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Triviality of scalar linear type isotropy subgroup by passing to an alternative canonical form of a hypersurface

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Abstract. The Chern–Moser (CM) normal form of a real hypersurface in $\mathbb{C}^N$ can be obtained by considering automorphisms whose derivative acts as the identity on the complex tangent space. However, the CM normal form is also invariant under a larger group (pseudo-unitary linear transformations) and it is this property that makes the CM normal form special. Without this additional restriction, various types of normal forms occur. One of them helps to give a simple proof of a (previously complicated) theorem about triviality of the scalar linear type isotropy subgroup of a nonquadratic hypersurface. An example of an analogous nontrivial subgroup for a 2-codimensional CR surface in $\mathbb{C}^4$ is constructed. We also consider the question whether the group structure that is induced on the family of normalisations to the CM normal form via the parametrisation of the isotropy automorphism group of the underlining hyperquadric coincides with the natural composition operation on the biholomorphisms.

1. Alternative canonical form. Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic hypersurface. For the coordinates in $\mathbb{C}^{n+1}$ we set $z = (z^1, \ldots, z^n)$, $w = u + iv$. We assume that M has nondegenerate Levi form $\langle z, z\rangle$ at the origin.

Consider the space $\mathcal{F}$ of power series in $(z, \overline{z}, u)$ such that the series itself, its differential and the Hessian $\partial^2/\partial z\partial\overline{z}$ vanish at the origin. The hypersurface M can always be written in the form

$$v = \langle z, z\rangle + H, \tag{1}$$

where $H(z, \overline{z}, u) \in \mathcal{F}$.

Any $H \in \mathcal{F}$ can be written as a sum $H = \sum_{k,l} H_{k,l}$ of homogeneous polynomials of degree k in z and l in $\overline{z}$ with coefficients analytic in u. We denote by $\Pi_{(k_1,l_1),\ldots,(k_s,l_s)}(H)$ the sum $H_{k_1,l_1} + \ldots + H_{k_s,l_s}$. In $\mathcal{F}$ we consider

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the subspace

$$\mathcal{R} = \{2 \operatorname{Re} \chi(\langle z, z\rangle - v)|_{v=\langle z,z\rangle}\}, \tag{2}$$

where χ runs over holomorphic vector fields of the form

$$\chi = g(z,w)\frac{\partial}{\partial w} + \sum_{j=1}^{n} f(z,w)^j \frac{\partial^j}{\partial z^j}$$

with

$$f^j(0) = \frac{\partial f^j}{\partial z}\bigg|_0 = g(0) = \frac{\partial g}{\partial z}\bigg|_0 = \frac{\partial g}{\partial w}\bigg|_0 = 0.$$

Let $\kappa = \{(k,0),(k,1),(2,2),(3,2),(3,3)\}$ and Π_κ be the projection on the corresponding jet space. As Π_κ is an isomorphism on $\mathcal{R}$ (see [CM]), below we identify $\mathcal{R}$ and $\Pi_\kappa\mathcal{R}$. Let $\mathcal{N}$ be any direct complement of $\mathcal{R}$ in $\mathcal{F}$. The $\mathcal{R}$-component can always be eliminated from the equation (1) of M. The freedom in the choice of $\mathcal{N}$ leads to various canonical forms of the equation. One of such canonical forms, constructed by Chern and Moser [CM], has some natural advantages. Here we introduce an alternative canonical form to provide a simple proof of the Beloshapka and Loboda ([Be], [Lo1]) theorem about the triviality of the *scalar linear type* isotropy subgroup of a nonquadratic hypersurface M. The original proof of this theorem, based on the Chern–Moser normal form, is technically very hard.

It is convenient to consider two cases of the Levi form:

$$\begin{aligned} &\text{(i)} \quad \langle z,z\rangle = 2\operatorname{Re} z^1\overline{z^n} + \sum_{\alpha=2}^{n-1} \varepsilon_\alpha |z^\alpha|^2, \\ &\text{(ii)} \quad \langle z,z\rangle = |z^1|^2 + \sum_{\alpha=2}^{n} \varepsilon_\alpha |z^\alpha|^2, \end{aligned} \tag{3}$$

where $\varepsilon_\alpha = \pm 1$.

Let $H_{k,l}$ be an arbitrary polynomial of the type (k,l). Each time the product $z^1\overline{z^n}$ occurs in $H_{k,l}$ in case (i), or $z^1\overline{z^1}$ in case (ii), we replace it by $\langle z,z\rangle - \langle z,z\rangle'$, where

$$\begin{aligned} &\langle z,z\rangle' = z^n\overline{z^1} + \sum_{\alpha=2}^{n-1} \varepsilon_\alpha |z^\alpha|^2 \quad \text{or} \\ &\langle z,z\rangle' = \sum_{\alpha=2}^{n} \varepsilon_\alpha |z^\alpha|^2 \end{aligned} \tag{4}$$

in cases (i) and (ii), respectively. Thus, we obtain a decomposition

$$H_{k,l} = \sum_m \sum_\alpha u^{m-\alpha} \langle z,z\rangle^\alpha Q_{k-\alpha,l-\alpha}(z,\overline{z}) \tag{5}$$

where no monomial in the polynomial $Q_{k-\alpha,l-\alpha}$ contains $z^1\overline{z^n}$, or $z^1\overline{z^1}$, respectively. Set $\nu = \langle z, z\rangle$ and write the equation of M in the form

$$v = \nu + \sum_{k,l} H_{k,l}(\nu, z, \overline{z}, u),$$

where the polynomials $H_{k,l}$ are written in the form (5) and ν is considered as an independent variable.

