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The holomorphic extension of C^k CR functions on tube submanifolds

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Abstract. We show that a CR function of class C^k, $0 \le k < \infty$, on a tube submanifold of $\mathbb{C}^n$ holomorphically extends to the convex hull of the submanifold. The extension and all its derivatives through order k are shown to have nontangential pointwise boundary values on the original tube submanifold. The C^k-norm of the extension is shown to be no bigger than the C^k-norm of the original CR function.

1. Definitions and main results. Recently, Boivin and Dwilewicz [BD] have generalized Bochner's Tube Theorem by showing that continuous CR functions on a tube submanifold of $\mathbb{C}^n$ holomorphically extend to its convex hull. In this paper, we generalize this result to CR functions of class C^k, for k a nonnegative integer. The technique we use (analytic discs) is different than the one used in [BD], and in the case of bounded CR functions provides an estimate on the norm of the extension.

For any submanifold S in $\mathbb{C}^n = \mathbb{R}^{2n}$, let $C^k(S)$ be the space of complex-valued functions on S with continuous derivatives through order k. We will identify an element in $C^k(S)$ with the restriction of an element of $C^k(\mathbb{R}^{2n})$ to S. For $f \in C^k(S)$, let

$$\|f\|_{k(S)} = \sup_{t\in S} \sum_{|\alpha|\le k} |D^\alpha f(t)|.$$

Here,

$$D^\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_{2n}}}{\partial t_1^{\alpha_1}\ldots\partial t_{2n}^{\alpha_{2n}}}$$

for any multi-index $\alpha = (\alpha_1,\ldots,\alpha_{2n})$. Of course, if S is not compact, $\|f\|_{k(S)}$ could be infinite.

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We will be working in $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ with coordinates $x + iy$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. The set S of interest will be a connected tube of the form $M = N + i\mathbb{R}^n$ where N is a smooth submanifold of $\mathbb{R}^n$ (not necessarily compact).

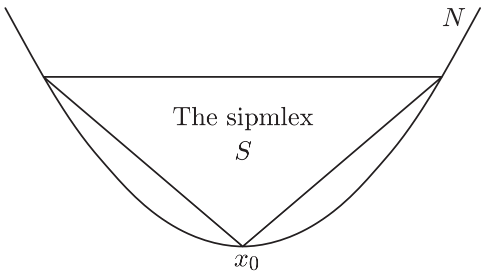
Our main theorem is the following.

Theorem 1 (Extension Theorem). *Suppose N is a connected submanifold of $\mathbb{R}^n$, and let $M = N + i\mathbb{R}^n$ be the tube over N. Let $\widehat{N}$ and $\widehat{M} = \widehat{N} + i\mathbb{R}^n$ denote the interior of the convex hull of N and M, respectively. If $\widehat{M}$ is nonempty then each CR function f on M of class C^k $(0 \le k < \infty)$ extends to a unique holomorphic function F on $\widehat{M}$ with $\|F\|_{k(\widehat{M})} \le \|f\|_{k(M)}$.*

Remark 1. To say that F extends f means that the boundary values of F on M equal f in the following nontangential sense: let S be any convex simplex contained in $\widehat{N}$ with vertex $x_0 \in N$; then

$$\lim_{S \ni x \to x_0,\, y \to y_0} D^\alpha F(x+iy) = D^\alpha f(x_0 + iy_0) \quad \text{for } |\alpha| \le k.$$

The above result for $k = 0$ with this notion of nontangential boundary values (but without the estimate on the extension) is the main result in [BD].



Remark 2. We are not assuming $\|f\|_{k(M)} < \infty$ for the above theorem. Of course, if $\|f\|_{k(M)} = \infty$, then the estimate on the extension is trivial.

A key result that is used in the proof of the above theorem is the following global C^k-version of Baouendi and Treves' Approximation Theorem for CR functions on tubes. For completeness, we include a proof.

Theorem 2 (Approximation Theorem, [BT]). *Let N be a connected submanifold of $\mathbb{R}^n$ and let $M = N + i\mathbb{R}^n$ be the tube over N. Let $0 \le k < \infty$ be an integer. If f is a CR function of class C^k on M then there exists a sequence of entire functions F_j on $\mathbb{C}^n$ such that for each compact set $K \subset M$,*

$$\lim_{j \mapsto \infty} \|F_j - f\|_{k(K)} = 0.$$

2. Proof of the Approximation Theorem. Fix any compact set $K \subset M$. Choose $R > 0$ and $A > 0$ so that $K \subset \{z = x + iy : |x| \le A,\ |y| \le R\}$.

