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On a problem of Seiberg and Witten

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Abstract. We describe alternate methods of solution for a model arising in the work of Seiberg and Witten on $N=2$ supersymmetric Yang–Mills theory and provide a complete argument for the characterization put forth by Argyres, Faraggi, and Shapere of the curve $\operatorname{Im} a_D/a = 0$.

1. The problem. In their work on $N = 2$ supersymmetric Yang–Mills theory, Seiberg and Witten pose the following problem [SW, §6].

PROBLEM. *Find a holomorphic section* $\binom{a_D(u)}{a(u)}$ *of the flat* $\mathbb{C}^2$ *bundle over* $\mathbb{C} \setminus \{\pm 1\}$ *with holonomy*

$$\begin{aligned} &\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} && \textit{counterclockwise about } u = \infty, \\ &\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} && \textit{counterclockwise about } u = 1, \\ &\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} && \textit{counterclockwise about } u = -1 \end{aligned} \tag{1.1}$$

satisfying the asymptotics

$$\begin{aligned} &\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \approx \begin{pmatrix} \frac{i}{\pi}\sqrt{2u}\log u \\ \sqrt{2u} \end{pmatrix} && \textit{near } u = \infty, \\ &\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \approx \begin{pmatrix} c_+(u-1) \\ a_+ + \frac{i}{\pi}c_+(u-1)\log(u-1) \end{pmatrix} && \textit{near } u = 1, \end{aligned} \tag{1.2}$$

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$$(1.2)_{\text{cont.}} \qquad \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \approx \begin{pmatrix} a(u) + c_-(u+1) \\ a_- + \frac{i}{\pi} c_-(u+1)\log(u+1) \end{pmatrix} \qquad \textit{near } u = -1$$

and the positivity condition

$$(1.3) \qquad \tau := \frac{da_D}{da} \quad \textit{has positive imaginary part.}$$

(In (1.2) we are viewing $\left(\begin{smallmatrix} a_D(u) \\ a(u) \end{smallmatrix}\right)$ as a multi-valued section of the trivial $\mathbb{C}^2$ bundle over $\mathbb{C} \setminus \{\pm 1\}$. Also, see §2 below for an indication of the significance of the matrices in (1.1).)

Seiberg and Witten use elliptic integrals to construct a solution of this problem. Bilal [Bil] uses a differential equations approach to construct the same solution.

In the physical application of this problem the "curve of marginal stability" γ defined by $\operatorname{Im} a_D/a = 0$ plays an important role. Seiberg and Witten suggest that this curve should look "something like $|u| = 1$." Fayyazuddin [Fay] shows that γ is a disjoint union of simple closed curves and that the puncture points ± 1 lie in the same component of γ. Argyres, Faraggi, and Shapere [AFS] provide a conformal mapping interpretation of γ implying that γ is indeed a single simple closed curve. (Their argument relies on an *ad hoc* assumption that a fundamental region maps onto a union of deck transformations of the same fundamental region.)

In §§2 through 7 below we provide an alternate method of solving the problem by applying very elementary complex-analytic arguments to suitably chosen single-valued mappings and differentials manufactured from the section $\left(\begin{smallmatrix} a_D(u) \\ a(u) \end{smallmatrix}\right)$. §§5, 10, and 11 combine to provide another method of solution via conformal mapping. Both methods should in particular serve to clarify uniqueness issues connected with this problem.

In §9 we show that a_D and a must indeed satisfy the differential equation used by Bilal and several other authors. §11 below contains a complete argument for the Argyres–Faraggi–Shapere description of γ.