By a *canonical form* we mean an equation of the type

$$v = \nu + \sum_{k,l\geq 2} N_{k,l}(\nu, z, \overline{z}, u), \tag{6}$$

where

$$\frac{\partial}{\partial\nu} N_{22} = \frac{\partial^2}{\partial\nu^2} N_{32} = \frac{\partial^3}{\partial\nu^3} N_{33} = 0.$$

The following is an analogue to the Chern–Moser theorem [CM]:

THEOREM 1. (i) *Any hypersurface can be locally biholomorphically transformed to a canonical form.*

(ii) *Let $\Phi = (F(z,w), G(z,w))$ be a transformation to a canonical form. Let J_Φ be the Jacobian of Φ:*

$$J_\Phi = \begin{pmatrix} \lambda U & \lambda U a \\ 0 & \varepsilon\lambda^2 \end{pmatrix},$$

where $\varepsilon = \pm 1$, U is a pseudo-unitary matrix such that $\langle Uz, Uz\rangle = \varepsilon\langle z, z\rangle$, $a \in \mathbb{C}^n$, $\lambda \geq 0$, and let

$$r = \operatorname{Re} \frac{\varepsilon}{\lambda^2} \frac{\partial^2 G}{\partial w^2}\bigg|_0 .$$

Then the set of initial data $I_\Phi = \{U, a, \lambda, r\}$ determines Φ uniquely, and, conversely, any set of parameters $\{U, a, \lambda, r\}$ with the above properties corresponds to a transformation of M to some canonical form.

(iii) *Any formal transformation that transforms M to a canonical form is automatically holomorphic.*

REMARK. For $n = 2$ and for quadrics $v = \langle z, z\rangle$ the described canonical form coincides with the Chern–Moser normal form.

We denote by G_0 the group of locally defined isotropic holomorphic automorphisms of M at 0. If M is written in a canonical form then any $\Phi \in G_0$ can be considered as a transformation between canonical forms, so the set of initial data $\{U, a, \lambda, r\}$ uniquely determines Φ. If M is a quadric then all parameters of initial data are free, while for M nonquadratic, i.e. not locally equivalent to a quadric, Beloshapka and Loboda ([Be], [Lo1]) proved that the matrix U uniquely determines the other parameters $\{a, \lambda, r\}$. This result can be reformulated as follows. We denote by $G_{0,\lambda\,\mathrm{id}}$ the subgroup of G_0 consisting of the automorphisms with $U = \mathrm{id}$; it is called the *scalar linear type* isotropy subgroup.

THEOREM 2. *If M is a nonquadratic hypersurface then* $G_{0,\lambda\,\mathrm{id}} = \{\mathrm{id}\}$.

REMARK. An analogous theorem does not hold for higher codimensional CR surfaces as the example in the last section of the paper shows.

2. Triviality of conformal isotropies. We start with the proof of Theorem 1, which basically follows the proof of the Chern–Moser theorem [CM].

Proof. We check that the space $\mathcal{N}$ described is complementary to $\mathcal{R}$. Expanding the right hand side of the expression for $\mathcal{R}$ we observe that

$$\mathcal{R} = \Big\{ \sum_{k=0}^{\infty}(H_{k,0} + H_{0,k}) + \sum_{k=2}^{\infty}(H_{k,1} + H_{1,k}) + ({}^{\tau}zB_1(u)\overline{z}) + \langle z, z\rangle({}^{\tau}zB_2(u)\overline{z}) + \langle z, z\rangle^2(\langle z, B_3(u)\rangle + \langle B_3(u), z\rangle) + b_4(u)\langle z, z\rangle^3 \Big\},$$

where $B_1(u)$ and $B_2(u)$ are hermitian matrices, $B_1(0) = 0$, $B_3(u)$ is an arbitrary vector in $\mathbb{C}^n$, and $b_4(u)$ is a real-valued function.

It is clear that the subspace

$$\mathcal{N} = \left\{ H \in \mathcal{F} \;\middle|\; H_{k,0} = H_{k,1} = \frac{\partial}{\partial\nu}H_{22} = \frac{\partial^2}{\partial\nu^2}H_{32} = \frac{\partial^3}{\partial\nu^3}H_{33} = 0 \right\}$$

is a direct complement to $\mathcal{R}$ in $\mathcal{F}$. So we can follow the Chern–Moser scheme to eliminate the $\mathcal{R}$-component in the equation of M. By the "convergence" part of [CM] the holomorphic transformations that do this job are parametrised by $\{U, a, \lambda, r\}$. By the "formal theory" of [CM], any formal transformation with fixed initial data $\{U, a, \lambda, r\}$ that eliminates the $\mathcal{R}$-component is unique, so it has to be holomorphic. This completes the proof of Theorem 1. ∎

We continue with the proof of Theorem 2.