Choose a smooth function g depending only on y with $g(y) = 1$ on $\{|y| \le R + 2A + 1\}$ and $g(y) = 0$ for $\{|y| \ge R + 2A + 2\}$. For any $z = x + iy$, let

$$T_z = \{x + it : t \in \mathbb{R}^n\}$$

(i.e. the tube over the point x passing through z). For $z = x + iy \in M$, let

$$G_\varepsilon(z) = \frac{1}{\varepsilon^n \pi^{n/2}} \int\limits_{\zeta \in T_z} f(\zeta) g(\zeta) e^{\varepsilon^{-2}[\zeta - z]^2} \, d\zeta$$

where $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$ and where for $w \in \mathbb{C}^n$, $[w]^2 = w_1^2 + \ldots + w_n^2$. Another description of G_ε is given by

$$G_\varepsilon(z) = \frac{1}{\varepsilon^n \pi^{n/2}} \int\limits_{t \in \mathbb{R}^n} (fg)(x + it) e^{-\varepsilon^{-2}|t-y|^2} \, dt.$$

Viewed this way, G_ε is the convolution of fg in the tube direction with an approximation to the identity (given by the spatial slices of the heat kernel). Since $g = 1$ on K, the following lemma can easily be established using standard techniques (e.g. see p. 13 in [SW]).

Lemma 1. *Let k be a nonnegative integer and let $f : M \to C$ be of class C^k (not necessarily CR). For the compact set K as above,*

$$\lim_{\varepsilon \to 0} \|G_\varepsilon - f\|_{k(K)} = 0.$$

Since the domain of integration defining $G_\varepsilon(z)$ depends on z, this function is not, in general, analytic in z. However, if f is CR on M, then G_ε can be modified so that its domain of integration can be made independent of z as the next lemma shows. By a translation, assume that the origin 0 belongs to N.

Lemma 2. *For $z \in \mathbb{C}^n$, let*

$$F_\varepsilon(z) = \frac{1}{\varepsilon^n \pi^{n/2}} \int\limits_{\zeta \in T_0} f(\zeta) g(\zeta) e^{\varepsilon^{-2}[\zeta - z]^2} \, d\zeta.$$

For each $\varepsilon > 0$, F_ε is entire. If f is CR on M, then $\|F_\varepsilon - f\|_{k(K)} \to 0$ as $\varepsilon \to 0$.

Proof. $F_\varepsilon(z)$ is analytic in z in view of the following observations: the domain of integration, $T_0 = \{0 + it : t \in \mathbb{R}^n\}$, is independent of z; the kernel $e^{\varepsilon^{-2}[it-z]^2}$ is analytic in z and has exponential decay in t uniformly in z belonging to a compact set in $\mathbb{C}^n$; and the function $t \mapsto f(0 + it) g(t)$ is continuous and compactly supported.

For each $x \in N$, let $\gamma : [0, 1] \to N$ be a smooth path which connects $0 = \gamma(0)$ to $x = \gamma(1)$ (recall that N is connected by assumption). Let

$$\widetilde{T}_z = \{\gamma(u) + it : t \in \mathbb{R}^n,\ 0 \le u \le 1\}.$$

The (manifold) boundary of $\widetilde{T}_z$ is $T_z - T_0$. So by Stokes' theorem

$$F_\varepsilon(z) = G_\varepsilon(z) - \frac{1}{\varepsilon^n \pi^{n/2}} \int\limits_{\zeta \in \widetilde{T}_z} d_\zeta \{g(\zeta) f(\zeta) e^{\varepsilon^{-2}[\zeta - z]^2}\, d\zeta\}. \tag{1}$$

