ANNALES
POLONICI MATHEMATICI
LXX (1998)

Extension of separately analytic functions and applications to range characterization of the exponential Radon transform

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Abstract. We consider the problem of characterizing the range of the exponential Radon transform. The proof uses extension properties of separately analytic functions, and we prove a new theorem about extending such functions.

1. Introduction. Given a function $h \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R})$, we define the *exponential Radon transform* $R_\mu(h)$ of h as

$$R_\mu(h)(\omega, p) := \int\limits_{x\cdot\omega=p} h(x)e^{\mu x\cdot\omega^\perp}\,dm(x).$$

In the definition above, μ is a non-zero fixed real number, $\omega := (\cos\alpha, \sin\alpha)$ for some $0 \le \alpha < 2\pi$ with $\omega^\perp := (-\sin\alpha, \cos\alpha)$ and dm is the 1-dimensional Lebesgue measure on the line $x\cdot\omega = p$. Problems of interest within integral geometry are to invert, study uniqueness properties, and characterize the image of the operator R_μ when it acts on various spaces. We will confine ourselves to the last problem, more precisely to characterizing the image of R_μ when it acts on $\mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{C})$.

1.1. *Summary of the results.* The first result that completely characterizes the range of R_μ was proved in [5]. In that article, an infinite set of rather strange differential conditions were given. In [8] there is a geometric description of the range conditions occurring in [5] and also a new set of range conditions. Later, in [1], a new description of the range, that was geometrically pleasing and natural, was given (see (2.2)). In this paper, we give a new proof of the theorem of [1] that describes the range. The proof is based on Theorem 3.1, which relies heavily on Theorem 4.2, which in turn is a new result about extending separately analytic functions.

1991 *Mathematics Subject Classification*: 32D, 44A15, 32A.
Key words and phrases: exponential Radon transform, range characterization, separately analytic functions.

1.2. *Comments about applications of range characterization.* In applications one usually measures $g = R_\mu(h)$ and the object one would like to recover is h (see e.g. [3]), i.e. we would like to invert R_μ. Range characterization of R_μ imposes conditions on g and these conditions can be used to *correct* measured data and to *restore* incomplete data.

Correction of measured data. Usually the data we measure, i.e. g, does not satisfy the range conditions because the mathematical model does not represent all real aspects of the collection of the data. As an example one has noisy data. One can then use the range conditions in order to correct the measured data, i.e. to decrease the discrepancy between the measured and ideal data. Using corrected data can bring us a more precise reconstruction of h (see [10] and [6]).

Restoration of incomplete data. If we consider the emission tomography problem when the data are known only for $0 \le \alpha \le \alpha_0$ where $\alpha_0 < 2\pi$, the range conditions give us some additional information which helps to restore the unknown part of the data. This enables us to find h based on incomplete data (see [7]).

2. The main theorem.

The well known relation between the exponential Radon transform and the Fourier transform is known as the *Fourier slice theorem* and reads as follows:

THEOREM 2.1. *Let g be the exponential Radon transform of a test function in $\mathbb{R}^2$, i.e. $g = R_\mu(h)$ where $h \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{C})$. Then*

$$\widehat{g}(\omega, \zeta) = \widehat{h}(\zeta\omega + i\mu\omega^\perp). \tag{2.1}$$

The proof is simply to use Fubini's theorem and the definitions (see e.g. [7]).

By taking $\zeta = it$ in (2.1), we observe that the existence of $\widehat{h}$ is not possible unless the following holds: for all $\sigma, \omega \in S^1$ and $t \in \mathbb{R}$,

$$\widehat{g}(\omega, it) = \widehat{g}(\sigma, -it) \quad \text{whenever } t\omega + \mu\omega^\perp = -t\sigma + \mu\sigma^\perp. \tag{2.2}$$

This condition has the following simple geometric interpretation. In the $\mathrm{Im}(\mathbb{C}^2)$-plane we have a circle with radius μ centered at the origin, and the family $\mathcal{L} := \{\ell_\omega\}_{\omega \in S^1}$ of lines tangent to this circle. On each line ℓ_ω we have an entire function $\mathbb{C} \ni \zeta \mapsto \widehat{g}(\omega, \zeta)$. The condition in (2.2) simply states that the values of the functions $\zeta \mapsto \widehat{g}(\omega, \zeta)$ must agree at the points where the lines intersect.

