The Bergman kernel functions of certain unbounded domains

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Abstract. We compute the Bergman kernel functions of the unbounded domains
\( \Omega_p = \{ (z', z) \in \mathbb{C}^2 : 3z > p(z') \} \), where \( p(z') = |z'|^{\alpha} / \alpha \).
It is also shown that these kernel functions have no zeros in \( \Omega_p \).
We use a method from harmonic analysis to reduce the computation of the 2-dimensional case
to the problem of finding the kernel function of a weighted space of entire functions in
one complex variable.

1. Introduction. Let \( \Omega_p \) be a domain in \( \mathbb{C}^{n+1} \) of the form
\( \Omega_p = \{ (z', z) : z' \in \mathbb{C}^n, z \in \mathbb{C}, 3z > p(z') \} \).
Such domains can be viewed as generalizations of the Siegel upper half space,
where \( p(z') = |z'|^2 \) (see [S]).
Weakly pseudoconvex domains of this kind were investigated by Bonami
and Lohoué [BL], Boas, Straube and Yu [BSY], McNeal [McN1], [McN2],
[McN3] and Nagel, Rosay, Stein and Wainger [NRSW1], [NRSW2]. For the
case where \( p(z') = |z'|^k \), \( k \in \mathbb{N} \), Greiner and Stein [GS] found an explicit
expression for the Szegő kernel of \( \Omega_p \).

If \( p \) is a subharmonic function on \( \mathbb{C} \) which depends only on the real or
only on the imaginary part of \( z' \), then one can find analogous expressions
and estimates in [N] (see also [Has1]). In [D] and in [K] properties of the
Szegő projection for such domains are studied. The asymptotic behavior
of the corresponding Szegő kernel was investigated in [Han] and [Has2].

There have been several recent papers obtaining explicit formulas for
the Bergman and Szegő kernel function on various weakly pseudoconvex
domains ([D'A], [BFS], [FH1], [FH2], [FH3] and [OPY]). From the explicit
formulas one can find examples of bounded convex domains whose Bergman
kernel functions have zeros (see [BSF]).

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of Sciences.
In this paper we compute the Bergman kernel functions of the unbounded domains \( \Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\} \), where \( p(z') = |z'|^\alpha / \alpha \), and we also show that these kernel functions have no zeros in \( \Omega_p \).

2. Computation of the Bergman kernel. We suppose that the weight function \( p : \mathbb{C}^n \rightarrow \mathbb{R}_+ \) is (pluri)subharmonic and of a growth behavior guaranteeing that the corresponding Bergman spaces \( H_\tau \) of entire functions are nontrivial, where \( H_\tau (\tau > 0) \) consists of all entire functions \( \phi : \mathbb{C}^n \rightarrow \mathbb{C} \) such that
\[
\int_{\mathbb{C}^n} |\phi(z')|^2 e^{-4\pi \tau p(z')} d\lambda(z') < \infty.
\]
The Bergman kernels of these spaces are denoted by \( K_\tau (z', w') \). A result on parameter families of Bergman kernels of pseudoconvex domains of Diederich and Ohsawa [DO] can be adapted to our case, showing that for fixed \((z', w')\) the function \( \tau \mapsto K_\tau (z', w') \) is continuous. Then we can apply a method from [Has1] to obtain the following formulas for the Szegő kernel \( S \) of the Hardy space \( H^2(\partial \Omega_p) \) and the Bergman kernel \( B \) of the domain \( \Omega_p \) (see [Has3]):

**Proposition 1.**

(a) If \( \partial \Omega_p \) is identified with \( \mathbb{C}^n \times \mathbb{R} \), then the Szegő kernel on \( \partial \Omega_p \times \partial \Omega_p \) has the form
\[
S((z', t), (w', s)) = \int_0^\infty K_\tau (z', w') e^{-2\pi \tau (p(z') + p(w'))} e^{-2\pi i \tau (s-t)} d\tau,
\]
where \( z', w' \in \mathbb{C}^n \) and \( s, t \in \mathbb{R} \).

(b) For \((z', z), (w', w) \in \Omega_p \) \((z', w' \in \mathbb{C}^n; z, w \in \mathbb{C})\) the Szegő kernel can be expressed in the form
\[
S((z', z), (w', w)) = \int_0^\infty K_\tau (z', w') e^{-2\pi i \tau (w-z)} d\tau.
\]

(c) The Bergman kernel of \( \Omega_p \) is
\[
B((z', z), (w', w)) = 4\pi \int_0^\infty \tau K_\tau (z', w') e^{-2\pi i \tau (w-z)} d\tau.
\]

We first compute the Bergman kernel \( K_\tau (z', w') \) of the weighted spaces of entire functions \( H_\tau \). Here we only consider the one-dimensional case. There are several possibilities to generalize to the higher dimensional case, where the corresponding formulas become quite complicated.

