ANNALES
POLONICI MATHEMATICI
LXXI.1 (1999)

On discontinuous implicit differential equations in ordered Banach spaces with discontinuous implicit boundary conditions

by S. Carl (Halle) and S. Heikkilä (Oulu)

Abstract. We consider the existence of extremal solutions to second order discontinuous implicit ordinary differential equations with discontinuous implicit boundary conditions in ordered Banach spaces. We also study the dependence of these solutions on the data, and cases when the extremal solutions are obtained as limits of successive approximations. Examples are given to demonstrate the applicability of the method developed in this paper.

1. Introduction. Given a real interval $I = [t_0, t_1]$ and a Banach space E, consider the boundary value problem (BVP)

$$\begin{cases} f(t, x(t), x'(t), -x''(t)) = 0 & \text{a.e. on } I, \\ C_i(x(t_0), x(t_1), x'(t_0), x'(t_1)) = 0, & i = 0, 1, \end{cases} \tag{1.1}$$

where $f : I \times E^3 \to E$ and $C_0, C_1 : E^4 \to E$. These functions are assumed to satisfy certain monotonicity conditions to be specified later, but they may be discontinuous in all their arguments.

Implicit second order differential equations of the form appearing in (1.1) with nonlinearity $f : I \times \mathbb{R}^3 \to \mathbb{R}$ being continuous at least in its last variable, and under standard linear or nonlinear but continuous boundary conditions, have been studied by various authors (cf. e.g. [4, 5, 7, 9, 11, 13]). The main approach to the implicit "continuous" differential equations as in (1.1) is to first solve the equation $f(t, x, p, z) = 0$ for z. However, this approach requires a global continuous solution $z = \widehat{F}(t, x, p)$ to be defined for all $(t, x, p) \in I \times \mathbb{R}^2$, which can rarely be obtained.

1991 *Mathematics Subject Classification*: 34A09, 34G20, 47H07, 47H10.

Key words and phrases: discontinuous implicit differential equations, discontinuous implicit boundary conditions, ordered Banach spaces, fixed point principles in partially ordered sets.

The work on this paper has been supported by the Deutsche Akademische Austauschdienst (DAAD) and by the Finnish Academy.

A second approach reduces the "continuous" implicit differential equation (1.1) to a differential inclusion of the form

$$x''(t) \in \varphi(t, x(t), x'(t)), \qquad t \in I \tag{1.2}$$

(cf. e.g. [5, 7, 9]). Existence results for differential inclusion problems have been obtained by various authors (e.g. [2, 3, 4, 9, 10, 12]). However, in all those papers the minimal assumption to treat an inclusion of the form (1.2) is the lower semicontinuity of the multifunction φ or at least the existence of a lower semicontinuous multiselection. Moreover, the assumptions on the original nonlinearity f are rather implicit and hard to verify.

Due to the discontinuous nonlinearity f in our implicit differential equation (1.1) none of the existing theories can be applied, even in the case when the functions f, C_0 and C_1 are real-valued. In fact, the authors are not aware of any reference dealing with the discontinuous implicit nonlinear boundary value problem of the form (1.1) in Banach spaces. On the other hand, it is well known that continuity of the data does not guarantee the solvability of differential equations in Banach spaces even in the explicit case. In the implicit case regularity conditions alone do not imply the existence of solutions for (1.1).

In this paper we develop a technique that provides existence and comparison results for the implicit boundary value problem (1.1) under explicit and readily verifiable assumptions on the data. This technique is based on the method of upper and lower solutions and a fixed point result in ordered metric spaces, obtained by a generalized iteration method (cf. [6]). Moreover, in some special cases our method allows getting solutions as limits of successive approximations, although the nonlinearities f, C_0 and C_1 may be discontinuous in all their arguments. Examples are given to demonstrate applications of the results obtained in this paper.

Throughout this paper we assume that

E is an ordered Banach space with regular order cone K, and the partial ordering $\leq$ in E, induced by K, is defined as $u \leq v$ if and only if $v - u \in K$.

Regularity of K means (cf. [8], p. 36) that each order bounded and monotone sequence of E has a limit.

A function x which belongs to the set

$$\mathrm{AC}^1(I, E) = \{x : I \to E \mid x' \text{ is absolutely continuous and a.e. differentiable on } I\}$$

is called a *lower solution* of (1.1) if

$$\begin{cases} f(t, x(t), x'(t), -x''(t)) \leq 0 & \text{a.e. on } I, \\ C_i(x(t_0), x(t_1), x'(t_0), x'(t_1)) \leq 0, & i = 0, 1, \end{cases}$$

and an *upper solution* if the reversed inequalities hold. If the equalities hold, we say that x is a *solution* of (1.1).

For each function $x \in \mathrm{AC}^1(I,E)$ we define $x''(t) = 0$ at those points $t \in I$ where the second derivative of x does not exist.

We equip the space $\mathrm{AC}^1(I,E)$ with the partial ordering $\preceq$ defined by

$$(1.3) \qquad x \preceq y \quad \text{iff} \quad \begin{aligned} &x(t) \le y(t) \text{ on } I,\ x'(t) \le y'(t) \text{ on } I \text{ and} \\ &x''(t) \ge y''(t) \text{ a.e. on } I. \end{aligned}$$

If $y, z \in \mathrm{AC}^1(I,E)$ and $y \preceq z$, define $[y,z] = \{x \in \mathrm{AC}^1(I,E) \mid y \preceq x \preceq z\}$.

We say that a solution x_* of (1.1) is the *minimal solution* of (1.1) in an order interval $[y,z]$ if $y \preceq x_* \preceq x$ for any other solution $x \in [y,z]$ of (1.1), and x^* is the *maximal solution* if $x \preceq x^* \preceq z$ for any other solution $x \in [y,z]$ of (1.1). If (1.1) has both the maximal and the minimal solution in $[y,z]$, they are called the *extremal solutions* of (1.1) in $[y,z]$.

We impose the following hypotheses on (1.1) and on the functions $f : I \times E^3 \to E$ and C_0, $C_1 : E^4 \to E$:

(f0) There exist a lower solution y and an upper solution z of (1.1) such that $y \preceq z$.

(f1) There is $\mu : I \times E^3 \to (0,\infty)$ such that $\mu \cdot f$ is strongly sup-measurable, and the function $w - (\mu \cdot f)(t,u,v,w)$ is nondecreasing in u, v and w for a.e. $t \in I$.

(C) There is $\nu_i : E^4 \to (0,\infty)$ such that $u_i - (\nu_i \cdot C_i)(u_0,v_0,v_1,u_1)$ is nondecreasing in all its arguments for $i = 0,1$.