Proof. STEP 1. Here we introduce a *weighted* form of the normal equation of M, that is convenient to analyse the action of infinitesimal automorphisms.

Consider $\Phi \in G_{0,\lambda\,\mathrm{id}}$. Since I_Φ determines Φ uniquely, it suffices to show that $I_\Phi = (\mathrm{id}, 0, 1, 0)$, which corresponds to the identity. Let $a \in I_\Phi$. We consider two cases:

(7)
(i) a is an isotropic vector for $\langle z, z\rangle$, i.e. $\langle a, a\rangle = 0$.
(ii) $\langle a, a\rangle = 1$ (without loss of generality this represents the situation $\langle a, a\rangle \neq 0$).

Consider a new basis $\{a, e_2, \ldots, e_n\}$ in $\mathbb{C}^n$ such that in case (i) of (7),

$$\langle e_j, a\rangle = 0, \quad \langle e_j, e_k\rangle = \pm\delta_{j,k},$$

while in case (ii),

$$\begin{gathered}\langle e_n, a\rangle = 1, \quad \langle e_j, a\rangle = 0, \quad \langle e_j, e_n\rangle = 0,\\ \langle e_j, e_k\rangle = \pm\delta_{j,k}, \quad j,k = 2,\ldots,n,\end{gathered}$$

where $\delta_{j,k}$ is the Kronecker delta.

Let z^* be coordinates in $\mathbb{C}^n$ in the new basis and C be the corresponding n-matrix that determines the coordinate change in $\mathbb{C}^{n+1}$ via $z = Cz^*,\ w = w^*$. In (z^*, w^*) coordinates the Jacobian of Φ equals

$$\begin{aligned} J_\Phi &= \begin{pmatrix} C^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda\,\mathrm{id} & \lambda a\\ 0 & \lambda^2\end{pmatrix}\begin{pmatrix} C & 0\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}\lambda\,\mathrm{id} & \lambda C^{-1}a\\ 0 & \lambda^2\end{pmatrix} = \begin{pmatrix} & \lambda\\ & 0\\ \lambda\,\mathrm{id} & \vdots\\ & 0\\ 0 & \lambda^2\end{pmatrix}.\end{aligned}$$

The equation of M takes the form

$$v = \langle z, z\rangle + N(z, \overline{z}, u),$$

where $\langle z, z\rangle$ has the form (i) or (ii) of (3) in case (i) or (ii) of (7), respectively.

We introduce a weight $\varkappa$ for the variables and for the coordinate differential operators in a standard way:

$$\begin{aligned}\varkappa(w) = \varkappa(\overline{w}) = -\varkappa\left(\frac{\partial}{\partial w}\right) = -\varkappa\left(\frac{\partial}{\partial \overline{w}}\right) = 2,\\ \varkappa(z^\alpha) = \varkappa(\overline{z^\alpha}) = -\varkappa\left(\frac{\partial}{\partial z^\alpha}\right) = -\varkappa\left(\frac{\partial}{\partial \overline{z^\alpha}}\right) = 1,\end{aligned}$$

and extend this weight to polynomials in $(z, \overline{z}, w, \overline{w})$ by linearity and homogeneity.

We expand an equation of M in a canonical form (6) as the sum of weighted homogeneous polynomials in $(z, \overline{z}, u)$:

$$v = \nu + \sum_{\gamma\ge 4} H_\gamma(\nu, z, \overline{z}, u). \tag{8}$$

Step 2. In this step we determine differential operators associated with infinitesimal automorphisms of M and their action on the weighted components.

Let γ_0 be the first value of γ such that $H_{\gamma_0} \neq 0$. As $\Phi \in G_{0,\lambda\,\mathrm{id}}$, there exists a vector field

$$\chi = \left((\ln\lambda)z\frac{\partial}{\partial z} + 2(\ln\lambda)w\frac{\partial}{\partial w}\right) + \left(w\frac{\partial}{\partial z^1} + 2iz^\varepsilon\frac{\partial}{\partial z} + 2iz^\varepsilon w\frac{\partial}{\partial w}\right) + \left(rwz\frac{\partial}{\partial z} + rw^2\frac{\partial}{\partial w}\right) + \left(f_{\gamma_0}\frac{\partial}{\partial z} + 2ig_{\gamma_0+1}\frac{\partial}{\partial w}\right) + \ldots \tag{9}$$

such that $\Phi = \exp(\chi)$. In the definition of χ we used the notations

$$\frac{\partial}{\partial z} = \sum_\alpha \frac{\partial}{\partial z^\alpha}, \tag{10}$$

$$z\frac{\partial}{\partial z} = \sum_\alpha z^\alpha\frac{\partial}{\partial z^\alpha}; \tag{11}$$

here $\varepsilon = n$ or $\varepsilon = 1$ in cases (i) or (ii) of (7), respectively. The collected terms in (9) represent the components $\chi_0,\ \chi_1,\ \chi_2$ and χ_{γ_0-1} of χ of weights $0,\ 1,\ 2$ and $\gamma_0 - 1$, respectively.

