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The Bergman kernel functions of certain unbounded domains

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Abstract. We compute the Bergman kernel functions of the unbounded domains $\Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\}$, where $p(z') = |z'|^\alpha/\alpha$. It is also shown that these kernel functions have no zeros in Ω_p. We use a method from harmonic analysis to reduce the computation of the 2-dimensional case to the problem of finding the kernel function of a weighted space of entire functions in one complex variable.

1. Introduction. Let Ω_p be a domain in $\mathbb{C}^{n+1}$ of the form

$$\Omega_p = \{(z', z) : z' \in \mathbb{C}^n,\ z \in \mathbb{C},\ \Im z > p(z')\}.$$

Such domains can be viewed as generalizations of the Siegel upper half space, where $p(z') = |z'|^2$ (see [S]).

Weakly pseudoconvex domains of this kind were investigated by Bonami and Lohoué [BL], Boas, Straube and Yu [BSY], McNeal [McN1], [McN2], [McN3] and Nagel, Rosay, Stein and Wainger [NRSW1], [NRSW2]. For the case where $p(z') = |z'|^k$, $k \in \mathbb{N}$, Greiner and Stein [GS] found an explicit expression for the Szegő kernel of Ω_p.

If p is a subharmonic function on $\mathbb{C}$ which depends only on the real or only on the imaginary part of z', then one can find analogous expressions and estimates in [N] (see also [Has1]). In [D] and in [K] properties of the Szegő projection for such domains are studied. The asymptotic behavior of the corresponding Szegő kernel was investigated in [Han] and [Has2].

There have been several recent papers obtaining explicit formulas for the Bergman and Szegő kernel function on various weakly pseudoconvex domains ([D'A], [BFS], [FH1], [FH2], [FH3] and [OPY]). From the explicit formulas one can find examples of bounded convex domains whose Bergman kernel functions have zeros (see [BSF]).

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In this paper we compute the Bergman kernel functions of the unbounded domains $\Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\}$, where $p(z') = |z'|^\alpha/\alpha$, and we also show that these kernel functions have no zeros in Ω_p.

2. Computation of the Bergman kernel. We suppose that the weight function $p : \mathbb{C}^n \to \mathbb{R}_+$ is (pluri)subharmonic and of a growth behavior guaranteeing that the corresponding Bergman spaces H_τ of entire functions are nontrivial, where H_τ $(\tau > 0)$ consists of all entire functions $\phi : \mathbb{C}^n \to \mathbb{C}$ such that

$$\int_{\mathbb{C}^n} |\phi(z')|^2 e^{-4\pi\tau p(z')}\, d\lambda(z') < \infty.$$

The Bergman kernels of these spaces are denoted by $K_\tau(z', w')$. A result on parameter families of Bergman kernels of pseudoconvex domains of Diederich and Ohsawa [DO] can be adapted to our case, showing that for fixed (z', w') the function $\tau \mapsto K_\tau(z', w')$ is continuous. Then we can apply a method from [Has1] to obtain the following formulas for the Szegő kernel S of the Hardy space $H^2(\partial\Omega_p)$ and the Bergman kernel B of the domain Ω_p (see [Has3]):

Proposition 1. (a) *If $\partial\Omega_p$ is identified with $\mathbb{C}^n \times \mathbb{R}$, then the Szegő kernel on $\partial\Omega_p \times \partial\Omega_p$ has the form*

$$S((z', t), (w', s)) = \int_0^\infty K_\tau(z', w') e^{-2\tau(p(z')+p(w'))} e^{-2\pi i\tau(s-t)}\, d\tau,$$

where $z', w' \in \mathbb{C}^n$ and $s, t \in \mathbb{R}$.