Condition (f0) is necessary for the existence of a solution of (1.1), since if x is a solution of (1.1), then (f0) holds when $y = z = x$. In Section 3 we show that conditions (f0), (f1) and (C) are sufficient for the existence of the extremal solutions of (1.1) in the order interval $[y,z]$. Before the proof we derive some auxiliary results.

2. Auxiliaries. We shall first define an operator whose fixed points are solutions of (1.1).

LEMMA 2.1. *Given* $f : I \times E^3 \to E$, $C_0, C_1 : E^4 \to E$, $\mu : I \times E^3 \to (0,\infty)$, *and* $\nu_0, \nu_1 : E^4 \to (0,\infty)$, *define*

$$(2.1) \quad \begin{cases} Fx(t) = -x''(t) - (\mu \cdot f)(t, x(t), x'(t), -x''(t)), & t \in I, \\ A_i x(t_i) = x^{(i)}(t_i) - (\nu_i \cdot C_i)(x(t_0), x(t_1), x'(t_0), x'(t_1)), & i = 0,1. \end{cases}$$

Then $x \in \mathrm{AC}^1(I,E)$ *is a solution of* (1.1) *if and only if* Fx *is Bochner integrable on* I *and* $x = Gx$, *where*

$$(2.2) \qquad \begin{aligned} Gx(t) = {} & A_0 x(t_0) + (t - t_0) A_1 x(t_1) \\ & + \int_{t_0}^{t} (s - t_0) Fx(s)\, ds + (t - t_0) \int_{t}^{t_1} Fx(s)\, ds. \end{aligned}$$

Proof. Assume first that Fx is Bochner integrable on I. From (2.2) it follows by differentiation that

$$(Gx)'(t) = A_1 x(t_1) + \int_t^{t_1} Fx(s)\,ds \tag{2.3}$$

for a.e. $t \in I$. The right-hand side of (2.3) is absolutely continuous and its integral function is Gx, which implies that (2.3) holds for all $t \in I$. In particular, $(Gx)'$ is absolutely continuous and a.e. differentiable on I. From (2.3) it follows by differentiation that

$$(Gx)''(t) = -Fx(t) \quad \text{a.e. on } I. \tag{2.4}$$

This and (2.1) imply that if $x = Gx$, then x satisfies the differential equation of (1.1), and by (2.1)–(2.3) the boundary conditions of (1.1) also hold. Thus x is a solution of (1.1).

Conversely, assume that $x \in \mathrm{AC}^1(I, E)$ is a solution of (1.1). Then x' is absolutely continuous and a.e. differentiable on I, so that x'' is Bochner integrable on I. Because $-x''(t) = Fx(t)$ a.e. on I, it follows that Fx is Bochner integrable on I. Replacing $Fx(s)$ on the right-hand side of (2.2) by $-x''(s)$, calculating the integrals obtained and using (2.1) and the boundary conditions of (1.1), one can show that the right-hand side of (2.2) equals $x(t)$ for each $t \in I$. Thus $x = Gx$. ∎

Recall that since the order cone K of E is regular, there exists $\gamma > 0$ such that

$$\|u\| \le \gamma\|v\| \quad \text{whenever } u, v \in K \text{ and } u \le v. \tag{2.5}$$

This property is needed in the proof of the following lemma.

Lemma 2.2. *Assume that the hypotheses* (C), (f0) *and* (f1) *are valid. Then* (2.1) *and* (2.2) *define a nondecreasing mapping* $G : [y, z] \to [y, z]$.

Proof. Let $x \in [y, z]$. Conditions (f0) and (f1) imply that the function Fx given by (2.1) satisfies

$$-y''(t) \le Fy(t) \le Fx(t) \le Fz(t) \le -z''(t) \quad \text{for a.e. } t \in I. \tag{a}$$

From (a) and (2.5) it follows that

$$\|Fx(t)\| \le (1+\gamma)(\|y''(t)\| + \|z''(t)\|) \quad \text{for a.e. } t \in I. \tag{2.6}$$

This and condition (f1) imply that Fx is Bochner integrable. Thus (2.1), (2.2) define a mapping $Gx \in \mathrm{AC}^1(I, E)$. From the definition of upper and lower solutions it follows by routine calculations that

$$y \preceq Gy \quad \text{and} \quad Gz \preceq z. \tag{b}$$

Assume next that $x, \widehat{x} \in [y, z]$ and $x \preceq \widehat{x}$. By (1.3) this means that

$$x(t) \le \widehat{x}(t) \text{ and } x'(t) \le \widehat{x}'(t) \text{ on } I, \text{ and } x''(t) \ge \widehat{x}''(t) \text{ a.e. on } I. \tag{c}$$

From (2.1) it follows by (c), (f1) and (C) that

$$\text{(d)}\qquad \begin{cases} Fx(t) \le F\widehat{x}(t) & \text{for a.e. } t \in I,\\ A_i x(t_i) \le A_i\widehat{x}(t_i), & i = 0,1. \end{cases}$$

In view of (2.2)–(2.4) and (d) we have

$$Gx(t) \le G\widehat{x}(t) \quad \text{and} \quad (Gx)'(t) \le (G\widehat{x})'(t) \quad \text{on } I,$$

and

$$(Gx)''(t) \ge (\widehat{G}x)''(t) \quad \text{a.e. on } I.$$

This and (1.3) imply that $Gx \preceq G\widehat{x}$. Moreover, $Gx \in \mathrm{AC}^1(I,E)$ for each $x \in [y,z]$. ∎

(Hint: Equations in a proof marked by small letters, e.g. (a), (b), ..., are referred to only in that proof, so that there will be no confusion if one and the same label appears in different proofs.)

Define a norm $\|\cdot\|_{012}$ in $\mathrm{AC}^1(I,E)$ by

$$\|x\|_{012} = \sup_{t\in I}\|x(t)\| + \sup_{t\in I}\|x'(t)\| + \int_{t_0}^{t_1}\|x''(t)\|\,dt. \tag{2.7}$$

Lemma 2.3. *Assume that the hypotheses* (C), (f0) *and* (f1) *hold. If* $(x_n)_{n=0}^\infty$ *is a monotone sequence in* $[y,z]$, *then* $(Gx_n)_{n=0}^\infty$ *converges in* $[y,z]$ *with respect to the norm* $\|\cdot\|_{012}$.

