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Concave iteration semigroups of linear set-valued functions

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Abstract. We consider a concave iteration semigroup of linear continuous set-valued functions defined on a closed convex cone in a separable Banach space. We prove that such an iteration semigroup has a selection which is also an iteration semigroup of linear continuous functions. Moreover it is majorized by an "exponential" family of linear continuous set-valued functions.

Let X be a real normed space. We denote by $n(X)$ the family of all nonempty subsets of X. The families $c(X)$ and $cc(X)$ consist of all compact and all compact convex members of $n(X)$, respectively. We consider the space $cc(X)$ with the Hausdorff metric h induced by the norm in X. For the properties of the Hausdorff metric and the convergence in the space $(cc(X), h)$ see [1], [2] or [3]. Some of them needed here are also collected in [5].

If X, Y, Z are nonempty sets and $F : X \to n(Y)$ is any set-valued function (s.v. function for brevity) we define the sets

$$F(A) := \bigcup \{F(x) : x \in A\},$$
$$F^{-}(B) := \{x \in X : F(x) \cap B \neq \emptyset\},$$
$$F^{+}(B) := \{x \in X : F(x) \subset B\},$$

for every $A \subset X$ and $B \subset Y$.

The composition $G \circ F$ of $F : X \to n(Y)$ and $G : Y \to n(Z)$ is the s.v. function given as follows:

$$(G \circ F)(x) := G(F(x)) \quad \text{ for } x \in X.$$

Assume that X, Y are metric spaces. We say that an s.v. function $F : X \to n(Y)$ is *lower semicontinuous* (resp. *upper semicontinuous*) iff

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the set $F^{-}(U)$ (resp. $F^{+}(U)$) is open for every open set U in Y. F is said to be *continuous* iff it is both lower and upper semicontinuous.

A family $\{F^t : t \geq 0\}$ of s.v. functions $F^t : X \to n(X)$ is called an *iteration semigroup* iff

$$F^t \circ F^s = F^{t+s} \quad \text{for all } s, t \geq 0.$$

We say that an iteration semigroup $\{F^t : t \geq 0\}$ is *continuous* iff for every $x \in X$ the s.v. function $t \mapsto F^t(x)$ is continuous.

An iteration semigroup $\{F^t : t \geq 0\}$ is *concave* iff

$$F^{\lambda s+(1-\lambda)t}(x) \subset \lambda F^s(x) + (1-\lambda)F^t(x)$$

for all $s, t \geq 0$, $\lambda \in [0,1]$ and $x \in X$.

EXAMPLE 1. The family $\{F^t : t \geq 0\}$ of set-valued functions $F^t : [0,\infty) \to cc([0,\infty))$ given by

$$F^t(x) = e^t[0,x], \qquad x \in [0,\infty),$$

is a concave iteration semigroup of linear continuous s.v. functions.

EXAMPLE 2. Let $G : [0,\infty)^2 \to cc([0,\infty)^2)$ be the s.v. function given by $G((x,y)) = [0,x] \times [0,y]$. Then the family $\{F^t : t \geq 0\}$ of s.v. functions $F^t : [0,\infty)^2 \to cc([0,\infty)^2)$ defined by

$$F^t((x,y)) = e^t G((x,y))$$

is a concave iteration semigroup of linear continuous s.v. functions.

Before we give the next example we present the correct version of Remark 2 of [5].

REMARK 1. *Let X be a Banach space, $C \subset X$ be a closed convex cone and let $G : C \to c(C)$ be a linear s.v. function satisfying*

$$G^2(x) = G(x) \quad \textit{for } x \in C, \tag{1}$$

$$x \in G(x) \quad \textit{for } x \in C. \tag{2}$$

Then the family $\{F^t : t \geq 0\}$ of s.v. functions

$$F^t(x) := \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$$

is an iteration semigroup of linear continuous s.v. functions.