We must show the integral on the right converges to zero uniformly for $z \in K$ as $\varepsilon \to 0$. In view of the presence of $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, the d_ζ reduces to $\overline{\partial}_\zeta$ in the last integral. Since f is CR, the $\overline{\partial}_\zeta$ only applies to $g(\zeta)$ which has support in the set

$$\{R + 2A + 1 \le |\operatorname{Im} \zeta| \le R + 2A + 2\}.$$

For $\zeta = s + it$ in this set and for $z = x + iy \in K = \{z = x + iy : |x| \le A, |y| \le R\}$, we have

$$\operatorname{Re}[\zeta - z]^2 = |x - s|^2 - |t - y|^2 \le 4A^2 - (2A + 1)^2 \le -1.$$

Therefore, the exponential term in the integral is dominated above (in absolute value) by $e^{-\varepsilon^{-2}}$ which converges to zero as $\varepsilon \to 0$. In a similar manner, derivatives of this term can be easily shown to converge to zero as $\varepsilon \to 0$. This completes the proof of the lemma.

By taking an increasing sequence of compact sets and a corresponding decreasing sequence of ε's, we can construct the desired sequence F_j for the Approximation Theorem.

Remark 1. If f is bounded (or more generally if f is L^p in the tube direction, $1 \le p \le \infty$), then the integral in (1) converges to zero as $R \to \infty$. Thus under these assumptions, we may replace g by 1 and conclude that $F_\varepsilon(z) = G_\varepsilon(z)$ for each $\varepsilon > 0$.

Remark 2. Technically speaking, the proof of the above lemma assumes that f is continuously differentiable for the Stokes theorem step (which is not assumed if $k = 0$). However, Stokes' theorem applies to currents (in fact Stokes' theorem becomes the definition of the exterior derivative of a current) and so the above argument can be dualized and applied to our context where f is assumed to be a distribution given by a continuous function.

3. Analytic discs. To prove the Extension Theorem, we will show that the sequence of entire functions constructed in the Approximation Theorem converges on the interior of the convex hull of the given tube, $M = N + i\mathbb{R}^n$. This is accomplished by showing that the interior of the convex hull of M can be realized as the union of centers of analytic discs with boundaries that are contained in M. The Mean Value Principle for analytic functions then implies that the sequence of entire functions converges on the interior of the convex hull of M.

To be precise, an *analytic disc* is an analytic map $A : D = \{\zeta \in \mathbb{C} : |\zeta|<1\} \to \mathbb{C}^n$ with boundary values $A|_{\{|\zeta|=1\}}$ in $L^2(\{|\zeta| = 1\})$. The *boundary* of the analytic disc A is its restriction to the unit circle. We will often identify the boundary of the disc with its image. The key lemma on the existence of analytic discs for tubes is the following.

LEMMA 3. *Suppose $e_0, \ldots, e_m$ are vectors in N that span a convex simplex S with nonempty interior in $\mathbb{R}^n$ $(m \ge n)$. Then each point $z = x + iy$ with $x \in S$ and $y \in \mathbb{R}^n$ can be realized as the center of an analytic disc*, A (*i.e.* $z = A(\zeta = 0)$), *whose boundary is contained in $\bigcup_{j=0}^m T_{e_j} \subset M$. Moreover*,

(1) *If*

$$x = \sum_{j=0}^{m} \lambda_j e_j \in S \quad \text{with } \lambda_j \ge 0 \text{ and } \sum_j \lambda_j = 1$$

then the boundary of the analytic disc $A(\cdot) = A(\lambda, y)(\cdot)$ depends continuously on $\lambda = (\lambda_0, \ldots, \lambda_m)$ and y in the $L^2(\{|\zeta| = 1\})$-norm.

(2) *Given $\delta > 0$, there exists a constant $C > 0$, depending only on δ and the vertices $e_0, \ldots, e_m$, such that the measure of the set $\{\zeta : |\zeta| = 1, |\operatorname{Im} A(\lambda, y)(\zeta) - y| \ge C\}$ is less than δ.*

P r o o f. To establish this lemma, we will specify the desired analytic disc $A = u + iv : D \to \mathbb{C}^n$ by specifying A on the boundary $\{e^{2\pi it} : 0 \le t < 1\}$ which we identify with the unit interval $I = [0, 1)$. Partition the unit interval I into a disjoint union of intervals, I_j, of length λ_j, $j = 0, \ldots, m$, where $I_0 = [0, \lambda_0)$, $I_1 = [\lambda_0, \lambda_0 + \lambda_1)$, etc. Let χ_{I_j} be the characteristic function of the interval I_j (one on I_j, zero everywhere else). Define $u : I \to N$ by

$$u(t) = \sum_{j=0}^{m} e_j \chi_{I_j}(t).$$

For $y \in \mathbb{R}^n$, let

$$v = v(\lambda, y) = T(u(\lambda)) + y,$$

where T is the Hilbert transform.