Since the iterations of Φ preserve the equation (8), it follows that χ satisfies the equation

$$\mathcal{H} = 2\operatorname{Re}\chi\Big(\langle z, z\rangle - v + \sum_{\gamma\ge\gamma_0} H_\gamma(\nu, z, \bar z, u)|_{v=\langle z,z\rangle+\sum_{\gamma\ge\gamma_0} H_\gamma(\nu,z,\bar z,u)}\Big) \equiv 0. \tag{12}$$

Considering the component $\mathcal{H}_{\gamma_0}$ of weight γ_0 in (12) we obtain

$$(\gamma_0 - 2)\ln\lambda H_{\gamma_0} = 0,$$

and since $\gamma_0 \ge 4$, this implies that $\lambda = 1$.

To show that $a = 0$ we consider the next component, $\mathcal{H}_{\gamma_0+1}$. By the *defect* of a polynomial in $(z, \bar z)$ we mean the difference in its degrees in holomorphic and anti-holomorphic variables. Let $H_{\gamma_0,d,m}$ be the component of H_{γ_0} of maximal defect and minimal degree in (ν, u) (see (5)).

By $\varrho \to_{\alpha,\beta,\delta,\ldots}$ we mean the contribution of an expression ϱ to the component of the type $(\alpha, \beta, \delta, \ldots)$, having weight α, defect β, degree δ in (ν, u), and other specifications denoted by the dots.

To simplify further computations we introduce the variables

$$\omega = u + i\nu, \qquad \overline{\omega} = u - i\nu.$$

Thus, the degree of a polynomial in (u, ν) equals the degree in $(\omega, \overline{\omega})$. So, the expression (5) for $H_{\gamma_0,d,m}$ takes the form

$$H_{\gamma_0,d,m} = \sum_m \sum_{\alpha=0}^m \omega^{m-\alpha}\overline{\omega}^\alpha Q_{m-\alpha}(z, \bar z). \tag{13}$$

Consider first case (i) of (7). Collecting in $\mathcal{H}$ the terms of type $(\gamma_0+1, d+1, m)$ we obtain

$$\mathcal{H}_{\gamma_0+1,d+1,m} = \left\{ \left(iz^n \left(2z\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial w} \right) + \overline{\omega}\frac{\partial}{\partial \overline{z^1}} \right) H_{\gamma_0,d,m} \right. \\ \left. - 2iz^n H_{\gamma_0,d,m} + 2\operatorname{Re}(\langle f_{\gamma_0}, z\rangle - g_{\gamma_0+1}|_{v=\nu}) \right\} \to_{\gamma_0+1,d+1,m} . \tag{14}$$

REMARK. The differential operator in the latter formula is applied to $H_{\gamma_0,d,m}$ only, the term of minimal degree in $(\omega, \overline{\omega})$, because neither of

$$2iz^n\frac{\partial}{\partial z}, \quad iz^n\omega\frac{\partial}{\partial w}, \quad \overline{\omega}\frac{\partial}{\partial \overline{z^1}} \tag{15}$$

decreases the degree in $(\omega, \overline{\omega})$. Moreover, the operators $iz^n\omega\partial/\partial w$ and $\overline{\omega}\partial/\partial\overline{z^1}$ increase the degree in $(\omega, \overline{\omega})$ unless applied to $\omega^{m-\alpha}\overline{\omega}^\alpha$ in (13).

STEP 3. In this final part of the proof, we consider in detail the action of infinitesimal automorphisms on weighted components.

Exact computation shows that in terms of the contribution to the $(\gamma_0+1, d+1, m)$ type component, the operators (15) can be substituted, respectively, by

$$\begin{gathered} iz^n\left(2h\,\mathrm{id} + (\omega - \overline{\omega})\left(\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\overline{\omega}} \right) \right), \\ iz^n\omega\left(\frac{\partial}{\partial\omega} + \frac{\partial}{\partial\overline{\omega}} \right), \quad iz^n\overline{\omega}\left(\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\overline{\omega}} \right). \end{gathered} \tag{16}$$

Summing up the operators in (16) we obtain a diagonal operator D:

$$2iz^n D = 2iz^n\left(h\,\mathrm{id} + \omega\frac{\partial}{\partial\omega} \right). \tag{17}$$

So far, in (14) we obtain

$$\mathcal{H}_{\gamma_0+1,d+1,m} = 2iz^n\sum_{\alpha=0}^{m}(m-\alpha+h-1)\omega^{m-\alpha}\overline{\omega}^\alpha Q_{m-\alpha}(z,\overline{z}) \\ + (2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \to_{\gamma_0+1,d+1,m}). \tag{18}$$