(b) *For $(z', z), (w', w) \in \Omega_p$ $(z', w' \in \mathbb{C}^n;\ z, w \in \mathbb{C})$ the Szegő kernel can be expressed in the form*

$$S((z', z), (w', w)) = \int_0^\infty K_\tau(z', w') e^{-2\pi i\tau(\overline{w}-z)}\, d\tau.$$

(c) *The Bergman kernel of Ω_p is*

$$B((z', z), (w', w)) = 4\pi \int_0^\infty \tau K_\tau(z', w') e^{-2\pi i\tau(\overline{w}-z)}\, d\tau.$$

We first compute the Bergman kernel $K_\tau(z', w')$ of the weighted spaces of entire functions H_τ. Here we only consider the one-dimensional case. There are several possibilities to generalize to the higher dimensional case, where the corresponding formulas become quite complicated.

We suppose that the weight function p has the property that the Taylor series of an entire function in H_τ is convergent in H_τ. For instance, these assumptions are satisfied in the following case:

PROPOSITION 2 (see [T]). *Suppose that p is a convex function on $\mathbb{R}^2 = \mathbb{C}$ such that H_τ contains the polynomials. Then the polynomials are dense in H_τ.*

We further suppose that p depends only on $|z|$ and has a continuously differentiable inverse ϱ as a function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then the Bergman kernel of H_τ can be computed as follows:

PROPOSITION 3.

$$K_\tau(z', w') = \frac{1}{2\pi\tau} \sum_{n=0}^{\infty} \frac{n+1}{a_n(\tau)}\, z'^n \overline{w}'^n,$$

where $a_n(\tau) = \mathcal{L}(\varrho^{2n+2})(4\pi\tau)$ is the Laplace transform of ϱ^{2n+2} at the point $(4\pi\tau)$:

$$\mathcal{L}(\varrho^{2n+2})(4\pi\tau) = \int_0^\infty (\varrho(s))^{2n+2} e^{-4\pi\tau s}\, ds.$$

Proof. Since the monomials $(z'^n)_{n\ge 0}$ constitute a complete orthogonal system in H_τ the Bergman kernel can be expressed in the form

$$K_\tau(z', w') = \sum_{n=0}^{\infty} \frac{z'^n \overline{w}'^n}{c_n(\tau)},$$

where

$$c_n(\tau) = \int_{\mathbb{C}} |z'|^{2n} \exp(-4\pi\tau p(z'))\, d\lambda(z')$$

(see [Kr] or [R]). Using polar coordinates we get

$$c_n(\tau) = 2\pi \int_0^\infty r^{2n+1} \exp(-4\pi\tau p(r))\, dr,$$

and after substituting $p(r) = s$ we obtain

$$c_n(\tau) = 2\pi \int_0^\infty (\varrho(s))^{2n+1} \exp(-4\pi\tau s) \varrho'(s)\, ds.$$

Now partial integration gives

$$2\pi \int_0^\infty (\varrho(s))^{2n+1} \exp(-4\pi\tau s)\varrho'(s)\, ds = \frac{2\pi\tau}{n+1} \int_0^\infty (\varrho(s))^{2n+2} \exp(-4\pi\tau s)\, ds,$$

which proves the proposition. ■

In the next step we compute the Bergman kernel of $\Omega_p \subset \mathbb{C}^2$:

PROPOSITION 4. *Let the weight function p be as in Proposition 3. Then the Bergman kernel $B((z', z), (w', w))$ of $\Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\}$*

can be written in the form

$$B((z',z),(w',w)) = 2\int_0^\infty \Big(\sum_{n=0}^\infty (n+1)\frac{e^{-2\pi i(\overline{w}-z)\tau}}{\mathcal{L}(\varrho^{2n+2})(4\pi\tau)}\, z'^n\ \overline{w}'^n\Big)\, d\tau.$$

Proof. Combine Propositions 1(c) and 3. ■

In the sequel we concentrate on weight functions of the form $p(z') = |z'|^\alpha/\alpha$, where $\alpha\in\mathbb{R}$, $\alpha \ge 1$. It is easily seen that in this case the assumptions of Propositions 2 and 3 are satisfied. Hence we can apply Proposition 4 to get