2. The ratio τ. The domain $\mathbb{C} \setminus \{\pm 1\}$ is covered by the upper half-plane $\{\zeta : \operatorname{Im} \zeta > 0\}$; this covering can be chosen to map the hyperbolic triangle with vertices $0, 1, \infty$ to the lower half-plane with $0, 1, \infty$ mapping respectively to $1, -1, \infty$ [Ahl, 7.3.5].

a_D, a, and τ can be viewed as single-valued functions of ζ. The problem is set so that each matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ in the holonomy subgroup of $\mathrm{SL}(2, \mathbb{R})$ comes from the corresponding deck transformation $\zeta \mapsto (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$. Thus the deck transformation $\zeta \mapsto (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$ takes τ to $(\alpha\tau + \beta)/(\gamma\tau + \delta)$. In view of (1.3) it follows that τ induces a self-map of $\mathbb{C} \setminus \{\pm 1\}$ homotopic to the identity. But such a self-map must in fact be the identity (see Appendix A) so that ζ and τ are related by a deck transformation. But the asymptotic conditions (1.2) require ζ and τ to agree at $1, -1, \infty$, so that finally $\tau \equiv \zeta$.

Henceforth we let τ denote the covering variable, but in the spirit of [SW] we continue to write a_D and a as multiple-valued functions of u.

3. The differentials da_D, da. Now we can decouple the transformation laws for da_D and da. In particular, da transforms to $\gamma da_D + \delta da = (\gamma\tau + \delta)da$ as $d\tau$ transforms to $d\frac{\alpha\tau+\beta}{\gamma\tau+\delta} = (\gamma\tau+\delta)^{-2}d\tau$ so that $da^2 d\tau$ defines a single-valued cubic differential $h(u)du^3$ on $\mathbb{C} \setminus \{\pm 1\}$. (See [Leh, IV.1.4] for terminology.)

We have

$$h(u) \approx \begin{cases} \dfrac{i}{2\pi u^2} & \text{near } u = \infty, \\ \dfrac{-c_+^2 i}{\pi(u-1)} & \text{near } u = 1, \\ \dfrac{-c_-^2 i}{\pi(u+1)} & \text{near } u = -1. \end{cases}$$

Thus

$$c_+^2 = -c_-^2 = -\frac{1}{4} \quad \text{and} \quad h(u) = \frac{i}{2\pi(u^2-1)}.$$

This gives

$$a_D(U) = \int_1^U \tau\, da = \int_1^U \tau \sqrt{\frac{i}{2\pi(u^2-1)(d\tau/du)}}\, du$$

with a similar formula for a.

The next three sections outline a method for representing a_D and a in terms of τ without integration.

In [SW] the positivity condition (1.3) is motivated by the requirement that

$$ds_{\mathrm{SW}} := \sqrt{\operatorname{Im}\tau}\, |da|$$

defines a metric on $\mathbb{C} \setminus \{\pm 1\}$. Note that

$$ds_{\mathrm{SW}} = \left(\frac{|d\tau|}{\operatorname{Im}\tau}\right)^{-1/2} |h(u)du^3|^{1/2};$$

here $|d\tau|/\operatorname{Im}\tau$ is the Poincaré metric and $|h(u)du^3|$ is the cube of an incomplete flat metric on $\mathbb{C} \setminus \{\pm 1\}$.

4. The mapping a_D/a. The transformation laws for $\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}$ guarantee that

$$g(u)du := \frac{2a d\tau}{a_D - \tau a} + d \log \frac{d\tau}{du}$$

is a single-valued meromorphic differential on $\mathbb{C} \setminus \{\pm 1\}$ with asymptotics

$$g(u) \approx \begin{cases} \dfrac{-2}{u} & \text{near } u = \infty, \\ \dfrac{-1}{u-1} & \text{near } u = 1, \\ \dfrac{-1}{u+1} & \text{near } u = -1. \end{cases}$$

Hence $g(u)du$ is in fact meromorphic on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Away from $\infty, \pm 1$ the poles of $g(u)du$ must come from zeros of $a_D - \tau a$; since $\frac{d}{d\tau}(a_D - \tau a) = -a$, a standard computation shows that such poles must be simple with residue $-2 \operatorname{ord}_\tau(a_D - \tau a)$. Thus

$$\begin{aligned} 0 &= \sum_{u_0 \in \widehat{\mathbb{C}}} \operatorname*{Res}_{u_0} g(u)du \\ &= \operatorname*{Res}_{\infty} g(u)du + \operatorname*{Res}_{1} g(u)du + \operatorname*{Res}_{-1} g(u)du + \sum_{u_0 \in \mathbb{C} \setminus \{\pm 1\}} \operatorname*{Res}_{u_0} g(u)du \\ &= 2 - 1 - 1 - 2 \cdot \#(\text{zeros of } a_D - \tau a); \end{aligned}$$