Proof. Since G satisfies (1), we have

$$\begin{aligned} F^t(x) &= \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x) = x + \frac{t}{1!} G(x) + \frac{t^2}{2!} G^2(x) + \ldots \\ &= x + \Big(\sum_{i=1}^{\infty} \frac{t^i}{i!} \Big) G(x) = x + (e^t - 1) G(x), \end{aligned}$$

for all $t \geq 0$ and $x \in C$. Thus the s.v. functions F^t $(t \geq 0)$ are linear and continuous with values in C. Moreover, by the Theorem of [5], for all $t, s \geq 0$ and $x \in C$ we have

$$(F^t \circ F^s)(x) \subset F^{t+s}(x).$$

On the other hand, if $y \in F^{t+s}(x)$, then there exists $z \in G(x)$ such that $y = x + (e^{t+s} - 1)z$. Therefore

$$\begin{aligned} y &= x + [(e^t - 1)(e^s - 1) + (e^s - 1) + (e^t - 1)]z \\ &= [x + (e^t - 1)z] + (e^s - 1)[z + (e^t - 1)z]. \end{aligned}$$

By (2) we can write

$$\begin{aligned} y &\in [x + (e^t - 1)G(x)] + (e^s - 1)[z + (e^t - 1)G(z)] \\ &\subset [x + (e^t - 1)G(x)] + (e^s - 1) \bigcup_{z \in G(x)} F^t(z) \\ &= F^t(x) + (e^s - 1)F^t(G(x)) = (F^t \circ F^s)(x). \end{aligned}$$

Finally $(F^t \circ F^s)(x) = F^{t+s}(x)$. ∎

EXAMPLE 3. Let $G : [0,\infty)^2 \to cc([0,\infty)^2)$ be the s.v. function given by $G((x,y)) = [0,x] \times [0,y]$. Consider the s.v. functions $F^t : [0,\infty)^2 \to cc([0,\infty)^2)$, $t \geq 0$, given by

$$F^t((x,y)) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i((x,y)).$$

Then $\{F^t : t \geq 0\}$ is a concave iteration semigroup of linear continuous s.v. functions of the form

$$F^t((x,y)) = (x,y) + (e^t - 1)G((x,y)), \qquad (x,y) \in [0,\infty)^2,\ t \geq 0.$$

LEMMA 1 (Lemma 3 of [7]). *Let C be a closed convex cone with nonempty interior in a real Banach space X and let Y be a normed space. If $\{A_n : n \in \mathbb{N}\}$ is a sequence of continuous additive s.v. functions $A_n : C \to cc(Y)$ such that $A_{n+1}(x) \subset A_n(x)$ for $x \in C$ and $n \in \mathbb{N}$, then the formula*

$$A(x) := \bigcap_{n=1}^{\infty} A_n(x)$$

defines a continuous additive s.v. function $A : C \to cc(Y)$. Moreover, the sequence $\{A_n : n \in \mathbb{N}\}$ is uniformly convergent to A on every compact subset of C.

From now on, Id denotes the set-valued identity, that is, the s.v. function $x \mapsto \{x\}$.

THEOREM 1. *Assume that C is a closed convex cone with nonempty interior in a Banach space X. Let $\{F^t : t \geq 0\}$ be a concave iteration* *semigroup of linear continuous s.v. functions* $F^t : C \to c(C)$ *such that* $F^0 = \mathrm{Id}$. *Then there exists an s.v. function* $G : C \to cc(C)$ *such that the family of s.v. functions* $\left\{\frac{1}{t}(F^t - \mathrm{Id}) : t > 0\right\}$ *uniformly converges to* G *on every compact subset of* C. *Moreover*, G *is linear and continuous and*

$$G(x) = \bigcap_{t>0} \frac{F^t(x) - x}{t} \quad \text{for every } x \in C. \tag{3}$$

Proof. Observe that for all $t, s \geq 0$, $0 \leq t < s$ and for every $x \in C$,

$$F^t(x) = F^{\frac{t}{s}s + (1-\frac{t}{s})0}(x) \subset \frac{t}{s}F^s(x) + \left(1 - \frac{t}{s}\right)F^0(x).$$

Hence

$$F^t(x) \subset \frac{t}{s}F^s(x) + \left(1 - \frac{t}{s}\right)x, \quad 0 \leq t < s,\ x \in C,$$

and consequently

$$\frac{F^t(x) - x}{t} \subset \frac{F^s(x) - x}{s}, \quad 0 \leq t < s,\ x \in C. \tag{4}$$

This means that $\left\{\frac{1}{t}(F^t(x) - x) : t > 0\right\}$ is an increasing family of sets, for every $x \in C$. Therefore, by Lemma 1, the s.v. function G given by (3) is linear, continuous and takes nonempty compact convex values in the space X. Moreover, for every $x \in C$,

$$G(x) = \lim_{t \to 0} \frac{F^t(x) - x}{t}$$

and the convergence is uniform on each compact subset of C.