Suppose that $\chi_{\gamma_0-1} \neq 0$. Then the last term in (18), i.e.

$$2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \to_{\gamma_0+1,d+1,m},$$

contains a nontrivial $\mathcal{R}$-component. So, $DH_{\gamma_0,d,m}$ must contain $\mathcal{R}$-terms as well. It is easy to check that the two operators that compose $2iz^nD$, namely, $2iz^n\partial/\partial z$ and $iz^n\omega\partial/\partial w$, preserve the space $\mathcal{N}$. So, the $\mathcal{R}$-component in $DH_{\gamma_0,d,m}$ may only arise from

$$\overline{\omega}\frac{\partial}{\partial\overline{z^1}}H_{\gamma_0,d,m} \to_{\gamma_0+1,d+1,m} . \tag{19}$$

The degree m in $(\omega, \overline{\omega})$ does not increase if $\overline{\omega}\partial/\partial\overline{z^1}$ is applied only to ν (and not to $Q_{m-\alpha}(z,\overline{z})$).

What sort of $\mathcal{R}$-component may arise?

The term $\mathcal{R}_{2,1}$ cannot appear in (19) from $\overline{\omega}\frac{\partial}{\partial \overline{z^1}} H_{2,2}$ because, by (6), $\frac{\partial}{\partial \nu} H_{2,2} = 0$.

Analogously, $\mathcal{R}_{3,2} = \langle z, b\rangle \nu^2$ cannot arise from (18) because $\frac{\partial^3}{\partial \nu^3} H_{3,3} = 0$.

The term $\mathcal{R}_{3,1}$ may arise from $u^{m-1}\nu Q_{2,1}(z,\overline{z})$ in $H_{3,2}$ as

$$\overline{\omega}\frac{\partial}{\partial \overline{z^1}}(u^{m-1}\nu Q_{2,1}) \to_{\mathcal{R}} u^m z^n Q_{2,1} = u^m z^n \langle Q_{2,0}, z\rangle.$$

This implies that for the compensation of the $\mathcal{R}$-component, the vector field $\chi_{\gamma_0 - 1}$ should have the form

$$\chi_{\gamma_0-1} = -\omega^m z^n Q_{2,0}\frac{\partial}{\partial z}.$$

Therefore, by (18) and (12) we have

$$\mathcal{H}_{\gamma_0+1,2,m} = -2i\omega^m \langle z^n Q_{2,0}, z\rangle z^n \sum_{\alpha=0}^{m} (m-\alpha+2-1)\omega^{m-\alpha}\overline{\omega}^{\alpha} Q_{m-\alpha}(z,\overline{z}) = 0.$$

Thus,

$$\mathcal{H}_{\gamma_0,2,m} = \frac{1}{2i(m+1)}\omega^m \langle Q_{2,0}, z\rangle.$$

But in the usual coordinates the latter expression contains a $(3,1)$ term

$$\frac{1}{2i(m+1)} u^m \langle Q_{2,0}, z\rangle,$$

which does not lie in the space $\mathcal{N}$. Contradiction. So, $\mathcal{R}_{3,1}$ cannot arise either.

The term $\mathcal{R}_{k,1}$, $k \geq 4$, can arise either from $u^{m-2}\nu^2 Q_{k-2,0}$ or from $u^{m-1}\nu Q_{k-1,1}(z,\overline{z})$. The second option can be considered as above. Consider the first option:

$$\overline{\omega}\frac{\partial}{\partial \overline{z^1}}(u^{m-2}\nu Q_{k-2,0}) \to_{\mathcal{R}} u^m z^n Q_{2,1} = 2u^{m-1}\nu z^n Q_{k-2,0}.$$

It follows as above that for the compensation of the $\mathcal{R}$-component we must have

$$\chi_{\gamma_0-1} = -2\omega^{m-1}\nu z^n Q_{k-2,0} z\frac{\partial}{\partial z}$$

and so,

$$\begin{aligned} 2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} &= i\omega^{m-1}(\omega-\overline{\omega})z^n Q_{k-2,0} \\ &= i\omega^m z^n Q_{k-2,0} - i\omega^{m-1}\overline{\omega} z^n Q_{k-2,0}. \end{aligned}$$

Equations (12) and (18) imply that

$$\begin{aligned}0 &\equiv \mathcal{H}_{\gamma_0+1,k-1,m}\\ &= i\omega^m z^n Q_{k-2,0} - i\omega^{m-1}\overline{\omega} z^n Q_{k-2,0}\\ &\quad + 2iz^n \sum_{\alpha=0}^{m} (m-\alpha+k-3)\omega^{m-\alpha}\overline{\omega}^{\alpha} Q_{m-\alpha}(z,\overline{z}).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{H}_{\gamma_0,k-2,m} &= -\frac{i}{m+k-3}\omega^m Q_{k-2,0} + \frac{i}{m+k-4}\omega^{m-1}\overline{\omega} Q_{k-2,0}\\ &= Q_{k-2,0}\Big(-\frac{i}{m+k-3}\omega^m + \frac{i}{m+k-4}\omega^{m-1}\overline{\omega}\Big).\end{aligned}$$

The latter does not lie in $\mathcal{N}$ since it contains a $(k-1,0)$ term. Contradiction.