hence $a_D - \tau a$ does not vanish on $\mathbb{C} \setminus \{\pm 1\}$ and

$$g(u)du = \frac{-2u du}{u^2 - 1} = -d \log(u^2 - 1).$$

It follows that

$$\frac{a_D}{a} = \tau - \frac{2 d\tau}{d \log \left((u^2 - 1) \frac{d\tau}{du} \right)}.$$

5. The Wrońskian. The Wrońskian

$$\mathcal{W} = a\, da_D - a_D\, da$$

is also single-valued on $\mathbb{C} \setminus \{\pm 1\}$ with asymptotics

$$\mathcal{W} \approx \begin{cases} \dfrac{2i}{\pi} du & \text{near } u = \infty, \\ c_+ a_+ du & \text{near } u = 1, \\ c_- a_- du & \text{near } u = -1. \end{cases}$$

Thus

$$c_+ a_+ = \frac{2i}{\pi} = c_- a_- \quad \text{and} \quad \mathcal{W} = \frac{2i}{\pi} du.$$

6. The section itself. From the previous three sections we have

$$da^2 d\tau = \frac{idu^3}{2\pi(u^2-1)},$$
$$\frac{2ad\tau}{a_D - \tau a} + d\log\frac{d\tau}{du} = -d\log(u^2-1)$$

and

$$(a_D - \tau a)da = \frac{2}{i\pi}du.$$

Thus

$$\begin{aligned} a &= \frac{1}{2\sqrt{d\tau}}\left(\frac{2ad\tau}{a_D-\tau a}\right)\frac{(a_D-\tau a)da}{\sqrt{da^2 d\tau}} \qquad (6.1)\\ &= \sqrt{\frac{2i(u^2-1)}{\pi d\tau du}}\, d\log\left((u^2-1)\frac{d\tau}{du}\right), \end{aligned}$$

$$\begin{aligned} a_D &= \frac{2du}{i\pi da} + \tau a \qquad (6.2)\\ &= \sqrt{\frac{2i(u^2-1)}{\pi d\tau du}}\Bigg(-2d\tau + \tau d\log\left((u^2-1)\frac{d\tau}{du}\right)\Bigg). \end{aligned}$$

7. Verification. Direct calculation shows that the functions a, a_D defined in (6.1), (6.2) satisfy the asymptotic conditions (1.2) (for the branches set up in §2) and that they satisfy the holonomy conditions (1.1) up to sign; to check that the signs work out correctly it suffices to examine, say, the asymptotics of a at $u = \infty$ and of a_D at $u = 1$.

To check the positivity condition (1.3) note that $d(a_D - \tau a) = -ad\tau$ so that da_D does indeed equal τda.

8. Function theory. The multiple-valued differential $d(a/(a_D - \tau a))$ satisfies the asymptotics

$$d(a/(a_D-\tau a)) \approx \begin{cases} \dfrac{Cdu}{u^2} & \text{near } u = \infty,\\ \dfrac{du}{i\pi(u-1)} & \text{near } u = 1,\\ \dfrac{du}{i\pi(u+1)} & \text{near } u = -1. \end{cases}$$

Note that due to residue considerations no single-valued differential can satisfy these conditions (nor could such a differential exist on an amenable covering of $\mathbb{C} \setminus \{\pm 1\}$). (Compare [Bar], [Dil], [BD].)