Proof. Let $(x_n)_{n=0}^\infty$ be a nondecreasing sequence in the order interval $[y,z]$ of $(\mathrm{AC}^1(I,E),\preceq)$. By Lemma 2.2 the sequence $(Gx_n)_{n=0}^\infty$ is nondecreasing and belongs to $[y,z]$. In view of (1.3) this means that the sequences $(Gx_n(t))_{n=0}^\infty$, $t \in I$, are nondecreasing in the order interval $[y(t),z(t)]$ of $(E,\le)$, the sequences $((Gx_n)'(t))_{n=0}^\infty$, $t\in I$, are nondecreasing in $[y'(t),z'(t)]$ and the sequence $((-Gx_n)''(t))_{n=0}^\infty = (Fx_n(t))_{n=0}^\infty$ is nondecreasing in $[-y''(t),-z''(t)]$ for a.e. $t \in I$. These results, (2.1) and (C) also imply that the sequences $(A_i x_n(t_i))_{n=0}^\infty$ are nondecreasing in $[A_i y(t_i), A_i z(t_i)]$ for $i=0,1$. Since the order cone of E is regular, the following limits exist:

$$\text{(a)}\qquad x(t) = \lim_{n\to\infty} Gx_n(t), \quad v(t) = \lim_{n\to\infty}(Gx_n)'(t), \quad t \in I,$$

$$\text{(b)}\qquad w(t) = \lim_{n\to\infty}(Gx_n)''(t) = \lim_{n\to\infty} -Fx_n(t) \quad \text{for a.e. } t\in I,$$

$$\text{(c)}\qquad c_i = \lim_{n\to\infty} A_i x_n(t_i), \quad i = 0,1.$$

Because of (2.2) and (2.3) we have, for each $t \in I$,

$$\text{(d)}\qquad Gx_n(t) = A_0 x_n(t_0) + (t-t_0)A_1 x_n(t_1) + \int_{t_0}^{t}(s-t_0)Fx_n(s)\,ds + (t-t_0)\int_t^{t_1} Fx_n(s)\,ds,$$

and

(e) $$(Gx_n)'(t) = A_1 x_n(t_1) + \int_t^{t_1} Fx_n(s)\,ds.$$

Applying the dominated convergence theorem we then obtain

$$x(t) = c_0 + (t-t_0)c_1 + \int_{t_0}^{t} (s-t_0)(-w(s))\,ds + (t-t_0)\int_t^{t_1}(-w(s))\,ds, \qquad t \in I,$$

and

$$v(t) = c_1 + \int_t^{t_1}(-w(s))\,ds, \qquad t \in I.$$

This implies that $x \in \mathrm{AC}^1(I,E)$, $x'(t) = v(t)$ on I, and $x''(t) = w(t)$ a.e. on I. In view of (d), (e) and (2.6) the sequences $(Gx_n)_{n=0}^{\infty}$ and $((Gx_n)')_{n=0}^{\infty}$ are equicontinuous. From these facts and from (a) it follows that $(Gx_n)(t) \to x(t)$ and $(Gx_n)'(t) \to x'(t)$ uniformly on I. The dominated convergence theorem also implies that

$$\lim_{n\to\infty} \int_{t_0}^{t_1} \|(Gx_n)''(t) - x''(t)\|\,dt = 0.$$

These results ensure that $Gx_n \to x$ with respect to the norm $\|\cdot\|_{012}$, defined by (2.7).

The proof in the case when the sequence $(x_n)_{n=0}^{\infty}$ is nonincreasing is similar. ■

As a special case of Theorem 1.2.2 of [6] we obtain the following result:

LEMMA 2.4. *Let* $[\alpha,\beta]$ *be a nonempty order interval in* $(\mathrm{AC}^1(I,E),\preceq)$, *and assume that* $G : [\alpha,\beta] \to [\alpha,\beta]$ *is nondecreasing. If* $(Gx_n)_{n=0}^{\infty}$ *converges in* $(\mathrm{AC}^1(I,E), \|\cdot\|_{0,1,2})$ *whenever* $(x_n)_{n=0}^{\infty}$ *is a monotone sequence in* $[\alpha,\beta]$, *then* G *has the least fixed point* x_* *and the greatest fixed point* x^*, *and*

(2.8) $$x_* = \min\{w \in [\alpha,\beta] \mid Gw \preceq w\}, \qquad x^* = \max\{w \in [\alpha,\beta] \mid w \preceq Gw\}.$$

3. Existence and comparison results

3.1. *Existence results.* Now we are ready to prove our main existence result.

THEOREM 3.1. *If conditions* (C), (f0) *and* (f1) *are valid*, *then the BVP* (1.1) *has the extremal solutions in* $[y,z]$.

Proof. Assume conditions (C), (f0) and (f1). Lemmas 2.2 and 2.3 ensure that the hypotheses of Lemma 2.4 hold for the operator G defined by (2.1), (2.2), and $[\alpha,\beta] = [y,z]$. Thus G has the least fixed point x_* and the greatest fixed point x^*. In view of Lemma 2.1 this means that x_* and x^* are the least and the greatest solutions of (1.1) in the order interval $[y,z]$. ■

REMARK 3.1. The hypotheses of Theorem 3.1 allow the functions f, C_0 and C_1 in (1.1) to be discontinuous with respect to all their arguments.

Consider next the case when certain one-sided continuity hypotheses are satisfied. Given a sequence (u_n) in E converging to u, write $u_n \nearrow u$ if (u_n) is nondecreasing and $u_n \searrow u$ if (u_n) is nonincreasing.

PROPOSITION 3.1. *If conditions* (f0), (f1) *and* (C) *hold*, *then the successive approximations*

$$x_n = Gx_{n-1}, \qquad n = 1, 2, \ldots, \tag{3.1}$$

where G is defined by (2.1), (2.2), *converge in* $(\mathrm{AC}^1(I), \|\cdot\|_{012})$ *to*

(a) *the minimal solution x_* of the BVP* (1.1) *in* $[y,z]$ *if $x_0 = y$ and if* $(\mu \cdot f)(t, u_n, v_n, w_n) \to (\mu \cdot f)(t, u, v, w)$ *for a.e.* $t \in I$ *and*

$$(\nu_i \cdot C_i)(s_n, u_n, v_n, w_n) \to (\nu_i \cdot C_i)(s, u, v, w) \quad \textit{for } i = 0, 1$$

whenever $s_n \nearrow s$, $u_n \nearrow u$, $v_n \nearrow v$ *and* $w_n \nearrow w$;

(b) *the maximal solution x^* of the BVP* (1.1) *in* $[y,z]$ *if $x_0 = z$ and if* $(\mu \cdot f)(t, u_n, v_n, w_n) \to (\mu \cdot f)(t, u, v, w)$ *for a.e.* $t \in I$ *and*

$$(\nu_i \cdot C_i)(s_n, u_n, v_n, w_n) \to (\nu_i \cdot C_i)(s, u, v, w) \quad \textit{for } i = 0, 1$$

whenever $s_n \searrow s$, $u_n \searrow u$, $v_n \searrow v$ *and* $w_n \searrow w$.