Suppose now that $\chi_{\gamma_0-1} = 0$. Then (12) implies that

$$DH_{\gamma_0,d,m} = 2iz^n \sum_{\alpha=0}^{m} (m-\alpha+h-1)\omega^{m-\alpha}\overline{\omega}^{\alpha} Q_{m-\alpha} = 0.$$

This can only happen in two cases:

(i) $h = 1, \alpha = m$,
(ii) $h = 0, \alpha = m-1$.

In case (i),

$$H_{\gamma_0,d,m} = \omega^m Q_{1,0},$$

which contains $u^m Q_{1,0} \in \mathcal{R}$; in case (ii),

$$H_{\gamma_0,d,m} = \omega^{m-1}\overline{\omega} Q_{1,0},$$

which contains $u^m Q_{1,0} \in \mathcal{R}$ as well. Contradiction. Hence, $a = 0$ in the set of initial data of Φ in case (i) of (7).

Case (ii) of (7) is similar: instead of $H_{\gamma_0,d,m}$ we take

$$H_{\gamma_0,d,m,d_{\bar{1}}} = \sum_m \sum_{\alpha=0}^{m} \omega^{m-\alpha}\overline{\omega}^{\alpha}\overline{z^1}^{d_{\bar{1}}} Q_{m-\alpha-d_{\bar{1}}}(z,\overline{z}),$$

the component of $H_{\gamma_0,d,m}$ of minimal degree $d_{\bar{1}}$ in $\overline{z^1}$.

If $d_{\bar{1}} = 0$ then the proof goes literally as above. Let $d_{\bar{1}} \neq 0$. Then the operators

$$2iz^1\frac{\partial}{\partial z}, \qquad iz^1\omega\frac{\partial}{\partial w}, \qquad \overline{\omega}\frac{\partial}{\partial \overline{z^1}},$$

being applied to $\omega^{m-\alpha}\overline{\omega}^{\alpha}$, contribute to the components of degrees m and $m+1$ in $(\omega,\overline{\omega})$. We are interested in the contribution to the minimal degree m:

$$
\begin{aligned}
(20)\qquad \mathcal{H}_{\gamma_0+1,d+1,m,d_{\bar 1}-1} &= 2i\frac{\langle z,z\rangle'}{\overline{z^1}}(D-\mathrm{id})H_{\gamma_0,d,m,d_{\bar 1}} \\
&\quad + (2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \to_{\gamma_0+1,d+1,m,d_{\bar 1}-1}) \\
&= 2i\langle z,z\rangle' \sum_{\alpha=0}^{m}(m-\alpha+h-1)\omega^{m-\alpha}\overline{\omega}^{\alpha}\overline{z^1}^{d_{\bar 1}-1}Q_{m-\alpha-d_{\bar 1}}(z,\bar z) \\
&\quad + (2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \to_{\gamma_0+1,d+1,m,d_{\bar 1}-1})
\end{aligned}
$$

(see (4) and (17) for $\langle z,z\rangle'$ and D). The latter expression does not vanish for the same reasons as above. This proves that $a=0$ in both cases (i) and (ii) of (7), and so $\chi_1=0$.

It remains to show that $r=0$. Consider

$$\chi_2 = rwz\frac{\partial}{\partial z} + rw^2\frac{\partial}{\partial w}.$$

It surely preserves the space $\mathcal{N}$, so $\chi=\chi_2$. But it is clear that $2\operatorname{Re}\chi_2$ does not annihilate the equation of M (it suffices to consider the component of H_{γ_0,d,m,d_u} of maximal degree d_u in u and to observe that χ_2 on this component is just multiplication by u). ∎

3. Counterexample for 2-codimensional CR manifolds. Let $M\in\mathbb{C}^4$ be a real-analytic Levi-nondegenerate CR surface of codimension 2. Set $z=(z^1,z^2)$, $w=(w^1=u^1+iv^1,\ w^2=u^2+iv^2)$ for coordinates in $\mathbb{C}^4$. Up to a nondegenerate linear transformation in (z,w) the Levi form of M may have one of three types, namely, elliptic, hyperbolic or parabolic (see [Lo2]).

The isotropy group G_0M is defined as for hypersurfaces. Set

$$G_{0,\mathrm{id}}M = \{\Phi\in G_0M : d\Phi(0)|_{T_0^{\mathbb{C}}M} = \{\mathrm{id}\}\}.$$

If M is a quadratic surface, i.e. it is locally equivalent to a CR quadric, its automorphism group and, in particular, $G_{0,\mathrm{id}}M$ are described in [ES]. If M is nonquadratic the description of G_0M remains an open question.