9. The differential equation. Our evaluation of the Wrońskian may be written

$$a\frac{da_D}{du} - a_D\frac{da}{du} = \frac{2i}{\pi}.$$

Differentiating with respect to u we have

$$a\frac{d^2a_D}{du^2} - a_D\frac{d^2a}{du^2} = 0$$

so that

$$\frac{1}{a}\frac{d^2a}{du^2} = \frac{1}{a_D}\frac{d^2a_D}{du^2}.$$

The left-hand side of this equation is single-valued near $u = \infty$ while the right-hand side is single-valued near $u = 1$, so together the two sides define a meromorphic function ϕ on $\mathbb{C} \setminus \{\pm 1\}$ with asymptotics

$$\phi(u) \approx \begin{cases} -\dfrac{1}{4u^2} & \text{near } u = \infty, \\ \dfrac{ic_+}{\pi a_+}\dfrac{1}{u-1} = -\dfrac{1}{8(u-1)} & \text{near } u = 1, \\ \dfrac{ic_-}{\pi a_-}\dfrac{1}{u-1} = \dfrac{1}{8(u+1)} & \text{near } u = -1; \end{cases}$$

in view of §4, a_D and a have no common zeros and thus ϕ is in fact holomorphic on $\mathbb{C} \setminus \{\pm 1\}$. It follows that $\phi(u) = \frac{1}{4(1-u^2)}$ and that

$$\frac{d^2a_D}{du^2} = \frac{a_D}{4(1-u^2)}, \qquad \frac{d^2a}{du^2} = \frac{a}{4(1-u^2)}.$$

Bilal [Bil] uses the differential equations for a_D and a to represent them in terms of hypergeometric functions; this leads in turn to the Seiberg–Witten representation in terms of elliptic integrals. We wish to note, however, that contrary to a statement in §6.1 of [Bil], the possibility of finding differential equations of the form $a_D'' = \psi_D a_D, a'' = \psi a$ (ψ_D and ψ single-valued) depends on the prescribed asymptotics for a_D, a, not just the holonomy conditions; if a_D and a are replaced, respectively, by ηa_D and ηa for some η holomorphic and non-constant on $\mathbb{C} \setminus \{\pm 1\}$ then the holonomy conditions will still hold but a_D''/a_D and a''/a will now be multiple-valued.

10. The Schwarzian derivative. Since neither $d(a_D/a) = \mathcal{W}/a^2$ nor $d(a/a_D) = -\mathcal{W}/a_D^2$ ever vanishes, the map

$$a_D/a : \widetilde{\mathbb{C} \setminus \{\pm 1\}} \to \widehat{\mathbb{C}}$$

is unbranched and thus the Schwarzian derivative

$$\mathcal{S} = \frac{d^2}{du^2} \log \frac{d(a_D/a)}{du} - \frac{1}{2}\left(\frac{d}{du} \log \frac{d(a_D/a)}{du}\right)^2$$

is holomorphic. Since all branches of a_D/a are related by post-composition with linear fractional transformations, $\mathcal{S}$ is single-valued on $\mathbb{C} \setminus \{\pm 1\}$ [Leh, II.1.1]. From (1.2) we have

$$\mathcal{S} \approx \begin{cases} \dfrac{1}{2u^2} & \text{near } u = \infty, \\ -\dfrac{2ic_+}{\pi a_+(u-1)} = \dfrac{1}{4(u-1)} & \text{near } u = 1, \\ -\dfrac{2ic_-}{\pi a_-(u+1)} = -\dfrac{1}{4(u+1)} & \text{near } u = -1. \end{cases}$$

Thus

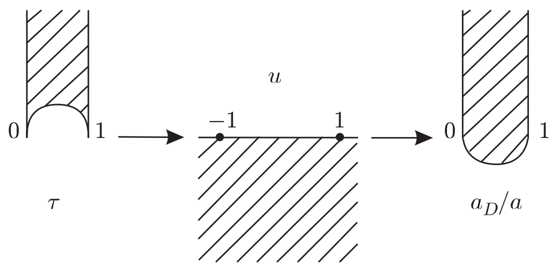
$$\mathcal{S} = \frac{1}{2(u^2-1)}.$$

Alternately, $\mathcal{S}$ can be determined from the differential equations in §9 by a standard argument [Mat], [FeBi], [Leh, II.1.2]. ([Mat] similarly uses the differential equations to evaluate the Wrońskian.)