Proof. (a) Choose $x_0 = y$. From Lemma 2.2 it follows that (3.1) defines a nondecreasing sequence $(x_n)_{n=0}^{\infty}$ in $[y,z]$. Since $x_n = Gx_{n-1}$, $n = 1, 2, \ldots$, by Lemma 2.3 there exists $x_* \in [y,z]$ such that $x_n \to x_*$ in the norm $\|\cdot\|_{012}$. In particular,

$$\begin{aligned} &x_n(t) \nearrow x_*(t) \quad \text{and} \quad x_n'(t) \nearrow x_*'(t) \quad \text{uniformly on } I, \text{ and} \\ &x_n''(t) \searrow x_*''(t) \quad \text{a.e. on } I. \end{aligned} \tag{a}$$

From this and (2.1) it follows by the given continuity hypotheses that

$$\lim_{n\to\infty} Fx_n(t) = Fx_*(t) \qquad \text{for a.e. } t \in I,$$
$$\lim_{n\to\infty} A_i x_n(t_i) = A_i x_*(t_i) \qquad \text{for } i = 0, 1.$$

These relations, (a), (2.2) and the dominated convergence theorem imply that

$$x_*(t) = \lim_{n\to\infty} x_{n+1}(t) = \lim_{n\to\infty} Gx_n(t) = Gx_*(t), \qquad t \in I.$$

Thus $x_* = Gx_*$, so that x_* is by Lemma 2.1 a solution of (1.1).

If x is any solution of (1.1) in $[y,z]$, we know by Lemma 2.1 that x is a fixed point of G, so that $x_0 = y \preceq x = Gx$. Since G is nondecreasing in $[y,z]$ by Lemma 2.2, it is then easy to see by induction that $x_n \preceq x$ for each $n \in \mathbb{N}$, which implies, as $n \to \infty$, that $x_* \preceq x$. This proves that x_* is the minimal solution of (1.1) in $[y,z]$.

(b) The proof is similar to case (a). ∎

3.2. *A comparison result.* As for the dependence of the extremal solutions of (1.1) on the functions f, C_0 and C_1 we have the following result.

PROPOSITION 3.2. *If conditions* (C), (f0) *and* (f1) *hold*, *then the extremal solutions of the BVP* (1.1) *are nonincreasing with respect to the functions* f, C_0 *and* C_1.

Proof. Assume that conditions (f0), (f1) and (C) are valid for the functions $f, \widehat{f} : I \times E^3 \to E$ and $C_i, \widehat{C}_i : E^4 \to E$. Assume also that

$$\text{(a)} \qquad f(t,u,v,w) \geq \widehat{f}(t,u,v,w) \quad \text{for a.e. } t \in I \text{ and all } u,v,w \in E,$$

and that

$$\text{(b)} \qquad C_i(u_0,v_0,v_1,u_1) \geq \widehat{C}_i(u_0,v_0,v_1,u_1), \quad u_i, v_i \in E,\ i = 0,1.$$

Thus the hypotheses of Theorem 3.1 also hold when f, C_0 and C_1 are replaced by $\widehat{f}$, $\widehat{C}_0$ and $\widehat{C}_1$, respectively, so that the boundary value problems (1.1) and

$$\text{(3.2)} \qquad \begin{cases} \widehat{f}(t,x(t),x'(t),-x''(t)) = 0 & \text{a.e. on } I, \\ \widehat{C}_i(x(t_0),x(t_1),x'(t_0),x'(t_1)) = 0, & i = 0,1, \end{cases}$$

have the minimal solutions x_*, $\widehat{x}_*$ and the maximal solutions x^*, $\widehat{x}^*$ in the order interval $[y,z]$. In view of Lemma 2.1 we have

$$\text{(c)} \qquad \widehat{x}_*(t) = \widehat{G}\widehat{x}_*(t), \quad t \in I,$$

where

$$\text{(3.3)} \qquad \widehat{G}x(t) = \widehat{A}_0 x(t_0) + (t-t_0)\widehat{A}_1 x(t_1) + \int_{t_0}^{t} (s-t_0)\widehat{F}x(s)\,ds + (t-t_0)\int_{t}^{t_1} \widehat{F}x(s)\,ds,$$

with

$$\text{(3.4)} \quad \begin{cases} \widehat{F}x(t) = -x''(t) - (\mu \cdot \widehat{f})(t,x(t),x'(t),-x''(t)), & t \in I, \\ \widehat{A}_i x(t_i) = x^{(i)}(t_i) - (\nu_i \cdot \widehat{C}_i)(x(t_0),x(t_1),x'(t_0),x'(t_1)), & i = 0,1. \end{cases}$$

From (a), (b), (2.1) and (3.4) it follows that

$$\begin{cases} F\widehat{x}_*(s) \leq \widehat{F}\widehat{x}_*(s) & \text{for a.e. } s \in I, \\ A_i\widehat{x}_*(t_i) \leq \widehat{A}_i\widehat{x}_*(t_i), & i = 0,1. \end{cases}$$

This, (2.2), (3.3), (3.4) and (c) imply that $G\widehat{x}_* \preceq \widehat{x}_*$. Thus $\widehat{x}_* \in [y,z]$ and $G\widehat{x}_* \preceq \widehat{x}_*$, so that $x_* \preceq \widehat{x}_*$, by the first formula of (2.8).

Similarly, it can be shown, by applying the second formula of (2.8), that $x^* \preceq \widehat{x}^*$. ■

4. Special cases. In this section we consider

$$(4.1)\qquad \begin{cases} -x''(t) = g(t, x(t), x'(t), -x''(t)) & \text{a.e. on } I,\\ x^{(i)}(t_i) = D_i(x(t_0), x(t_1), x'(t_0), x'(t_1)), & i = 0, 1, \end{cases}$$

where $g : I \times E^3 \to K$ and $D_0, D_1 : E^4 \to K$, $i = 0, 1$. We make the following assumptions:

(g0) g is a strongly sup-measurable function, and there is $p \in L^1_+(I)$ such that $g(t, u, v, w) + p(t)w$ is nondecreasing in u, v and w for a.e. $t \in I$.

(g1) $g(t, u, v, w) \le M(t)(u + v) + h(t) + \lambda(t)w$ for a.e. $t \in I$ and for all $u, v, w \in K$, where $h : I \to K$ is Bochner integrable, $M/(1 - \lambda) \in L^1_+(I)$, and

$$(4.2)\qquad \max_{t\in I}\Big\{\int_{t_0}^{t} \frac{(s-t_0)M(s)}{1-\lambda(s)}\,ds + \int_{t}^{t_1} \frac{(t-t_0)M(s)}{1-\lambda(s)}\,ds\Big\} + \int_{t_0}^{t_1} \frac{M(s)}{1-\lambda(s)}\,ds < 1.$$

(D0) There exist $m_i \ge 0$ such that $D_i(u_0, v_0, v_1, u_1) + m_i u_i$ are nondecreasing in u_0, u_1, v_0 and v_1 for $i = 0, 1$.