We suppose that the weight function \( p \) has the property that the Taylor series of an entire function in \( H_\tau \) is convergent in \( H_\tau \). For instance, these assumptions are satisfied in the following case:
Proposition 2 (see [T]). Suppose that $p$ is a convex function on $\mathbb{R}^2 = \mathbb{C}$ such that $H_\tau$ contains the polynomials. Then the polynomials are dense in $H_\tau$.

We further suppose that $p$ depends only on $|z|$ and has a continuously differentiable inverse $q$ as a function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then the Bergman kernel of $H_\tau$ can be computed as follows:

**Proposition 3.**

$$K_\tau(z', w') = \frac{1}{2\pi \tau} \sum_{n=0}^{\infty} \frac{n+1}{a_n(\tau)} z'^n w'^n,$$

where $a_n(\tau) = \mathcal{L}(q^{2n+2})(4\pi \tau)$ is the Laplace transform of $q^{2n+2}$ at the point $(4\pi \tau)$:

$$\mathcal{L}(q^{2n+2})(4\pi \tau) = \int_0^\infty (q(s))^{2n+2} e^{-4\pi \tau s} \, ds.$$

**Proof.** Since the monomials $(z'^n)_{n \geq 0}$ constitute a complete orthogonal system in $H_\tau$ the Bergman kernel can be expressed in the form

$$K_\tau(z', w') = \sum_{n=0}^{\infty} \frac{z'^n w'^n}{c_n(\tau)},$$

where

$$c_n(\tau) = \int_{\mathbb{C}} |z'|^{2n} \exp(-4\pi \tau p(z')) \, d\lambda(z')$$

(see [Kr] or [R]). Using polar coordinates we get

$$c_n(\tau) = 2\pi \int_0^{\infty} r^{2n+1} \exp(-4\pi \tau p(r)) \, dr,$$

and after substituting $p(r) = s$ we obtain

$$c_n(\tau) = 2\pi \int_0^{\infty} (q(s))^{2n+1} \exp(-4\pi \tau s) q'(s) \, ds.$$

Now partial integration gives

$$2\pi \int_0^{\infty} (q(s))^{2n+1} \exp(-4\pi \tau s) q'(s) \, ds = \frac{2\pi \tau}{n+1} \int_0^{\infty} (q(s))^{2n+2} \exp(-4\pi \tau s) \, ds,$$

which proves the proposition. \[ \square \]

In the next step we compute the Bergman kernel of $\Omega_p \subset \mathbb{C}^2$:

**Proposition 4.** Let the weight function $p$ be as in Proposition 3. Then the Bergman kernel $B((z', z), (w', w))$ of $\Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\}$
can be written in the form
\[
B((z', z), (w', w)) = 2 \int_0^\infty \left( \sum_{n=0}^\infty (n+1) \frac{e^{-2\pi i (\overline{w} - z) \tau}}{L(g^{2n+2})(4\pi \tau)} z^n \overline{w}^n \right) d\tau.
\]

Proof. Combine Propositions 1(c) and 3.

In the sequel we concentrate on weight functions of the form \( p(z') = |z'|^\alpha/\alpha \), where \( \alpha \in \mathbb{R}, \alpha \geq 1 \). It is easily seen that in this case the assumptions of Propositions 2 and 3 are satisfied. Hence we can apply Proposition 4 to get

**Proposition 5.** Let \( p(z') = |z'|^\alpha/\alpha \), where \( \alpha \in \mathbb{R}, \alpha \geq 1 \). Then the Bergman kernel \( B((z', z), (w', w)) \) of \( \Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\} \) has the form
\[
B((z', z), (w', w)) = \frac{2}{\pi i (\overline{w} - z)^2} \frac{[\frac{\alpha i}{2} (\overline{w} - z)]^{2/\alpha} \left[ (2 + \alpha) \frac{\alpha i}{2} (\overline{w} - z) \right]^{2/\alpha} + (2 - \alpha) z' \overline{w}'}{\left[ \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{2/\alpha} - z' \overline{w}' \right]^3}.
\]

We always take the principal values of the multi-valued functions involved.

Proof. First we compute the Laplace transform \( L(g^{2n+2})(4\pi \tau) \). In our case we have \( g(s) = (\alpha s)^{1/\alpha} \), hence
\[
L(g^{2n+2})(4\pi \tau) = \int_0^\infty (\alpha s)^{(2n+2)/\alpha} e^{-4\pi \tau s} ds
\]
\[
= (4\pi \tau)^{-1-(2n+2)/\alpha} \alpha^{(2n+2)/\alpha} \int_0^\infty t^{(2n+2)/\alpha} e^{-t} dt
\]
\[
= (4\pi \tau)^{-1-(2n+2)/\alpha} \alpha^{(2n+2)/\alpha} \Gamma(1 + (2n + 2)/\alpha).
\]