PROPOSITION 5. *Let $p(z') = |z'|^\alpha/\alpha$, where $\alpha\in\mathbb{R}$, $\alpha\ge 1$. Then the Bergman kernel $B((z',z),(w',w))$ of $\Omega_p = \{(z',z)\in\mathbb{C}^2 : \Im z > p(z')\}$ has the form*

$$B((z',z),(w',w)) = \frac{2}{\pi(i(\overline{w}-z))^2}\, \frac{\big[\frac{\alpha i}{2}(\overline{w}-z)\big]^{2/\alpha}\big[(2+\alpha)\big[\frac{\alpha i}{2}(\overline{w}-z)\big]^{2/\alpha} + (2-\alpha)z'\overline{w}'\big]}{\big[\big[\frac{\alpha i}{2}(\overline{w}-z)\big]^{2/\alpha} - z'\ \overline{w}'\big]^3}.$$

We always take the principal values of the multi-valued functions involved.

Proof. First we compute the Laplace transform $\mathcal{L}(\varrho^{2n+2})(4\pi\tau)$. In our case we have $\varrho(s) = (\alpha s)^{1/\alpha}$, hence

$$\begin{aligned}\mathcal{L}(\varrho^{2n+2})(4\pi\tau) &= \int_0^\infty (\alpha s)^{(2n+2)/\alpha} e^{-4\pi\tau s}\, ds\\ &= (4\pi\tau)^{-1-(2n+2)/\alpha}\alpha^{(2n+2)/\alpha}\int_0^\infty t^{(2n+2)/\alpha}e^{-t}\, dt\\ &= (4\pi\tau)^{-1-(2n+2)/\alpha}\alpha^{(2n+2)/\alpha}\Gamma(1+(2n+2)/\alpha).\end{aligned}$$

In the sequel of the proof it will become apparent that summation and integration in Proposition 4 can be interchanged. We now obtain

$$\begin{aligned}B((z',z),(w',w)) = 2\sum_{n=0}^\infty &\frac{(n+1)(4\pi)^{1+(2n+2)/\alpha}}{\alpha^{(2n+2)/\alpha}\Gamma(1+(2n+2)/\alpha)}\\ &\times\Big(\int_0^\infty \tau^{1+(2n+2)/\alpha}e^{-2\pi i(\overline{w}-z)\tau}\, d\tau\Big) z'^n\overline{w}'^n.\end{aligned}$$

The integral in brackets can be expressed in the form

$$\begin{aligned}&\int_0^\infty \tau^{1+(2n+2)/\alpha}e^{-2\pi i(\overline{w}-z)\tau}\, d\tau\\ &\qquad = (2\pi i(\overline{w}-z))^{-2-(2n+2)/\alpha}\int_0^\infty \sigma^{1+(2n+2)/\alpha}e^{-\sigma}\, d\sigma,\end{aligned}$$

since $\Re(2\pi i(\overline{w}-z))>0$; this follows by Cauchy's theorem (see for instance [He], p. 33). Now we obtain

$$\begin{aligned}\int_0^\infty &\tau^{1+(2n+2)/\alpha}e^{-2\pi i(\overline{w}-z)\tau}\,d\tau\\ &=(2\pi i(\overline{w}-z))^{-2-(2n+2)/\alpha}\Gamma(2+(2n+2)/\alpha)\\ &=(2\pi i(\overline{w}-z))^{-2-(2n+2)/\alpha}(1+(2n+2)/\alpha)\Gamma(1+(2n+2)/\alpha).\end{aligned}$$

We can now continue computing the Bergman kernel:

$$\begin{aligned}B(&(z',z),(w',w))\\ &=2\sum_{n=0}^{\infty}\frac{(n+1)(1+(2n+2)/\alpha)(4\pi)^{1+(2n+2)/\alpha}}{\alpha^{(2n+2)/\alpha}(2\pi i(\overline{w}-z))^{2+(2n+2)/\alpha}}\,z'^n\overline{w}'^n\\ &=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{2^{(2n+2)/\alpha}[2(n+1)^2/\alpha+(n+1)]}{\alpha^{(2n+2)/\alpha}(i(\overline{w}-z))^{2+(2n+2)/\alpha}}\,z'^n\overline{w}'^n\\ &=\frac{2}{\pi(i(\overline{w}-z))^2}\sum_{n=0}^{\infty}\left[\frac{2(n+1)^2}{\alpha}+(n+1)\right]\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{-2(n+1)/\alpha}z'^n\overline{w}'^n.\end{aligned}$$