(D1) $D_i(u_0, v_0, v_1, u_1) \le d_i u_i + c_i$ for $u_0, u_1, v_0, v_1 \in K$, where $c_i/(1 - d_i) \in K$, $i = 0, 1$.

LEMMA 4.1. *Assume that conditions* (D1) *and* (g1) *are valid. If* $x \in \mathrm{AC}^1(I, E)$ *is a solution of the BVP* (4.1), *then* x *belongs to the order interval* $[0, z]$, *where* z *is the solution of the BVP*

$$(4.3)\qquad -z''(t) = \frac{M(t)(z(t) + z'(t)) + h(t)}{1 - \lambda(t)}, \qquad z^{(i)}(t_i) = \frac{c_i}{1 - d_i}.$$

P r o o f. It is an elementary matter to show (cf. the proof of Lemma 2.1) that $z \in \mathrm{AC}^1(I, E)$ is a solution of (4.3) if and only if z is a solution of the equation

$$(4.4)\qquad z(t) = z_0(t) + Lz(t), \qquad t \in I,$$

where $L : C^1(I, E) \to C^1(I, E)$ and $z_0 \in C^1(I, E)$ are defined by

$$(4.5)\qquad Lx(t) = \int_{t_0}^{t} \frac{(s-t_0)M(s)}{1-\lambda(s)}(x(s) + x'(s))\,ds + (t - t_0)\int_{t}^{t_1} \frac{M(s)}{1-\lambda(s)}(x(s) + x'(s))\,ds,$$

$$(4.6)\qquad z_0(t) = \frac{c_0}{1-d_0} + \frac{(t-t_0)c_1}{1-d_1} + \int_{t_0}^{t}(s-t_0)\frac{h(s)}{1-\lambda(s)}\,ds + (t-t_0)\int_{t}^{t_1}\frac{h(s)}{1-\lambda(s)}\,ds, \qquad t \in I.$$

Since

$$(Lx)'(t) = \int_t^{t_1} \frac{M(s)(x(s)+x'(s))}{1-\lambda(s)}\, ds, \qquad t \in I, \tag{4.7}$$

condition (4.2) and equations (4.5) and (4.7) imply that, with respect to the norm $\|x\| = \max_{t\in I} \|x(t)\| + \max_{t\in I} \|x'(t)\|$ of $C^1(I,E)$, the norm of the linear operator L is less than one. Thus (4.4) and hence also (4.3) has a unique solution $z \in \mathrm{AC}^1(I,E)$. This solution and its derivative can be obtained as the limits of the successive approximations

$$z_{n+1}(t) = Lz_n(t), \quad z'_{n+1}(t) = (Lz_n)'(t), \qquad t \in I.$$

Since h is K-valued and since $c_i/(1-d_i) \in K$, $i = 0,1$, it follows from (4.6) that $z_0(t) \in K$ and $z_0'(t) \in K$ for all $t \in I$. But $M/(1-\lambda) \in L^1_+(I)$, so from (4.5) and (4.7) it follows that L is a positive operator with respect to the partial ordering of $C^1(I,E)$ defined by

$$y \le z \quad \text{iff} \quad y(t) \le z(t) \text{ and } y'(t) \le z'(t) \text{ for all } t \in I.$$

Thus $z(t) \in K$ and $z'(t) \in K$ for all $t \in I$. This and (4.3) imply that $-z''(t) \in K$ a.e. on I. The above proof shows that (4.3) has a unique solution $z \in \mathrm{AC}^1(I,E)$, and that $0 \preceq z$.

Assume that $x \in \mathrm{AC}^1(I,E)$ is a solution of (4.1). Then x also satisfies the integral equation

$$\begin{aligned} x(t) &= x(t_0) + (t-t_0)x'(t_1) + \int_{t_0}^{t} (s-t_0) g(s,x(s),x'(s),-x''(s))\, ds \\ &\quad + (t-t_0) \int_t^{t_1} g(s,x(s),x'(s),-x''(s))\, ds, \qquad t \in I. \end{aligned} \tag{4.8}$$

This implies by differentiation that

$$x'(t) = x'(t_1) + \int_t^{t_1} g(s,x(s),x'(s),-x''(s))\, ds, \qquad t \in I. \tag{4.9}$$

Since g and D_i, $i = 0,1$, are K-valued, it follows from (4.1), (4.8) and (4.9) that $x(t) \in K$ and $x'(t) \in K$ on I, and $-x''(t) \in K$ a.e. on I. Thus we can apply condition (g1) to show that

$$-x''(t) = g(t,x(t),x'(t),-x''(t)) \le M(t)(x(t)+x'(t)) + h(t) + \lambda(t)\,(-x''(t))$$

for a.e. $t \in I$, or equivalently,

$$\begin{aligned} -x''(t) &= g(t,x(t),x'(t),-x''(t)) \\ &\le \frac{M(t)(x(t)+x'(t))+h(t)}{1-\lambda(t)} \qquad \text{for a.e. } t \in I. \end{aligned} \tag{a}$$

From condition (D1) and (4.1) it follows that

$$x^{(i)}(t_i) = D_i(x(t_0), x(t_1), x'(t_0), x'(t_1)) \le d_i x^{(i)}(t_i) + c_i, \qquad i = 0, 1,$$

so that

$$\text{(b)} \qquad x^{(i)}(t_i) \le \frac{c_i}{1 - d_i} = z^{(i)}(t_i), \qquad i = 0, 1.$$

The integral equations (4.8) and (4.9) and the inequalities (a) and (b) imply that

$$x(t) \le \frac{c_0}{1-d_0} + \frac{(t-t_0)c_1}{1-d_1} + \int_{t_0}^{t} (s-t_0) \frac{M(s)(x(s)+x'(s)) + h(s)}{1-\lambda(s)}\, ds$$
$$+ (t-t_0) \int_{t}^{t_1} \frac{M(s)(x(s)+x'(s)) + h(s)}{1-\lambda(s)}\, ds, \qquad t \in I,$$

and

$$\text{(c)} \qquad x'(t) \le \frac{c_1}{1-d_1} + \int_{t}^{t_1} \frac{M(s)(x(s)+x'(s)) + h(s)}{1-\lambda(s)}\, ds, \qquad t \in I.$$

In particular, x and x' satisfy the inequalities

$$x(t) \le z_0(t) + Lx(t), \quad x'(t) \le z_0'(t) + (Lx)'(t), \qquad t \in I,$$

where L and z_0 are defined by (4.5) and (4.6). On the other hand, z and z' are the solutions of the corresponding equalities. Since L is a positive operator, it follows by the Abstract Gronwall Lemma (cf. [14], Prop. 7.15) that $x(t) \le z(t)$ and $x'(t) \le z'(t)$ on I. In view of this and (a) we then have

$$-x''(t) \le \frac{M(t)(z(t)+z'(t)) + h(t)}{1-\lambda(t)} = -z''(t) \qquad \text{for a.e. } t \in I.$$

Thus

$$\begin{cases} x(t) \le z(t), \qquad x'(t) \le z'(t) \quad \text{on } I, \\ -x''(t) \le -z''(t) \quad \text{for a.e. } t \in I. \end{cases}$$

From these relations it follows by (1.3) that $x \preceq z$. Moreover, since g and D_i, $i = 0, 1$, are K-valued, it follows from (4.1), (4.8) and (4.9) that $0 \preceq x$. Thus $x \in [0, z]$. ∎

The next result is a consequence of Theorem 3.1, Proposition 3.2 and Lemma 4.1.