This is obviously a necessary condition if there is to be a function (in this case denoted by $\widehat{h}$), defined on $\mathrm{Im}(\mathbb{C}^2)$, whose restriction to the lines ℓ_ω is $\zeta \mapsto \widehat{g}(\omega, \zeta)$.

Thus, (2.2) gives us a *necessary* condition that a function g must satisfy in order to be in the range of R_μ. The question is if this condition is *sufficient*. The affirmative answer is given in the following theorem.

THEOREM 2.2 (Range characterization of R_μ). *Let* $g : S^1 \times \mathbb{R} \to \mathbb{C}$ *and* $\mu \in \mathbb{R} \setminus \{0\}$. *Then the following are equivalent*:

(i) *There exists* $h \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{C})$ *such that* $g = R_\mu(h)$.
(ii) $g \in \mathcal{C}_c^\infty(S^1 \times \mathbb{R}, \mathbb{C})$ *and* $\widehat{g}$ *satisfies the condition in* (2.2).

3. Proof of Theorem 2.2. First, without loss of generality, we can assume that $\mu > 0$.

DEFINITION 3.1. We define the set $M_\mu \subset \mathbb{C}^2$ as the union of the complexified lines in $\mathcal{L}$, i.e.

$$M_\mu := \{z \in \mathbb{C}^2 : z = \zeta\omega + i\mu\omega^\perp,\ \zeta \in \mathbb{C},\ \omega \in S^1\},$$

and set

$$K_\mu := \{x \in \mathbb{R}^2 : |x| \geq \mu\} \quad \text{and} \quad K_\mu^{\mathbb{C}} := \{z \in \mathbb{C}^2 : z = ix \text{ where } x \in K_\mu\}.$$

The proof of Theorem 2.2 boils down to proving the following two theorems, the first one about analytic extension of functions defined on lower dimensional sets in $\mathbb{C}^2$, and the second about extending growth properties valid on a submanifold in $\mathbb{C}^2$ to corresponding growth properties on $\mathbb{C}^2$.

THEOREM 3.1. *Let* $f : S^1 \times \mathbb{C} \to \mathbb{C}$ *and assume that* f *satisfies the same condition as* $\widehat{g}$ *does in* (2.2), *i.e.* $\zeta \mapsto f(\omega, \zeta)$ *is an entire function on* $\mathbb{C}$ *for all* $\omega \in S^1$ *and, for* $\ell_\omega, \ell_\sigma \in \mathcal{L}$ *where* $\omega \neq \sigma$ *and* $t\omega + \mu\omega^\perp = s\sigma + \mu\sigma^\perp$,

$$f(\omega, it) = f(\sigma, is). \tag{3.1}$$

Then the function $F : K_\mu^{\mathbb{C}} \to \mathbb{C}$ *defined as*

$$F(i(t\omega + \mu\omega^\perp)) := f(\omega, it) \tag{3.2}$$

extends to an entire function on $\mathbb{C}^2$.

REMARK 3.1. Observe that elements in the family $\mathcal{L} = \{\ell_\omega\}_{\omega \in S^1}$ are given as

$$\ell_\omega : t \mapsto t\omega + \mu\omega^\perp.$$

Here $\omega \in S^1$ denotes the parameter of this family.

THEOREM 3.2. *Let* $N \in \mathbb{N}$ *and* $F \in \mathcal{O}(\mathbb{C}^2, \mathbb{C})$ *where* $\zeta \mapsto F(\zeta\omega + i\mu\omega^\perp)$ *satisfies the one-dimensional Paley–Wiener growth estimates uniformly with respect to* ω, *i.e. there is a constant* $C_N > 0$ (*independent of* ω) *and* $r > 0$ *such that*

$$|F(\zeta\omega + i\mu\omega^\perp)| \leq C_N(1 + |\zeta|)^{-N} e^{r|\operatorname{Im}\zeta|} \quad \textit{for all } \zeta \in \mathbb{C}. \tag{3.3}$$

Then F satisfies the corresponding two-dimensional Paley–Wiener growth estimates, i.e. there is a constant $C'_N > 0$ and $r' > 0$ such that

$$|F(z)| \le C'_N (1+|z|)^{-N} e^{r'|\operatorname{Im} z|} \quad \text{for all } z \in \mathbb{C}^2. \tag{3.4}$$

Assuming the validity of Theorems 3.1 and 3.2 we can prove Theorem 2.2.