For the summation we use the formulas

$$\sum_{n=0}^{\infty}(n+1)^2x^n=\frac{1+x}{(1-x)^3}\quad\text{and}\quad\sum_{n=0}^{\infty}(n+1)x^n=\frac{1}{(1-x)^2},$$

where $|x|<1$. Sine $\Im z>|z'|^\alpha/\alpha$ and $\Im w>|w'|^\alpha/\alpha$ it follows that

$$|z'w'|<\left|\frac{\alpha i}{2}(\overline{w}-z)\right|^{2/\alpha}$$

and hence

$$\begin{aligned}B(&(z',z),(w',w))\\ &=\frac{2}{\pi(i(\overline{w}-z))^2}\,\frac{\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{-2/\alpha}\left[2+\alpha+(2-\alpha)\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{-2/\alpha}z'\overline{w}'\right]}{\left[1-\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{-2/\alpha}z'\overline{w}'\right]^3}\\ &=\frac{2}{\pi(i(\overline{w}-z))^2}\,\frac{\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{2/\alpha}\left[(2+\alpha)\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{2/\alpha}+(2-\alpha)z'\overline{w}'\right]}{\left[\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{2/\alpha}-z'\overline{w}'\right]^3},\end{aligned}$$

which proves Proposition 5. ∎

PROPOSITION 6. *Let* $p(z')=|z'|^\alpha/\alpha$, *where* $\alpha\in\mathbb{R}$, $\alpha\ge 1$. *Then the Bergman kernel* $B((z',z),(w',w))$ *of* $\Omega_p=\{(z',z)\in\mathbb{C}^2:\Im z>p(z')\}$ *has no zeros in* Ω_p.

Proof. By Proposition 5 the Bergman kernel $B((z',z),(w',w))$ has a zero if and only if

$$\left[\frac{\alpha i}{2}(\overline{w}-z)\right]^{2/\alpha} = \frac{\alpha-2}{\alpha+2}\, z'\overline{w}'.$$

Since $\Im z > 0$ and $\Im w > 0$, the factor $\overline{w}-z$ never vanishes on Ω_p. So we have a contradiction in the case $\alpha = 2$.

Now suppose that $\alpha \neq 2$. If the Bergman kernel has a zero, then

$$\left|\frac{\alpha i}{2}(\overline{w}-z)\right|^{2} = \left|\frac{\alpha-2}{\alpha+2}\right|^{\alpha} |z'|^{\alpha}\ |\overline{w}'|^{\alpha}.$$

We set $w = u+iv$, $z = x+iy$ and know that $\alpha y > |z'|^{\alpha}$ and $\alpha v > |w'|^{\alpha}$, hence

$$(u-x)^2 + (v+y)^2 < 4\left|\frac{\alpha-2}{\alpha+2}\right|^{\alpha} vy.$$

Since both v and y are positive and $4vy \leq (v+y)^2$, this inequality can only hold if at least

$$1 < \left|\frac{\alpha-2}{\alpha+2}\right|^{\alpha}.$$

It is clear that the last inequality is false, so the Bergman kernel has no zeros in Ω_p. ■

References

[BFS] H. P. Boas, S. Fu and E. J. Straube, *The Bergman kernel function: Explicit formulas and zeros*, Proc. Amer. Math. Soc. (to appear).

[BL] A. Bonami and N. Lohoué, *Projecteurs de Bergman et Szegő pour une classe de domaines faiblement pseudo-convexes et estimations L^p*, Compositio Math. 46 (1982), 159–226.