Proposition 4.1. *Assume that conditions* (g0), (g1), (D0) *and* (D1) *hold. Then the BVP* (4.1) *has the extremal solutions* x_* *and* x^* *in the sense that if* x *is a solution of* (4.1), *then* x *belongs to* $[x_*, x^*]$. *Moreover*, x_* *and* x^* *are nondecreasing with respect to* g, D_0 *and* D_1.

Proof. Let

$$
(4.10)\qquad \begin{cases} f(t,u,v,w) = w - g(t,u,v,w), \\ \mu(t,u,v,w) = \dfrac{1}{p(t)+1}, \qquad t \in I,\ u,v,w \in E, \\ C_i(u_0,v_0,v_1,u_1) = u_i - D_i(u_0,v_0,v_1,u_1), \qquad u_i, v_i \in E,\ i = 0,1, \\ \nu_i(u_0,v_0,v_1,u_1) = \dfrac{1}{m_i+1}, \qquad u_i, v_i \in E,\ i = 0,1. \end{cases}
$$

Since

$$
w - (\mu \cdot f)(t,u,v,w) = \frac{g(t,u,v,w) + p(t)w}{p(t)+1}, \qquad t \in I,\ u,v,w \in E,
$$

condition (g0) implies that condition (f1) is valid. From

$$
u_i - (\nu_i \cdot C_i)(u_0,v_0,v_1,u_1) = \frac{D_i(u_0,v_0,v_1,u_1) + m_i u_i}{1+m_i}, \qquad u_i, v_i \in E,\ i = 0,1,
$$

it follows by condition (D0) that (C) holds. We now show that the solution z of (4.3) and 0 are upper and lower solutions of (1.1) with f, μ, C_i and ν_i given by (4.10). Since $z(t) \in K$ on I, $z'(t) \in K$ on I and $-z''(t) \in K$ a.e. on I, it follows from (4.3), (4.10) and (g1) that

$$
\begin{aligned}
f(t, z(t), z'(t), -z''(t)) & \\
&= -z''(t) - g(t, z(t), z'(t), -z''(t)) \\
&\geq -z''(t) - M(t)(z(t) + z'(t)) - h(t) + \lambda(t) z''(t) \\
&= (1 - \lambda(t)) \left(-z''(t) - \frac{M(t)(z(t) + z'(t)) + h(t)}{1 - \lambda(t)} \right) = 0
\end{aligned}
$$

for a.e. $t \in I$. Since g is K-valued, we have

$$
f(t,0,0,0) = 0 - g(t,0,0,0) \leq 0, \qquad t \in I.
$$

Thus

(a) $\quad f(t,0,0,0) \leq 0 \quad$ and $\quad f(t, z(t), z'(t), -z''(t)) \geq 0 \quad$ for a.e. $t \in I$.

In view of (D1), (4.3) and (4.10) we have, for $i = 0,1$,

(b)
$$
C_i(z(t_0), z(t_1), z'(t_0), z'(t_1)) = \frac{c_i}{1-d_i} - D_i(z(t_0), z(t_1), z'(t_0), z'(t_1)) \geq 0.
$$

Because D_0 and D_1 are K-valued, we have

(c)
$$
C_i(0,0,0,0) = -D_i(0,0,0,0) \leq 0, \qquad i = 0,1.
$$

From (a)–(c) it follows that 0 is a lower solution and z is an upper solution of (1.1). Moreover, $0 \leq z(t)$ on I, $0 \leq z'(t)$ on I and $0 \leq -z''(t)$ a.e. on I. Thus condition (f0) holds when $y(t) \equiv 0$ and z is the solution of (4.3).

The above proof shows that all the hypotheses of Theorem 3.1 are valid, so that (1.1), or equivalently (4.1), has the minimal solution x_* and the maximal solution x^* in the order interval $[0, z]$. Since all the solutions x of (4.1) belong by Lemma 4.1 to $[0, z]$, it follows that x_* is the least and x^* is the greatest solution of (4.1).

The last conclusion is a consequence of Proposition 3.2. ■

REMARKS 4.1. (i) We would like to emphasize that the proof of Proposition 4.1 shows that the class of functions f, μ, C_i and ν_i having properties (f0), (f1) and (C) is wide. Namely, examples of such classes of functions are defined by (4.10) combined with conditions (g0), (g1), (D0) and (D1).

(ii) Condition (4.2) can be weakened, for instance by assuming that the operator $T : C^1(I, E) \to C^1(I, E)$ defined by

$$Tx(t) = \int_{t_0}^{t} \frac{(s-t_0)M(s)}{1-\lambda(s)}(x(s)+x'(s))\,ds + (t-t_0)\int_{t}^{t_1} \frac{M(s)}{1-\lambda(s)}(x(s)+x'(s))\,ds$$

satisfies $\|T\|^n < 1$ for some $n \in \mathbb{N}$.

If $M(t)/(1-\lambda(t)) \equiv r \geq 0$, then (4.2) reduces to $(r/2) \cdot ((t_1 - t_0)^2 + 2(t_1 - t_0)) < 1$. In particular, if $E = \mathbb{R}$, one can prove the following result:

PROPOSITION 4.2. *Let $g : I \times \mathbb{R}^3 \to \mathbb{R}$ and $D_i : \mathbb{R}^4 \to \mathbb{R}$, $i = 0, 1$, satisfy the conditions*

(g2) *g is a Shragin function* (*cf.* [1]) *and there is $p \in L^1_+(I)$ such that $g(t, u, v, w) + p(t)w$ is nondecreasing in u, v and w for a.e. $t \in I$*;

(g3) *$|g(t, u, v, w)| \leq h(t) + M(t)(|u| + |v|) + \lambda(t)|w|$ for all $u, v, w \in \mathbb{R}$ and for a.e. $t \in I$, where $h, M, \lambda \in L^1_+(I)$ and*

$$\left\| \frac{M}{1-\lambda} \right\|_\infty < \frac{2}{(t_1 - t_0)^2 + 2(t_1 - t_0)};$$

(D2) *there exists $m_i \geq 0$ such that $D_i(u_0, v_0, v_1, u_1) + m_i u_i$ is nondecreasing in u_0, u_1, v_0 and v_1 for $i = 0, 1$*;

(D3) *$|D_i(u_0, v_0, v_1, u_1)| \leq d_i|u_i| + c_i$ for all $u_0, u_1, v_0, v_1 \in \mathbb{R}$, where $c_i/(1-d_i) \geq 0$, $i = 0, 1$.*

Then the BVP (4.1) *has the extremal solutions x_* and x^*, and all the solutions of* (4.1) *belong to the order interval $[-z, z]$, where z is the solution of the BVP* (4.3).