Proof of Theorem 2.2. The proof is naturally divided into two parts.

(i)⇒(ii). Let g be in the range of R_μ, i.e. there exists $h \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{C})$ such that $g = R_\mu(h)$. Then it is clear that $g \in \mathcal{C}_c^\infty(S^1 \times \mathbb{R}, \mathbb{C})$ and Theorem 2.1 shows that (2.2) holds.

(ii)⇒(i). Let $g \in \mathcal{C}_c^\infty(S^1 \times \mathbb{R}, \mathbb{C})$ be given and assume that $\widehat{g}$ satisfies (2.2). Then, by Theorem 3.1, there exists $F \in \mathcal{O}(\mathbb{C}^2, \mathbb{C})$ such that

$$F(i(t\omega + \mu\omega^\perp)) = \widehat{g}(\omega, it) \tag{3.5}$$

for all $t \in \mathbb{R}$ and $\omega \in S^1$. By the 1-dimensional Paley–Wiener theorem $\widehat{g}$ satisfies the assumptions of Theorem 3.2. Thus, we can consider the inverse Fourier transform of F. Denote it by h. Moreover, by Theorem 3.2 and the 2-dimensional Paley–Wiener theorem, $h \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{C})$. Finally, Theorem 2.1 with $\zeta = it$ gives $g = R_\mu(h)$.

This concludes the proof of Theorem 2.2. ∎

Thus, it remains to prove Theorems 3.1 and 3.2. We begin with proving the former. For that we need results from the theory of several complex variables.

4. Results from the theory of several complex variables.

The proof of Theorem 3.1 relies heavily on extension properties of separately analytic functions. The result that we eventually need is Corollary 4.1.

We begin with the following standard result (see e.g. Lemma 1A in [2]) on the removal of singularities of analytic functions.

PROPOSITION 4.1. *Assume that $\Omega \subset \mathbb{C}^2$ is open and H is analytic in $\Omega \setminus \{z \in \Omega : \operatorname{Im} z = 0\}$. Then H extends to an analytic function on Ω.*

Now, let us turn to extension of separately analytic functions; we begin with defining the concept of a separately analytic function w.r.t. a decomposition of a set.

DEFINITION 4.1. Let $K_1, U_1 \subset \mathbb{C}^n$, $K_2, U_2 \subset \mathbb{C}^m$ and assume that U_1, U_2 are open and connected, and K_1, K_2 are subsets of U_1 and U_2, respectively. Let $X \subset \mathbb{C}^{n+m}$ be defined as

$$X := (K_1 \times U_2) \cup (U_1 \times K_2), \tag{4.1}$$

and assume that $H : X \to \mathbb{C}$. We say that H is *separately analytic* in X *w.r.t. the decomposition* (4.1) if

$$z \mapsto H(z,w) \text{ is analytic in } U_1 \text{ for all } w \in K_2,$$
$$w \mapsto H(z,w) \text{ is analytic in } U_2 \text{ for all } z \in K_1.$$

We are interested in the case where $n = m = 1$ and K_1, K_2 are intervals in $\mathbb{R}$. We need the following notation.

NOTATION 4.1. Let $U \subset \mathbb{C}$ be an open set. Then $\mathcal{HAR}(U,V)$ and $\mathcal{SH}(U,V)$ denote the sets of harmonic resp. subharmonic functions in U with values in V. If U is a closed set, we extend the definition to mean that the function is harmonic or subharmonic on some open neighborhood of U.

DEFINITION 4.2. Let $U \subset \mathbb{C}$ and $K \subset U$ be an interval. Define $h_{U,K} : U \to \mathbb{R}$ as the bounded function $0 \le h_{U,K} \le 1$ that solves the following generalized Dirichlet problem:

$$h_{U,K} \in \mathcal{HAR}(U \setminus K, \mathbb{R}), \qquad h_{U,K}|_K \equiv 0, \qquad h_{U,K}|_{\partial U} \equiv 1.$$

We say that the pair (K,U) is *regular* if the Dirichlet problem has a unique solution. It is clear that if U has a "sufficiently nice" boundary and K is an interval, then (K,U) is regular. $h_{U,K}$ is called the *zero-one maximal function* for the pair (U,K).