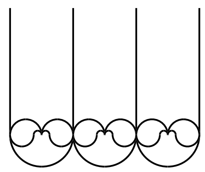
11. The curve

Theorem (cf. [AFS]). *The map a_D/a has a branch mapping the lower half-plane onto the region*

$$\Omega := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1,\ \operatorname{Im} z > 0\} \cup \left\{z \in \mathbb{C} : \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}.$$



All branches of a_D/a are obtained from this one by repeated Schwarz reflection.



Corollary. *The curve*

$$\gamma := \{u \in \mathbb{C} \setminus \{\pm 1\} : (a_D/a)(u) \in \mathbb{R} \cup \{\infty\}\} \cup \{\pm 1\}$$

is a connected simple closed curve of class C^1.

Proof of Theorem. We consider a branch of a_D/a on the lower half-plane satisfying (1.2) for u near ∞. Since $\mathcal{S}$ is real along each of the intervals

$$I_1 = (-\infty, -1), \qquad I_2 = (-1, 1), \qquad I_3 = (1, \infty),$$

this branch will map each I_j into a line or circle C_j. Examination of (1.2) reveals that

$$C_1 = \{z : \operatorname{Re} z = 1\} \cup \{\infty\} \quad \text{and} \quad C_3 = \{z : \operatorname{Re} z = 0\} \cup \{\infty\}.$$

Our branch also satisfies (1.2) for u near 1. (This follows from the set-up in §2, but we may also argue this without the use of (1.3): since a_D/a transforms like ζ, the structure of the deck group

$$\left\{\zeta \mapsto \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} : \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv I \bmod 2\right\}$$

reveals that the only branches of a_D/a satisfying

$$\frac{a_D}{a}(1) \in C_3 \cap (\mathbb{R} \cup \{\infty\}) = \{0, \infty\}$$

are those obtained by continuing the branch in (1.2) around $u = 1$. The determination of C_3 fixes the branch of the logarithm.)

Using (1.2) to analyze a/a_D for u near 1 we find that

$$C_2 = \left\{z : \left|z - \frac{1}{2}\right| = \frac{1}{2}\right\}.$$

We saw in the previous section that a_D/a has no branch points; thus, traveling from right to left, I_3 will map downwards along C_3 a total of $\nu_3 + 1/2$ times, I_2 will map counterclockwise along C_2 a total of $\nu_2 + 1/2$ times, and I_1 will map upwards along C_1 a total of $\nu_1 + 1/2$ times. An application of the argument principle reveals that the number of branch points of a_D/a in the lower half-plane is equal to $\nu_1 + \nu_2 + \nu_3$. Since there are no such branch points we must in fact have $\nu_1 = \nu_2 = \nu_3 = 0$. Another application of the argument principle reveals that a_D/a maps the lower half-plane bijectively to Ω.

Standard arguments show that repeated continuation of a_D/a across the intervals I_j is accomplished by Schwarz reflection. ∎

Remark. Differentiating a_D/a we find that the Theorem and the result of §5 are sufficient to determine a_D and a. It follows that the positivity assumption (1.3) is in fact redundant.

Proof of Corollary. The transformation laws for a_D/a show that γ is well defined, the absence of branch points for a_D/a implies that γ is smooth and real-analytic away from ± 1, and the asymptotics (1.2) reveal that γ is of class C^1 also at ± 1.

The Theorem shows that γ intersects the lower half-plane in a single arc joining the points ± 1, and a Schwarz reflection argument shows that the same holds for the upper half-plane. ∎

Appendix A. Homotopically trivial self-maps

THEOREM. *If X is a Riemann surface with non-abelian fundamental group and $f : X \to X$ is a holomorphic self-map that is homotopic to the identity then f is in fact the identity map of X.*

Proof. X is covered by the unit disk Δ, and the deck group of the covering contains infinitely many non-commuting hyperbolic elements [Bea, Thm. 5.1.3]. Then [Hub, Satz 2] implies that f is an automorphism of X lifting to an automorphism of Δ commuting with the deck group. But this implies [Bea, Thm. 4.3.6] that the lifted map is the identity map of Δ so that f is the identity map of X. ∎

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