In the sequel of the proof it will become apparent that summation and integration in Proposition 4 can be interchanged. We now obtain
\[
B((z', z), (w', w)) = 2 \sum_{n=0}^\infty \frac{(n+1)(4\pi)^{1+(2n+2)/\alpha}}{\alpha^{(2n+2)/\alpha} \Gamma(1 + (2n + 2)/\alpha)} \int_0^\infty \tau^{1+(2n+2)/\alpha} e^{-2\pi i (\overline{w} - z) \tau} d\tau z^n \overline{w}^n.
\]

The integral in brackets can be expressed in the form
\[
\int_0^\infty \tau^{1+(2n+2)/\alpha} e^{-2\pi i (\overline{w} - z) \tau} d\tau = (2\pi i (\overline{w} - z))^{-2-(2n+2)/\alpha} \int_0^\infty \sigma^{1+(2n+2)/\alpha} e^{-\sigma} d\sigma,
\]
since $\Re(2\pi i(\overline{w} - z)) > 0$; this follows by Cauchy’s theorem (see for instance [He], p. 33). Now we obtain
\[
\int_0^\infty \tau^{1+(2n+2)/\alpha} e^{-2\pi i(\overline{w} - z)\tau} d\tau
= (2\pi i(\overline{w} - z))^{-2-(2n+2)/\alpha} \Gamma(2 + (2n + 2)/\alpha)
= (2\pi i(\overline{w} - z))^{-2-(2n+2)/\alpha} (1 + (2n + 2)/\alpha) \Gamma(1 + (2n + 2)/\alpha).
\]

We can now continue computing the Bergman kernel:
\[
B((z', z), (w', w))
= 2 \sum_{n=0}^{\infty} \frac{(n+1)(1+ (2n+2)/\alpha)(4\pi)^{1+(2n+2)/\alpha}}{\alpha^{2n+2}/\alpha (2\pi i(\overline{w} - z))^{2+(2n+2)/\alpha}} z^n \overline{w}^n
= \frac{2}{\pi i(i(\overline{w} - z))^2} \sum_{n=0}^{\infty} \left[ \frac{2(n+1)^2}{\alpha} + (n+1) \right] \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{-2(n+1)/\alpha} z^n \overline{w}^n.
\]

For the summation we use the formulas
\[
\sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1+x}{(1-x)^3} \quad \text{and} \quad \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2},
\]
where $|x| < 1$. Since $\Im z > |z'|^{\alpha}/\alpha$ and $\Im w > |w'|^{\alpha}/\alpha$ it follows that
\[
|z'| < \left| \frac{\alpha i}{2} (\overline{w} - z) \right|^{2/\alpha}
\]
and hence
\[
B((z', z), (w', w))
= \frac{2}{\pi i(i(\overline{w} - z))^2} \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{-2/\alpha} \left[ 2 + \alpha + (2- \alpha) \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{-2/\alpha} z' \overline{w} \right]
= \frac{2}{\pi i(i(\overline{w} - z))^2} \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{2/\alpha} \left[ (2+\alpha) \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{2/\alpha} + (2-\alpha) z' \overline{w} \right] \left[ \left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{2/\alpha} - z' \overline{w} \right]^{-3/\alpha},
\]
which proves Proposition 5. ■

**Proposition 6.** Let $p(z') = |z'|^{\alpha}/\alpha$, where $\alpha \in \mathbb{R}$, $\alpha > 1$. Then the Bergman kernel $B((z', z), (w', w))$ of $\Omega_p = \{(z', z) \in \mathbb{C}^2 : \Im z > p(z')\}$ has no zeros in $\Omega_p$. 

Proof. By Proposition 5 the Bergman kernel $B((z', z), (w', w))$ has a zero if and only if

$$
\left[ \frac{\alpha i}{2} (\overline{w} - z) \right]^{2/\alpha} = \frac{\alpha - 2}{\alpha + 2} z' w'.
$$

Since $\Im z > 0$ and $\Im w > 0$, the factor $\overline{w} - z$ never vanishes on $\Omega_p$. So we have a contradiction in the case $\alpha = 2$.

Now suppose that $\alpha \neq 2$. If the Bergman kernel has a zero, then

$$
\left| \frac{\alpha i}{2} (\overline{w} - z) \right|^2 = \left| \frac{\alpha - 2}{\alpha + 2} \right|^\alpha |z'|^\alpha |w'|^\alpha.
$$

We set $w = u + iv$, $z = x + iy$ and know that $\alpha y > |z'|^\alpha$ and $\alpha v > |w'|^\alpha$, hence

$$(u - x)^2 + (v + y)^2 < 4 \left| \frac{\alpha - 2}{\alpha + 2} \right|^\alpha vy.$$

Since both $v$ and $y$ are positive and $4vy \leq (v + y)^2$, this inequality can only hold if at least

$$1 < \left| \frac{\alpha - 2}{\alpha + 2} \right|^\alpha.$$

It is clear that the last inequality is false, so the Bergman kernel has no zeros in $\Omega_p$. ■

References


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