The following extension theorem is a special case of a very general result (Theorem 7.1 of [11]) due to Siciak about extending separately analytic functions.

THEOREM 4.1. *Let* $K_1, K_2 \subset \mathbb{R}$ *be closed intervals*, $U_1, U_2 \subset \mathbb{C}$ *open and connected sets where* $K_j \subset U_j$ *and* (K_j, U_j) *are regular for* $j = 1,2$. *Define* $X \subset \mathbb{C}^2$ *as in* (4.1) *and let* $H : X \to \mathbb{C}$ *be separately analytic in* X *w.r.t. the decomposition* (4.1). *Then* H *extends analytically to* $\Omega \subset \mathbb{C}^2$ *where*

$$\Omega := \{z \in U_1 \times U_2 : h_1(z_1) + h_2(z_2) < 1\} \quad \textit{and} \quad h_j := h_{U_j,K_j} \textit{ for } j = 1,2.$$

Our aim is to prove Corollary 4.1 which is an extension theorem where we "remove a line" and at the same time have regularity at infinity. Let us first prove a simpler version of that result, namely Theorem 4.2, where we only "remove a curve".

DEFINITION 4.3. $\triangle_{\mathbb{R}}$ and $\triangle_{\mathbb{C}}$ denote the real and complex diagonals in $\mathbb{R}^2$ and $\mathbb{C}^2$ respectively, i.e.

$$\triangle_{\mathbb{R}} := \{x \in \mathbb{R}^2 : x_1 = x_2\} \quad \text{and} \quad \triangle_{\mathbb{C}} := \{z \in \mathbb{C}^2 : z_1 = z_2\}.$$

THEOREM 4.2. *Assume that* $H : \mathbb{R}^2 \setminus \triangle_{\mathbb{R}} \to \mathbb{C}$ *has the following properties*:

$$z_1 \mapsto H(z_1, x_2) \textit{ extends analytically to } \mathbb{C} \setminus \{x_2\} \textit{ for all } x_2 \in \mathbb{R},$$

$z_2 \mapsto H(x_1, z_2)$ *extends analytically to* $\mathbb{C} \setminus \{x_1\}$ *for all* $x_1 \in \mathbb{R}$.

Then H *extends analytically to* $\Omega := \mathbb{C}^2 \setminus \triangle_{\mathbb{C}}$.

4.1. *Proof of Theorem 4.2.* The idea is to use Theorem 4.1 locally at every point $w \in \Omega$ in order to show that H extends analytically to a neighborhood Ω_w of w in $\mathbb{C}^2$. In order to use Theorem 4.1 one needs sets where H is separately analytic. We choose these sets in such a way that we can use our assumption of separate analyticity of H, and at the same time show that the set [1] Ω_w contains w. The proof is divided into the following steps:

STEP 1. Choose the sets where we use our separate analyticity assumption.

STEP 2. Apply Theorem 4.1 to extend H to Ω_w.

STEP 3. Describe Ω_w.

STEP 4. Show that Ω_w contains w.

Fix $w = (a_1, a_2) + i(b_1, b_2) \in \Omega = \mathbb{C}^2 \setminus \triangle_{\mathbb{C}}$. Then we have two cases: $\operatorname{Re} w \notin \triangle_{\mathbb{R}}$ and $\operatorname{Im} w \in \mathbb{R}^2$ arbitrary, and $\operatorname{Re} w \in \triangle_{\mathbb{R}}$ and $\operatorname{Im} w \notin \triangle_{\mathbb{R}}$.

4.1.1. *The case* $\operatorname{Re} w \notin \triangle_{\mathbb{R}}$ *and* $\operatorname{Im} w \in \mathbb{R}^2$ *arbitrary.* Without loss of generality, we can assume that $a_1 > a_2$.

