Clearly $W_0^{0,p}(\Omega) = L^p(\Omega)$.

Let $E = C_0^1(\overline{\Omega})$ with its usual norm $|u|_E = \sum_{|\alpha|\le 1}\max_{x\in\overline{\Omega}}|D^\alpha u(x)|$. For $p > n$, $W_0^{2,p}(\Omega)$ is compactly imbedded in E. A couple $(\lambda, u) \in \mathbb{R}\times E$ is said to be a *solution* of (9) if $u \in W_0^{2,p}(\Omega)$ and (λ, u) satisfies (9).

If $f = 0$ then the linear problem $\mathcal{N}u = \mu u$ in Ω, $u|_{\partial\Omega} = 0$ has a smallest positive eigenvalue μ_1 which is simple.

Using the L^p theory for uniformly elliptic partial differential equations ([4], §2.5) we can convert (9) into an equivalent operator equation in $\mathbb{R}\times E$ of the form (1). Namely, let $(\lambda, u) \in \mathbb{R}\times E$ and consider the linear boundary value problem

$$\mathcal{N}w = \lambda u + f(\cdot, u, Du) \quad \text{in } \Omega, \qquad w|_{\partial\Omega} = 0. \tag{10}$$

Since the right-hand side of (10) as a continuous function on $\overline{\Omega}$ is in $L^p(\Omega)$ for all $p > 1$, there exists a unique solution $w = G(\lambda, u) \in W_0^{2,p}(\Omega)$ of (10) satisfying

$$|w|_{2,p} \le \text{const}\,|\lambda u + f(\cdot, u, Du)|_p. \tag{11}$$

Thus choosing any $p > n$ we obtain the compactness of $G : \mathbb{R}\times E \to E$. Any solution of (9) satisfies $u = G(\lambda, u)$ and conversely.

Let $z \in C(\overline{\Omega})$ and let $w = Lz$ denote the unique solution of $\mathcal{N}w = z$ in Ω, $w|_{\partial\Omega} = 0$. Then $L : L^p(\Omega) \supset C(\overline{\Omega}) \to W_0^{2,p}(\Omega)$ is well defined for any $p > 1$. In particular L is linear and symmetric in $H = L^2(\Omega)$ (since $\mathcal{N}$ is symmetric and Range $\mathcal{N}$ is dense in H) and $L : C(\overline{\Omega}) \to E$ is compact. Hence $G(\lambda, u) = \lambda Lu + L(F(u))$, where $F(u) = f(\cdot, u, Du)$.

Thus we have shown that problem (9) is equivalent to (1), so the next result is an immediate consequence of Theorem 2:

THEOREM 5. *If* $\operatorname{dist}(\mu_1, r(L) - \{\mu_1\}) > 2M$, *then* $[\mu_1 - M, \mu_1 + M]\times\{0\}$ *is a bifurcation interval for* (9).

REMARK 3. The same result may be proved for symmetric and positive definite operators $\mathcal{N} = \mathcal{N}_1 \circ \mathcal{N}_0$, where

$$\mathcal{N}_k u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial u}{\partial x_j}\right) + q^k u, \qquad k = 0, 1,$$

with boundary conditions $u|_{\partial\Omega} = 0$, $\mathcal{N}_0 u|_{\partial\Omega} = 0$.

Consider as an example the following problem:

$$\Delta^2 u = \lambda u + f(\cdot, u, Du) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad -\Delta u|_{\partial\Omega} = 0. \tag{12}$$

Let $E = \{u \in C^3(\overline{\Omega}) : u|_{\partial\Omega} = 0,\ -\Delta u|_{\partial\Omega} = 0\}$, $H = L^2(\Omega)$. We only need to verify that (12) is equivalent to (1). For fixed $(\lambda, u) \in \mathbb{R} \times E$ the linear problem

$$\Delta^2 v = \lambda u + f(\cdot, u, Du) \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0, \quad -\Delta v|_{\partial\Omega} = 0,$$

can be replaced by the following coupled problem:

$$\begin{gathered} -\Delta w = \lambda u + f(\cdot, u, Du) \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0, \\ -\Delta v = w \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0. \end{gathered}$$

The L^p theory implies the existence of a unique $w \in W_0^2(\Omega)$ satisfying (11). Arguing as above once again we obtain a unique solution $v = G(\lambda, u) \in W_0^4(\Omega)$ satisfying

$$|v|_{4,p} \le \text{const}\, |w|_{2,p} \le \text{const}\, |\lambda u + f(\cdot, u, Du)|_p.$$

Since $W_0^{4,p}(\Omega) \subset E$ is compact for $p > n$, problem (12) is equivalent to (1) with L defined in the same way as in the proof of Theorem 5.