Let $x \in C$. By (3), we have

$$G(x) \subset \frac{F^n(x) - x}{n} \subset C - \frac{1}{n}x$$

for every positive integer n. This implies that $G(x) \subset C$. Therefore $G(x) \in cc(C)$ for every $x \in C$. ■

THEOREM 2. *Suppose that* C *is a closed convex cone with nonempty interior in a separable Banach space* X. *Let* $\{F^t : t \geq 0\}$ *be a concave iteration semigroup of linear continuous s.v. functions* $F^t : C \to c(C)$ *such that* $F^0 = \mathrm{Id}$. *Then there exists a linear continuous s.v. function* $G : C \to cc(C)$ *such that for every linear continuous selection* g *of* G *each of the functions*

$$f^t(x) := \sum_{i=0}^{\infty} \frac{t^i}{i!} g^i(x), \quad x \in C,$$

is a linear continuous selection of F^t *for* $t \geq 0$ *and the family* $\{f^t : t \geq 0\}$ *is an iteration semigroup.*

P r o o f. Let $G : C \to cc(C)$ be given by (3). Then, by Theorem 1, G is linear and continuous. Since for every $x \in C$ and $t > 0$,

$$G(x) \subset \frac{F^t(x) - x}{t}$$

we have

$$x + tG(x) \subset F^t(x) \quad \text{for } x \in C,\ t > 0. \tag{5}$$

Let $\mathcal{F}_G$ be the family of all linear continuous selections of G. Then $\mathcal{F}_G \neq \emptyset$. Indeed, the Corollary in [6] shows that there exists a linear continuous selection $\widehat{a}$ of $\widehat{G} := G|_{\operatorname{Int} C}$. Let a be the linear continuous extension of $\widehat{a}$ onto the closed cone C. Then a is a linear continuous selection of G and consequently $a \in \mathcal{F}_G$.

Now, fix any $g \in \mathcal{F}_G$. By the Theorem of [5], we can define functions $f^t : C \to C$, for $t > 0$, as follows:

$$f^t(x) := \sum_{i=0}^{\infty} \frac{t^i}{i!} g^i(x), \quad x \in C. \tag{6}$$

For each $t > 0$, we also define

$$h_t(x) := x + tg(x) \in x + tG(x), \quad x \in C,$$

which is a linear continuous selection of F^t (see (5)).

Fix $t > 0$. Observe that for every $x \in C$ we have

$$h_t^2(x) = h_t(h_t(x)) = x + 2tg(x) + t^2 g^2(x) \in F^{2t}(x).$$

By induction one can prove that for $n \in \mathbb{N}$,

$$h_t^n(x) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} t^i g^i(x) \in F^{nt}(x), \quad x \in C.$$

Set $f_n^t := h_{t/n}^n$. Then, by the above,

$$f_n^t(x) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \frac{t^i}{n^i} g^i(x) \in F^t(x), \quad x \in C. \tag{7}$$

Since for all $n \in \mathbb{N}$ and $i \in \{2, \ldots, n\}$,

$$\frac{n!}{i!(n-i)!} \cdot \frac{1}{n^i} = \frac{1}{i!}\left(1 - \frac{i-1}{n}\right) \ldots \left(1 - \frac{1}{n}\right)$$

we can rewrite (7) as follows:

$$f_n^t(x) = x + tg(x) + \sum_{i=2}^{n} \frac{t^i}{i!}\left(1 - \frac{i-1}{n}\right) \ldots \left(1 - \frac{1}{n}\right) g^i(x) \in F^t(x), \quad x \in C.$$

In this way we get a sequence $\{f_n^t : n \in \mathbb{N}\}$ of linear continuous selections of F^t.

We now show that this sequence converges to the function (6). Let $x \in C$ and let $\varepsilon > 0$. Since the series $\sum_{i=0}^{\infty} \frac{(t\|g\|)^i}{i!}\|x\|$ is convergent there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=n}^{\infty} \frac{(t\|g\|)^i}{i!}\|x\| < \frac{\varepsilon}{2} \quad \text{for } n > n_0. \tag{8}$$

Define

$$a_n^i := \left(1 - \frac{i-1}{n}\right)\cdots\left(1 - \frac{1}{n}\right)$$

for $n \in \mathbb{N}$, $n \geq 2$ and $i \in \{2, \ldots, n\}$. It is easily seen that $0 < a_n^i < 1$ ($n \geq 2$, $i \in \{2, \ldots, n\}$). Moreover for every $i \geq 2$ the sequence $\{a_n^i\}_{n\geq i}$ converges to 1. Therefore there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$ and $i \in \{1, \ldots, n_0\}$,

$$\frac{(t\|g\|)^i}{i!}\|x\|(1 - a_n^i) < \frac{\varepsilon}{2(n_0 - 1)}. \tag{9}$$