This problem has recently been solved for CR manifolds with hyperbolic Levi form by Schmalz [Sch].

In contrast to Theorem 2 even for nonquadratic M the subgroup $G_{0,\mathrm{id}}M$ may be nontrivial as the example below shows:

Let M be a surface with Levi form of parabolic type, defined by the equations

$$v^1=|z^1|^2,\qquad v^2=2\operatorname{Re}z^1\overline{z^2}+|z^1|^8.$$

To check that M is nonquadratic one can estimate the dimension of the automorphism group.

Let χ be the vector field

$$\chi = i\Big(2(z^1)^2\frac{\partial}{\partial z^2} + 2z^1w^1\frac{\partial}{\partial w^2} - iw^1\frac{\partial}{\partial z^2}\Big). \tag{21}$$

It is easy to check that χ and M satisfy the identity

$$2\operatorname{Re}\chi\begin{pmatrix} |z^1|^2 - \frac{1}{2i}(w^1-\overline{w^1}) \\ 2\operatorname{Re} z^1\overline{z^2} - \frac{1}{2i}(w^2-\overline{w^2}) + |z^1|^8\end{pmatrix}\Bigg|_M \equiv 0.$$

Therefore, Φ generates a 1-parameter subgroup in $G_{0,\mathrm{id}}M$, namely, $\Phi^t = \exp(t\chi)$, since

$$d\Phi^t(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}.$$

The 1-parameter subgroup Φ^t consists of the polynomial transformations

$$z^1 \mapsto z^1, \qquad z^2 \mapsto z^2 + tw^1 + 2it(z^1)^2,$$
$$w^1 \mapsto w^1, \qquad w^2 \mapsto w^2 + 2itw^1z^1.$$

4. Group structure on the family of normalisations. In this section we consider M in a CM normal form at the origin. Initial data $\sigma = (U, a, \lambda, r)$ induce on the family $\mathcal{N}_0(M)$ of CM normalisations a group structure of $\operatorname{Aut}_0 Q$ by the relation

$$\Phi_{\sigma_1} * \Phi_{\sigma_2} = \Phi_\sigma,$$

where $\sigma_j = (U_j, a_j, \lambda_j, r_j)$, $j = 1, 2$, and

$$\sigma = (U_1U_2, a_1 + \lambda_1U_1^{-1}a_2, \lambda_1\lambda_2, r_1 + r_2 - 2\operatorname{Im}\langle\lambda_2U_2a_2, a_1\rangle).$$

A priori it is not clear whether the $*$-operation on $\mathcal{N}_0$ coincides with the natural composition in $\operatorname{Aut}_0(\mathbb{C}^n)$.

The following question is motivated by I. Lieb:

QUESTION 1. *For which M the family $\mathcal{N}_0(M)$ forms a subgroup in $\operatorname{Aut}_0(\mathbb{C}^n)$ under the composition of biholomorphisms?*

For example, projective transformations of the form

$$z \mapsto \frac{\lambda Uz}{1-rw}, \qquad w \mapsto \frac{\lambda^2 w}{1-rw}$$

preserve the CM normal form, so the subfamily $\{\Phi_{\sigma_{\mathbb{P}}}, \sigma_{\mathbb{P}} = \{(U,0,\lambda,r)\}\}$ forms a subgroup in $\operatorname{Aut}_0(\mathbb{C}^n)$.

It turns out, however, that the whole family $\mathcal{N}_0(M)$ forms a group only in the case of a hyperquadric.

THEOREM 3. *If $(\mathcal{N}_0(M), \circ)$ is a subgroup in $\operatorname{Aut}_0(\mathbb{C}^n)$ then $M = Q = \{\operatorname{Im} w = \langle z, z\rangle\}$ for a nondegenerate hermitian form $\langle z, z\rangle$.*

REMARK. Let $\phi \in \mathcal{N}_0(M)$ and consider $\mathcal{N}_0(\Phi(M))$. Under the assumption of the theorem we observe that $\mathcal{N}_0(\Phi(M)) = \mathcal{N}_0(M)$. Indeed, let $\Psi \in \mathcal{N}_0(\Phi(M))$. Then $\Psi \circ \Phi = \eta \in \mathcal{N}_0(M)$, and, therefore, $\Psi = \eta \circ \Psi^{-1} \in \mathcal{N}_0(M)$ since $\mathcal{N}_0(M)$ is a group.

Proof (of Theorem 3). Consider a subgroup in $\mathcal{N}_0(M)$, namely

$$\mathcal{N}_{0,\mathrm{id}}(M) = \{\Phi \in \mathcal{N}_0(M) : d\Phi(0)|_{w=0} = \mathrm{id}\}.$$

The corresponding subgroup $\mathrm{Aut}_{0,\mathrm{id}}(Q) \subset \mathrm{Aut}_0(Q)$ consists of the transformations

$$z \mapsto \frac{z + aw}{1-\delta}, \qquad w \mapsto \frac{w}{1-\delta},$$

where $\delta = 2i\langle z, a\rangle + (r + i\langle a, a\rangle)w,\ a \in \mathbb{C}^n,\ r \in \mathbb{R}$.