5. Convergence of successive approximations and examples. The next result is an application of Proposition 3.1 to the special case considered in Section 4. We use the following extra conditions:

(A) $g(t,u_n,v_n,w_n) \to g(t,u,v,w)$ for a.e. $t \in I$ and $D_i(s_n,u_n,v_n,w_n) \to D_i(s,u,v,w)$ for $i = 0,1$ whenever $s_n \nearrow s$, $u_n \nearrow u$, $v_n \nearrow v$ and $w_n \nearrow w$.

(B) $g(t,u_n,v_n,w_n) \to g(t,u,v,w)$ for a.e. $t \in I$ and $D_i(s_n,u_n,v_n,w_n) \to D_i(s,u,v,w)$ for $i = 0,1$ whenever $s_n \searrow s$, $u_n \searrow u$, $v_n \searrow v$ and $w_n \searrow w$.

PROPOSITION 5.1. *If conditions* (g0), (g1), (D0) *and* (D1) *are valid, then the successive approximations*

$$x_{n+1}(t) = A_0 x_n(t_0) + (t-t_0)A_1 x_n(t_1) + \int_{t_0}^{t} (s-t_0) F x_n(s)\,ds + (t-t_0)\int_{t}^{t_1} F x_n(s)\,ds, \tag{5.1}$$

with

$$F x_n(t) = \frac{g(t, x_n(t), x_n'(t), -x_n''(t)) - p(t) x_n''(t)}{1+p(t)}, \qquad t \in I, \tag{5.2}$$

and

$$A_i x_n(t_i) = \frac{D_i(x_n(t_0), x_n(t_1), x_n'(t_0), x_n'(t_1)) + m_i x_n^{(i)}(t_i)}{1+m_i}, \qquad i \in 0,1, \tag{5.3}$$

converge in $\mathrm{AC}^1(I,E)$, *with respect to the norm* $\|\cdot\|_{012}$ *defined by* (2.7), *to*

(a) *the minimal solution* x_* *of the BVP* (4.1) *if* $x_0 = 0$ *and if* (A) *holds*;

(b) *the maximal solution* x^* *of the BVP* (4.1) *if* $x_0 = z$, *where* z *is the solution of* (4.3), *and if* (B) *holds.*

Proof. (a) The hypotheses of Proposition 3.1 hold when $y = 0$ and when f, μ, C_i and ν_i, $i = 0,1$, are given by (4.10). Then the successive approximations (5.1) converge to the minimal solution of (4.1) in $[0,z]$. If x is any solution of (4.1), then $x \in [0,z]$, by Lemma 4.1. In view of the choices (4.10) this proves (a).

(b) The proof is similar. ■

REMARKS 5.1. If the assumptions that g and D_i are K-valued are replaced by the assumptions that $g(t,0,0,0) \in K$ for a.e. $t \in I$ and that $D_i(0,0,0,0) \in K$, then the results of Sections 4 and 5 hold when one restricts attention to the solutions of (4.1) which satisfy $0 \preceq x$.

If $E = \mathbb{R}$ and if the hypotheses (g2), (g3), (D2) and (D3) hold, then the conclusions of Proposition 5.1 are valid when we choose in the case (a) $x_0 = -z$, where z is the solution of (4.3).

The results derived in Sections 2–5 are valid when E is

(a) a finite-dimensional ordered Banach space (cf. [6], Prop. 1.3.1),

(b) l^p, $1 \le p < \infty$, with componentwise ordering (cf. [6], Ex. 5.8.3),

(c) an ordered Hilbert space with $(x|y) \geq 0$ for all $x, y \geq 0$ (cf. [6], Prop. 5.8.2),

(d) an ordered Banach space with uniformly monotone norm (cf. [6], Prop. 5.8.1),

or one of the following function spaces:

(e) $L^p(\Omega, Z)$, $1 \leq p < \infty$, with a.e. pointwise ordering, where $(\Omega, \mathcal{A}, \mu)$ is a measure space and Z is an ordered Banach space with regular order cone (cf. [6], Prop. 5.8.7),

(f) Orlicz space $L_M(\Omega)$, with a.e. pointwise ordering, where Ω is a bounded domain in $\mathbb{R}^m$, and M satisfies the Δ_2-condition (cf. [8]),

(g) the space c_0 of real sequences $x = (x_i)_{i=1}^{\infty}$ with $\lim_i x_i = 0$, equipped with the norm $\|x\| = \sup_i |x_i|$ and componentwise ordering (cf. [6], Ex. 5.8.1),

(h) the closure $E_M(\Omega)$ of $L^\infty(\Omega)$ in $L_M(\Omega)$, equipped with a.e. pointwise ordering (cf. [8]).

For other examples of ordered Banach spaces with regular order cones, see for instance [6], Ex. 5.8.4–5.8.6, Cor. 5.8.1–5.8.3 and Prop. 5.8.3.

Example 5.1. Define functions $h_i : I \to \mathbb{R}$ and $q_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots$, by

$$h_i(t) = \frac{1}{2^i} \sum_{m=1}^{i} \sum_{k=1}^{\infty} \frac{2 + [k^{1/m}t] - k^{1/m}t}{(km)^2} \left(2 + \sin\left(\frac{1}{1 + [k^{1/m}t] - k^{1/m}t}\right)\right)$$

for $t \in I$, $i = 1, 2, \ldots$, and

$$q_i(s) = \frac{1}{2^i} \sum_{m=1}^{i} \sum_{k=1}^{\infty} \frac{\pi/2 + \arctan([k^{1/m}s])}{(km)^2}, \qquad s \in \mathbb{R},\ i = 1, 2, \ldots,$$

where $[v]$ denotes the greatest integer $\leq v$. By choosing $E = l^1$, ordered by the cone l^1_+ of those elements of l^1 with nonnegative coordinates, consider the BVP

$$\text{(5.4)} \qquad \begin{cases} -x''(t) = g(t, x(t), x'(t), -x''(t)) & \text{a.e. on } I, \\ x(t_0) = c_0, \quad x'(t_1) = c_1, \end{cases}$$

where $c_0, c_1 \in l^1_+$, the components of $g = (g_1, g_2, \ldots)$ being defined by

$$g_i(t, u, v, w) = h_i(t) + q_i\Big(\sum_{j=1}^{i} (u_j + v_j)\Big) + \frac{\pi/2 + \arctan([-w_i])}{2^i}$$

for $i = 1, 2, \ldots$, $t \in I$ and $u = (u_1, u_2, \ldots), v = (v_1, v_2, \ldots), w = (w_1, w_2, \ldots) \in l^1$. It is easy to see that conditions (g0), (g1) and also (D0) and (D1) hold when $D_i(u_0, v_0, v_1, u_1) \equiv c_i$, $i = 0, 1$. Thus by Proposition 4.1 the BVP (5.4) has the extremal solutions x_* and x^*. Since the function $v \mapsto [v]$ is right-continuous, it follows from Proposition 5.1 that x^* is obtained as a limit of successive approximations.