Let $\{S_n(x) : n \in \mathbb{N}\}$ be the sequence of partial sums of the series (6). Take any $n > \max\{n_0, n_1\}$. Then (8) and (9) yield

$$\begin{aligned}
&\|S_n(x) - f_n^t(x)\| \\
&\quad = \left\| \sum_{i=0}^{n} \frac{t^i}{i!} g^i(x) - \left[x + tg(x) + \sum_{i=2}^{n} \frac{t^i}{i!}\left(1 - \frac{i-1}{n}\right)\cdots\left(1 - \frac{1}{n}\right) g^i(x)\right]\right\| \\
&\quad = \left\| \sum_{i=2}^{n} \frac{t^i}{i!} g^i(x)(1 - a_n^i)\right\| \leq \sum_{i=2}^{n} \frac{(t\|g\|)^i}{i!}\|x\|(1 - a_n^i) \\
&\quad = \sum_{i=2}^{n_0} \frac{(t\|g\|)^i}{i!}\|x\|(1 - a_n^i) + \sum_{i=n_0+1}^{n} \frac{(t\|g\|)^i}{i!}\|x\|(1 - a_n^i) \\
&\quad < \sum_{i=2}^{n_0} \frac{(t\|g\|)^i}{i!}\|x\|(1 - a_n^i) + \sum_{i=n_0+1}^{n} \frac{(t\|g\|)^i}{i!}\|x\| \\
&\quad < \sum_{i=2}^{n_0} \frac{\varepsilon}{2(n_0 - 1)} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, since (7) is satisfied and $F^t(x) \in cc(C)$ we conclude that $f^t(x) \in F^t(x)$. Moreover the family $\{f^t : t \geq 0\}$ is an iteration semigroup of linear continuous functions (see Theorem of [5]). ∎

The next theorem is a consequence of Theorem 1 and the Theorem of [4].

THEOREM 3. *Let X be a separable Banach space, and $C \subset X$ a closed convex cone with nonempty interior. If $\{F^t : t \geq 0\}$ is a concave iteration* *semigroup of linear continuous s.v. functions* $F^t : C \to c(C)$ *such that* $F^0 = \mathrm{Id}$, *then there exists a linear continuous s.v. function* $G : C \to cc(C)$ *such that*

$$F^t(x) \subset \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$$

for all $t \ge 0$ *and* $x \in \operatorname{Int} C$.

Proof. By Theorem 1, there exists $G : C \to cc(C)$ such that

$$\lim_{t\to 0} \frac{1}{t}(F^t(x) - x) = G(x), \quad x \in C. \tag{10}$$

Moreover, the convergence is uniform on every compact subset of C. Therefore the semigroup $\{F^t : t \ge 0\}$ satisfies assumptions (i) and (iii) of the Theorem of [4].

Fix $t > 0$. Take $n_0 \in \mathbb{N}$ with $t \le n_0$. By (3),

$$F^t(x) - x \subset t\frac{F^n(x) - x}{n} \subset C - \frac{t}{n}x$$

for all $n \ge n_0$ and $x \in C$. This implies that $F^t(x) - x \subset C$ for every $x \in C$. Hence condition (ii) of the Theorem of [4] is also satisfied.

Now we show that the semigroup $\{F^t : t \ge 0\}$ is continuous. Fix $x \in C$. By (10), there exists $T > 0$ such that for every $0 < t \le T$,

$$\frac{1}{t}(F^t(x) - x) \subset G(x) + S,$$

where S is the closed unit ball in X. Thus

$$F^t(x) - x \subset tG(x) + tS, \quad 0 \le t \le T,$$

and consequently

$$\|F^t(x) - x\| \le T(\|G(x)\| + 1) =: m, \quad t \in [0, T].$$

Therefore for all $t \in [0, T]$,

$$F^t(x) \subset x + mS.$$

The above considerations imply that the concave s.v. function $t \mapsto F^t(x)$ is bounded on the interval $[0, T]$, and finally it is continuous (see Theorem 4.4 of [3]). Since the semigroup $\{F^t : t \ge 0\}$ satisfies all the assumptions of the Theorem of [4] we have

$$F^t(x) \subset B^t(x) := \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$$

for all $t \ge 0$ and $x \in \operatorname{Int} C$. ∎

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