LEMMA 1. *Under the assumptions of Theorem* 3, $\mathcal{N}_{0,\mathrm{id}}(M) = \mathrm{Aut}_{0,\mathrm{id}}(Q)$.

Proof. Suppose that $0 \neq a \in \mathbb{C}^n$ and let $\Phi_a = \Phi_{(\mathrm{id},a,1,r)}$ and $\Phi_\lambda = \Phi_{(\mathrm{id},0,\lambda,0)}$. By the above remark, Φ_a is a normalisation of both M and $\Phi_\lambda(M)$. Suppose that the weighted expansion of a normal form of M is

$$M: \quad v = \langle z, z\rangle + N_{\gamma_0}(z, \overline{z}, u) + \ldots, \qquad N_{\gamma_0} \neq 0.$$

Then the equation of $M_\lambda = \Phi_\lambda(M)$ takes the form

$$M_\lambda: \quad v = \langle z, z\rangle + \lambda^{\gamma_0 - 2} N_{\gamma_0} + \ldots + \lambda^{\gamma-2} N_\gamma + \ldots,$$

since Φ_λ is just the scaling $z \mapsto \lambda z,\ w \mapsto \lambda^2 w$.

As in Theorem 1 we take the holomorphic vector field

$$\chi = (aw + 2i\langle z, a\rangle z)\frac{\partial}{\partial z} + 2i\langle z, a\rangle w \frac{\partial}{\partial w} + rw^2 \frac{\partial}{\partial w} + \chi_{\gamma_0 - 1} + \ldots + \chi_\gamma + \ldots,$$

where $\chi_\gamma = f_{\gamma+1}(z, w)\partial/\partial z + g_{\gamma+2}(z, w)\partial/\partial w$, such that $\Phi_a = \exp(\chi)$ and χ generates a 1-parameter subgroup $\Phi_t a = \exp(t\chi)$ in $\mathcal{N}(M)$ and also in $\mathcal{N}_\lambda(M)$.

This fact can be expressed analytically in the form of two identities:

$$\mathcal{R}\biggl(2\,\mathrm{Re}\,\chi\biggl(\langle z, z\rangle - \frac{1}{2i}(w - \overline{w}) + N_{\gamma_0} + \ldots\biggr)\bigg|_M\biggr) = 0$$

and

$$\mathcal{R}\biggl(2\,\mathrm{Re}\,\chi\biggl(\langle z, z\rangle - \frac{1}{2i}(w - \overline{w}) + \lambda^{\gamma_0 - 2} N_{\gamma_0} + \ldots\biggr)\bigg|_{M_\lambda}\biggr) = 0.$$

Let γ be the first integer such that $\chi_{\gamma-1} \neq 0$. Then the component $\mathcal{R}_\gamma$ of weight γ in both identities contains the polynomial $\langle f_\gamma(z, u), z\rangle + g_{\gamma+1}(z, u)$

that does not depend on λ. On the other hand, this polynomial must cancel with the $\mathcal{R}$-component of

$$2\operatorname{Re}\left((aw+2i\langle z,a\rangle z)\frac{\partial}{\partial z}+2i\langle z,a\rangle w\frac{\partial}{\partial w}\right)$$

applied to N_γ and to $\lambda^{\gamma-2}N_\gamma$, respectively. Obviously, such a cancellation cannot happen simultaneously for different λ. This proves that $\gamma=\infty$ and, therefore, that $\mathcal{N}_{0,\mathrm{id}}(M)$ is generated by the vector fields

$$(aw+2i\langle z,a\rangle z)\frac{\partial}{\partial z}+2i\langle z,a\rangle w)\frac{\partial}{\partial w}+rw^2\frac{\partial}{\partial w}$$

that form the Lie algebra of the group $\mathrm{Aut}_{0,\mathrm{id}}(Q)$. ■

It remains to show that if $\mathcal{N}_{0,\mathrm{id}}(M)=\mathrm{Aut}_{0,\mathrm{id}}(Q)$, then $M=Q$. Consider the family $\{M_a\}=\{\Phi_a(M)\}$ for all $a\in\mathbb{C}^n$. Assume that M is not quadratic and choose a_0 such that the equation of M_{a_0},

$$v=\langle z,z\rangle+N_{\varepsilon,\delta}+\ldots,$$

contains the term of the smallest degree ε in $(z,\overline{z},u)$ and of the least degree δ in $(z,\overline{z})$ among all other M_a. We may assume that $\partial N_{\varepsilon,\delta}/\partial z^1\neq 0$. Setting $a=(1,0,\ldots,0)$ and applying the corresponding Φ_a to M_{a_0} we see that $\Phi_a(M_{a_0})$ contains in the (ε,δ) nonquadratic component terms of degree $\delta-1$ in $(z,\overline{z})$, which contradicts the extremal property of M_{a_0}. This observation completes the proof of Theorem 3. ■

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