Now let $A(\lambda, y)(e^{2\pi it}) = u(\lambda)(t) + iv(\lambda, y)(t)$. Then $A(\lambda, y)(\zeta)$ for $|\zeta| = 1$ extends analytically to $\{|\zeta| < 1\}$ (by the definition of the Hilbert transform). Its boundary lies in $\bigcup_{j=0}^m T_{e_j} \subset M$ since $\operatorname{Re} A = u$ takes values in $\{e_0, \ldots, e_m\}$.

We claim $A(\zeta = 0) = x + iy$, where $x = \sum_j \lambda_j e_j$. Since $v = T(u) + y$ and since the Hilbert transform produces the unique harmonic conjugate which vanishes at the origin, clearly $\operatorname{Im}(A)(\zeta = 0) = y$. The real part, $\operatorname{Re} A(0)$, is obtained by averaging its boundary values:

$$\mathrm{Re}(A)(\zeta=0)=\int_0^1 u(\lambda)(t)\,dt=\sum_{j=0}^m\int_0^1 e_j\chi_{I_j}\,dt=\sum_{j=0}^m\lambda_j e_j=x.$$

The length of I_j is λ_j and so $u=u(\lambda)$ depends continuously on $\lambda=(\lambda_0,\ldots,\lambda_m)$ in the $L^2(I)$-norm. Since $T: L^2(I)\to L^2(I)$ is continuous, $v(\lambda,y)=T(u(\lambda,y))$ also depends continuously on λ and y in the $L^2(I)$-norm. This establishes (1). Note, however, that T is not continuous in the sup-norm. So Tu is unbounded near the points of discontinuity of u. To establish (2), let $t_0=0,t_1,t_2,\ldots,t_{m+1}=1$ be the endpoints of the intervals $I_0,\ldots,I_m$ (the discontinuities of u). By examining the kernel of the Hilbert transform and using the fact that the Hilbert transform maps constant functions to zero, the following estimate can easily be established:

$$\text{If}\quad t\notin\bigcup_{j=0}^{m+1}\{|t-t_j|\le\delta\}\quad\text{then}\quad |(Tu)(t)|\le C|\ln\delta|$$

where C is a uniform constant. Since the measure of the set on the left is $2(m+1)\delta$, the second claim in Lemma 3 is established (relabeling δ as $2(m+1)\delta$ and C as $C|\ln\delta|$).

This completes the proof of the lemma.

4. The extension. We wish to show that the sequence of entire functions F_j which converges to our given CR function f on M (from the Approximation Theorem) also converges uniformly on the compact subsets of the convex hull of M. It suffices to show that the F_j are uniformly Cauchy on the compact subsets of each tube of the form $S+i\mathbb{R}^n$ where S is the closed convex hull of any given set of vertices $e_0,\ldots,e_n\in N$. From here on, we fix a set of vertices $e_0,\ldots,e_n\in N$ and the resulting simplex S.

As is clear from the proof, the analytic discs constructed for the proof of Lemma 3 are unbounded in the tube (i.e. y) direction. In addition, the CR function and the sequence of entire functions F_j may be unbounded in the tube direction. Thus we will need the following lemma to control the growth of f and F_j in the tube direction.

LEMMA 4. *There exists a nonvanishing entire function E such that*

$$|E(z)|\ge\max\{|f(z)|,|F_j(z)|: j=0,1,\ldots\}$$

for all $z\in\bigcup_{k=0}^n T_{e_k}$.

We will postpone the proof of this lemma until after we complete the proof of the Extension Theorem.

The above lemma implies that the sequence $|F_j/E|$ is bounded by 1 on the set $\bigcup_{j=0}^n T_{e_j}$ which contains the boundaries of the family of discs constructed in Lemma 3.