[BSY] H. P. Boas, E. J. Straube and J. Yu, *Boundary limits of the Bergman kernel and metric*, Michigan Math. J. 42 (1995), 449–462.

[D'A] J. P. D'Angelo, *An explicit computation of the Bergman kernel function*, J. Geom. Anal. 4 (1994), 23–34.

[D] K. P. Diaz, *The Szegő kernel as a singular integral kernel on a family of weakly pseudoconvex domains*, Trans. Amer. Math. Soc. 304 (1987), 147–170.

[DO] K. Diederich and T. Ohsawa, *On the parameter dependence of solutions to the $\overline{\partial}$-equation*, Math. Ann. 289 (1991), 581–588.

[FH1] G. Francsics and N. Hanges, *Explicit formulas for the Szegő kernel on certain weakly pseudoconvex domains*, Proc. Amer. Math. Soc. 123 (1995), 3161–3168.

[FH2] —, —, *The Bergman kernel of complex ovals and multivariable hypergeometric functions*, J. Funct. Anal. 142 (1996), 494–510.

[FH3] —, —, *Asymptotic behavior of the Bergman kernel and hypergeometric functions*, in: Contemp. Math. (to appear).

[GS] P. C. Greiner and E. M. Stein, *On the solvability of some differential operators of type* $\Box_b$, Proc. Internat. Conf., (Cortona, 1976–1977), Scuola Norm. Sup. Pisa, 1978, 106–165.
[Han] N. Hanges, *Explicit formulas for the Szegő kernel for some domains in* $\mathbb{C}^2$, J. Funct. Anal. 88 (1990), 153–165.
[Has1] F. Haslinger, *Szegő kernels of certain unbounded domains in* $\mathbb{C}^2$, Rév. Roumaine Math. Pures Appl. 39 (1994), 914–926.
[Has2] —, *Singularities of the Szegő kernel for certain weakly pseudoconvex domains in* $\mathbb{C}^2$, J. Funct. Anal. 129 (1995), 406–427.
[Has3] —, *Bergman and Hardy spaces on model domains*, Illinois J. Math. (to appear).
[He] P. Henrici, *Applied and Computational Complex Analysis, II*, Wiley, New York, 1977.
[K] H. Kang, $\overline{\partial}_b$*-equations on certain unbounded weakly pseudoconvex domains*, Trans. Amer. Math. Soc. 315 (1989), 389–413.
[Kr] S. G. Krantz, *Function Theory of Several Complex Variables*, Wadsworth & Brooks/Cole, Pacific Grove, Calif., 1992.
[McN1] J. McNeal, *Boundary behavior of the Bergman kernel function in* $\mathbb{C}^2$, Duke Math. J. 58 (1989), 499–512.
[McN2] —, *Local geometry of decoupled pseudoconvex domains*, in: Complex Analysis, Aspects of Math. E17, K. Diederich (ed.), Vieweg 1991, 223–230.
[McN3] —, *Estimates on the Bergman kernels on convex domains*, Adv. Math. 109 (1994), 108–139.
[N] A. Nagel, *Vector fields and nonisotropic metrics*, in: Beijing Lectures in Harmonic Analysis, E. M. Stein (ed.), Princeton Univ. Press, Princeton, N.J., 1986, 241–306.
[NRSW1] A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger, *Estimates for the Bergman and Szegő kernels in certain weakly pseudoconvex domains*, Bull. Amer. Math. Soc. 18 (1988), 55–59.
[NRSW2] —, —, —, —, *Estimates for the Bergman and Szegő kernels in* $\mathbb{C}^2$, Ann. of Math. 129 (1989), 113–149.
[OPY] K. Oeljeklaus, P. Pflug and E. H. Youssfi, *The Bergman kernel of the minimal ball and applications*, Ann. Inst. Fourier (Grenoble) 47 (1997), 915–928.
[R] M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer, 1986.
[S] E. M. Stein, *Harmonic Analysis. Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.
[T] B. A. Taylor, *On weighted polynomial approximation of entire functions*, Pacific J. Math. 36 (1971), 523–539.

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