Example 5.2. Consider the BVP

$$(5.5)\qquad \begin{cases} -x''(t) = \dfrac{[x(t)-t^2]}{2+2|[x(t)-t^2]|} + \dfrac{[x'(t)]}{2+2|[x'(t)]|} + \dfrac{[-x''(t)]}{2+2|[-x''(t)]|} \\ \hfill \text{a.e. on } I, \\ x(t_0) = c_0, \quad x'(t_1) = c_1. \end{cases}$$

From Proposition 4.2 it follows that (5.5) has the extremal solutions. By Proposition 5.1 the maximal solution is obtained as a limit of successive approximations. These approximations can be calculated by computer, and afterwards one can in some special cases even infer the exact maximal solution. For instance, the maximal solution of (5.5) when $I = [0,1]$, $c_0 = 0$, and $c_1 = 3/4$, is

$$x(t) = \begin{cases} \dfrac{t}{2}, & 0 \le t < \dfrac{1}{2}, \\ \dfrac{t^2}{4} + \dfrac{t}{4} + \dfrac{1}{15}, & \dfrac{1}{2} \le t \le 1. \end{cases}$$

When $I = [0,1]$, $c_0 = -1$ and $c_1 = 3/4$, the maximal solution of (5.5) is

$$x(t) = \begin{cases} \dfrac{t^2}{4} + \dfrac{3t}{16} - 1, & 0 \le t < \dfrac{1}{4}, \\ \dfrac{7}{24}t^2 + \dfrac{t}{6} + \dfrac{383}{384}, & \dfrac{1}{4} \le t \le 1. \end{cases}$$

In the case when when $I = [0,1]$ and $c_i = 1$, $i = 0,1$, the maximal solution of (5.5) is

$$x(t) = \frac{-t^2}{4} + \frac{3}{2}t + 1, \qquad t \in I.$$

Example 5.3. In our last example

$$(5.6)\qquad \begin{cases} -x''(t) = \dfrac{[x(t)-t^2]}{2} + \dfrac{[x'(t)]}{4} + \dfrac{[-x''(t)]}{2} \qquad \text{a.e. on } I, \\ x(t_0) = \dfrac{x(t_1)}{2+2|[x(t_1)]|}, \qquad x'(t_1) = \dfrac{4[x(t_0)+1]}{1+|[x(t_0)+1]|}, \end{cases}$$

also the boundary conditions are implicit. In view of Proposition 5.1(b) and Remarks 5.1 the maximal solution of (5.6) can be obtained by a method of successive approximations. When $I = [0,1/2]$, it is

$$x(t) = \begin{cases} -\dfrac{t^2}{2} + \dfrac{11}{4}t + \dfrac{13}{32}, & 0 \le t < \dfrac{1}{4}, \\ -t^2 + 3t + \dfrac{3}{8}, & \dfrac{1}{4} \le t \le \dfrac{1}{2}. \end{cases}$$

In the case when $I = [1/2,1]$, the maximal solution of (5.6) is

$$x(t) = -\frac{t^2}{2} + 3t - 1, \qquad t \in I.$$

Acknowledgements. The authors are very grateful to the referee whose comments helped to improve the paper.

References

[1] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Univ. Press, Cambridge, 1990.
[2] G. Bartuzel and A. Fryszkowski, *Abstract differential inclusions with some applications to partial differential ones*, Ann. Polon. Math. 53 (1991), 67–78.
[3] L. H. Erbe and W. Krawcewicz, *Nonlinear boundary value problems for differential inclusions* $y'' \in F(t, y, y')$, ibid. 54 (1991), 195–226.
[4] L. H. Erbe, W. Krawcewicz and T. Kaczyński, *Solvability of two-point boundary value problems for systems of nonlinear differential equations of the form* $y'' = g(t, y, y', y'')$, Rocky Mountain J. Math. 20 (1990), 899–907.
[5] M. Frigon and T. Kaczyński, *Boundary value problems for systems of implicit differential equations*, J. Math. Anal. Appl. 179 (1993), 317–326.
[6] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, New York, 1994.
[7] T. Kaczyński, *Implicit differential equations which are not solvable for the highest derivative*, in: Delay Differential Equations and Dynamical Systems (Claremont, CA, 1990), S. Busenberg and M. Martelli (eds.), Lecture Notes in Math. 1475, Springer, Berlin, 1991, 218–224.
[8] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1961.
[9] S. A. Marano, *On a boundary value problem for the differential equation* $f(t, x, x', x'') = 0$, J. Math. Anal. Appl. 182 (1994), 309–319.
[10] —, *Implicit elliptic differential equations*, Set-Valued Anal. 2 (1994), 545–558.
[11] W. V. Petryshyn, *Solvability of various boundary value problems for the equation* $x'' = f(t, x, x', x'') - y$, Pacific J. Math. 122 (1986), 169–195.
[12] B. Ricceri, *Applications de théorèmes de semi-continuité inférieure*, C. R. Acad. Sci. Paris Sér. I 295 (1982), 75–78.
[13] S. Stanek, *On a class of functional boundary value problems for the equation* $x'' = f(t, x, x', x'', \lambda)$, Ann. Polon. Math. 59 (1994), 225–237.
[14] E. Zeidler, *Nonlinear Functional Analysis and its Applications. Vol. I: Fixed-Point Theorems*, Springer, Berlin, 1985.

Fachbereich Mathematik und Informatik
Institut für Analysis
Martin-Luther-Universität Halle-Wittenberg
06099 Halle, Germany
E-mail: carl@condor.mathematik.uni-halle.de

Department of Mathematical Sciences
University of Oulu
90570 Oulu, Finland
E-mail: sheikki@sun3.oulu.fi

Reçu par la Rédaction le 19.6.1996
Révisé le 15.5.1997 et 9.10.1997