3. Bifurcation from infinity. The theorems obtained may be transferred to the case of bifurcation from infinity.

Following Rabinowitz [6] we say (λ, ∞) is a *bifurcation point* for (1) if every neighbourhood of (λ, ∞) contains solutions of (1), i.e. there exists a sequence (λ_n, u_n) of solutions of (1) such that $\lambda_n \to \lambda$ and $\|u_n\|_E \to \infty$. We follow the standard pattern and perform the change of variable $w = u\|u\|_E^{-2}$ for $u \ne 0$. Dividing (1) by $\|u\|_E^2$ yields

$$w = L(\lambda w + \widetilde{F}(w)), \tag{13}$$

where

$$\widetilde{F}(w) = \begin{cases} \|w\|_E^2 F(w/\|w\|_E^2) & \text{for } w \ne 0, \\ 0 & \text{for } w = 0. \end{cases}$$

Thus $(\lambda, 0)$ is a bifurcation point for (13) if and only if (λ, ∞) is a bifurcation point for (1). Since $\widetilde{F}$ satisfies (2) we obtain the "infinite version" of Theorem 2:

THEOREM 6. *If $\mu \in r(L)$ is of odd multiplicity and* $\operatorname{dist}(\mu, r(L) - \{\mu\}) > 2M$, *then $[\mu - M, \mu + M] \times \{\infty\}$ is a bifurcation interval for* (1).

Applying Theorem 6 to problem (8) mentioned above we find that $[\mu_k - M, \mu_k + M] \times \{\infty\}$ is a bifurcation interval for k sufficiently large.

Remark 4. The same result may be formulated for compositions of Sturm–Liouville operators and, for the first eigenvalue μ_1, for elliptic operators $\mathcal{N}$ considered earlier.

Remark 5. It is easy to check that if we replace condition (6) by

$$\exists_{m\in\mathbb{R}} \qquad \lim_{\|u\|_E\to\infty} \frac{\|L(F(u)) - mLu\|_E}{\|u\|_E} = 0, \tag{14}$$

then $\widetilde{F}$ satisfies (6) and the "infinite version" of Theorem 3 also holds:

Theorem 7. *Under assumption* (14), *the bifurcation interval* $[\mu - M, \mu + M] \times \{\infty\}$ *degenerates to the point* $(\mu - m, \infty)$.

Example 1. Consider

$$-u'' = \lambda u + |u|\cos(u^2 + u'^2)^{1/2} \quad \text{in } (0,\pi), \quad u(0) = u(\pi) = 0. \tag{15}$$

We have the family of solutions $(\lambda_\gamma, u_\gamma) = (1 - (\operatorname{sgn}\gamma)\cos|\gamma|, \gamma\sin x) \in [0,2]\times E$. It is clear that $[0,2]\times\{0\}$ and $[0,2]\times\{\infty\}$ are bifurcation intervals for (15). The interval $[0,2]\times\{0\}$ contains two bifurcation points $(0,0)$ and $(2,0)$ while all the points of $[0,2]\times\{\infty\}$ are bifurcation points.

Example 2. Consider

$$\begin{aligned} &u^{(4)} = \lambda u + u\arctan[1 + (u'')^2 + (u''')^2] \quad \text{in } (0,\pi),\\ &u(0) = u(\pi) = u''(0) = u''(\pi) = 0. \end{aligned} \tag{16}$$

The equation has the family of solutions $(\lambda_\gamma, u_\gamma) = (1 - \arctan(1+\gamma^2), \gamma\sin x) \in [1-\pi/2, 1+\pi/2]\times E$. The bifurcation intervals $[1-\pi/2, 1+\pi/2]\times\{0\}$ and $[1-\pi/2, 1+\pi/2]\times\{\infty\}$ degenerate to the points $(1-\pi/4, 0)$ and $(1-\pi/2, \infty)$ respectively.

Acknowledgements. This work was partially supported by local grant 10.420.03.

References

[1] H. Berestycki, *On some Sturm–Liouville problems*, J. Differential Equations 26 (1977), 375–390.

[2] R. Chiappinelli, *On eigenvalues and bifurcation for nonlinear Sturm–Liouville operators*, Boll. Un. Mat. Ital. A (6) 4 (1985), 77–83.

[3] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[4] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, New York Univ. Lecture Notes, 1973–74.

[5] J. Przybycin, *Nonlinear eigenvalue problems for fourth order ordinary differential equations*, Ann. Polon. Math. 60 (1995), 249–253.

[6] P. H. Rabinowitz, *On bifurcation from infinity*, J. Differential Equations 14 (1973), 462–475.
[7] K. Schmitt and H. L. Smith, *On eigenvalue problems for nondifferentiable mappings*, J. Differential Equations 33 (1979), 294–319.

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Reçu par la Rédaction le 23.2.1998
Révisé le 14.10.1998