We will now show that the sequence $\widetilde{F}_j = F_j/E$ is uniformly Cauchy on any set of the form $K_1 = S + i\{|y| \le C_1\}$. Let $z = x + iy$ be any point in K_1 and let $A(\lambda, y)$ be the analytic disc given in Lemma 3 with $A(\lambda, y)(\zeta = 0) = z$. By the Mean Value Theorem

$$\widetilde{F}_j(z) = \int_0^1 \widetilde{F}_j(A(\lambda, y)(e^{2\pi it}))\, dt.$$

From Lemma 3(2), for any $\delta > 0$, there is a uniform constant $C > 0$ and a set, I_δ, of measure at most δ such that $|\operatorname{Im} A(\lambda, y)(e^{2\pi it}) - y| \le C$ for all $t \in [0, 1) - I_\delta$. Let $K = S + i\{|y| \le C + C_1\}$. On the compact set K, the sequence $\widetilde{F}_j$ is uniformly Cauchy (from the Approximation Theorem). Therefore, $\widetilde{F}_j(A(\lambda, y)(e^{2\pi it}))$ is uniformly Cauchy for t in $[0, 1) - I_\delta$. On I_δ, $|\widetilde{F}_j(A(\lambda, y)(e^{2\pi it}))| \le 1$ from Lemma 4. From this, the uniform convergence of the $\widetilde{F}_j$ on K_1 easily follows.

Proof of Lemma 4. The following proof only uses the fact that f is continuous and that F_j are continuous and converging uniformly on the compact subsets of M (the fact that f is CR and F_j is entire is not needed). Let a_k, $k = 1, 2, \ldots$, be a strictly increasing sequence of real numbers with $\lim_{k\to\infty} a_k = \infty$. Let $S^0 = \{e_0, \ldots, e_n\}$ be the set of vertices for the simplex S and let

$$M_k = \sup_{z \in S^0 + i\{|y| \le a_k\}} \{|f(z)|, |F_j(z)| : j = 0, 1, \ldots\}.$$

Since the sequence F_j is converging uniformly on the compact set $S_0 + i\{|y| \le a_k\}$, each M_k is finite. The sequence M_k is increasing. It suffices to find an entire function, E, with $|E(x + iy)| \ge M_{k+1}$ on $a_k \le |y| \le a_{k+1}$.

We first find a nonvanishing entire function of one complex variable, $E(z)$, with $|E(0 + iy)| \ge M_{k+1}$ for $a_k \le |y| \le a_{k+1}$. To this end, let b_k be the average of a_k and a_{k-1}. Let N_k be a sequence of integer multiples of 4 so that

$$\left(\frac{|z|}{b_k}\right)^{N_k} \le \frac{1}{2^k} \quad \text{for } |z| \le a_{k-1}$$

and

$$e^{(iy/b_k)^{N_k}} \ge M_{k+1} \quad \text{for } |y| \ge a_k.$$

Let

$$E(z) = \prod_{j=1}^{\infty} e^{(z/b_j)^{N_j}}.$$

The first inequality above guarantees that the infinite product defining E converges uniformly on compact subsets of C to a nonvanishing entire function. Since the N_k are integer multiples of 4, each factor in $E(z)$ is real and greater than or equal to 1 for z on the imaginary axis. The second inequality guarantees that $|E(iy)| \geq M_{k+1}$ on $a_k \leq |y| \leq a_{k+1}$.

Now we complete the proof of the lemma. By a complex linear change of variables, assume that e_0 is the origin and $\{e_j : j = 1, \ldots, n\}$ is the standard basis for $\mathbb{R}^n$ (i.e. e_j is the vector with one in the jth component and zero in all the rest). Let $\widetilde{E}(z_1, z_2, \ldots, z_n) = E(\sin(2\pi z_1))$. Then $\widetilde{E}(z)$ is periodic in z_1 with period 1 and independent of $z_2, \ldots, z_n$. Thus if we can show that $|E(\sin(2\pi i y_1))| \geq M_{k+1}$ for $a_k \leq |y_1| \leq a_{k+1}$, then $\widetilde{E}$ will be our desired entire function. Now for $y \in \mathbb{R}$, $\sin(2\pi i y) = i \sinh(2\pi y)$ whose magnitude increases much faster than $|y|$. Since the $\{M_k\}$ is an increasing sequence, clearly the inequality $|E(\sin(2\pi i y))| \geq M_{k+1}$ for $a_k \leq |y| \leq a_{k+1}$ holds. This completes the proof of the lemma.

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