Choosing the sets in Step 1. Our choices will depend on w. We start by defining $\varepsilon_0, q \in \mathbb{R}$, both depending on w. We know that $a_1 > a_2$, so choose q with

$$a_2 < q < a_1. \tag{4.2}$$

Also, choose $\varepsilon_0 > 0$ small [2] enough that

$$a_2 < q - \varepsilon_0 < q + \varepsilon_0 < a_1 \quad \text{and} \quad |q| + \varepsilon_0 < 1/\varepsilon_0.$$

We are now ready to define the sets in Step 1. For arbitrary $0 < \varepsilon < \varepsilon_0$, define

$$\begin{aligned} K_{1,\varepsilon} &:= [q+\varepsilon, 1/\varepsilon], & V_1 &:= \,]-\infty, q], & U_1 &:= \mathbb{C} \setminus V_1, \\ K_{2,\varepsilon} &:= [-1/\varepsilon, q-\varepsilon], & V_2 &:= \,]q, \infty[, & U_2 &:= \mathbb{C} \setminus V_2. \end{aligned}$$

Finally, define $X_\varepsilon \subset \mathbb{C}^2$ as

$$X_\varepsilon := (K_{1,\varepsilon} \times U_2) \cup (U_1 \times K_{2,\varepsilon}). \tag{4.3}$$

Applying Theorem 4.1 as described in Step 2. The idea is to apply Theorem 4.1 to H w.r.t. the sets defined in Step 1 above.

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(1) Ω_w is the analogue of the set Ω in Theorem 4.1.

(2) ε_0 will denote the upper bound for a parameter $\varepsilon > 0$ and the conditions given on ε_0 are there only to ensure that our sets, which depend on ε, are well defined. We will be interested in the limiting case $\varepsilon \to 0^+$, thus the value of the upper bound ε_0 is actually not that important, since the conditions for ε_0 are automatically satisfied for small ε.

For $j = 1, 2$, let $h_{j,\varepsilon} := h_{U_j,K_{j,\varepsilon}}$ and

$$h_\varepsilon(z) := h_{1,\varepsilon}(z_1) + h_{2,\varepsilon}(z_2).$$

Our function H is separately analytic in X_ε w.r.t. the decomposition in (4.3), and the sets in (4.3) depend only on w and ε. Thus, we can use Theorem 4.1 on H with

$$\begin{aligned} X\colon &= X_\varepsilon, \quad & K_1\colon &= K_{1,\varepsilon}, \quad & U_1\colon &= U_1, \\ & & K_2\colon &= K_{2,\varepsilon}, \quad & U_2\colon &= U_2. \end{aligned}$$

Hence, H extends analytically to $\Omega_{w,\varepsilon} \subset \mathbb{C}^2$ where $\Omega_{w,\varepsilon} := \{z \in U_1 \times U_2 : h_\varepsilon(z) < 1\}$. Observe that h_ε and $\Omega_{w,\varepsilon}$ depend only on w and ε. However, $0 < \varepsilon < \varepsilon_0$ is arbitrary, so if

$$\Omega_w := \bigcup_{0<\varepsilon<\varepsilon_0} \Omega_{w,\varepsilon},$$

then H extends analytically to Ω_w.

Describing Ω_w. We are interested in what happens when $\varepsilon \to 0^+$, so it is natural to define

$$K_1 := [q, \infty[\quad \text{and} \quad K_2 :=]-\infty, q].$$

For $j = 1, 2$, we also define $h_j := h_{U_j,K_j}$ and

$$h(z) := h_1(z_1) + h_2(z_2).$$

LEMMA 4.1. *With the sets and functions defined as above,*

$$\Omega_w = \{z \in U_1 \times U_2 : h(z) < 1\}.$$

To prove Lemma 4.1, it is enough to show that for $j = 1, 2$,

$$\lim_{\varepsilon \to 0} h_{j,\varepsilon}(z) = h_j(z).$$

This is proved by using the Phragmén–Lindelöf principle. See [9, Lemma 4.1] for the details.

Showing that $w \in \Omega_w$. It is easy to see that

$$h_1(x + iy) = \frac{1}{\pi}|\arg(x - q + iy)|, \tag{4.4}$$

$$h_2(x + iy) = 1 - \frac{1}{\pi}|\arg(x - q + iy)|, \tag{4.5}$$

where $\arg(z)$ is the argument of z in $[-\pi, \pi]$.

REMARK 4.1. Table 1 summarizes the observations that we make from (4.5) and (4.4) about h_1 and h_2.

Table 1. Observations about the values of the functions h_1 and h_2

Location in (x, y)-space	Value of $h_1(x+iy)$	Value of $h_2(x+iy)$
$x = q$	$1/2$	$1/2$
$x < q$, i.e. $x+iy$ is on the left hand side of the line $x = q$	$> 1/2$	$< 1/2$
$x > q$, i.e. $x+iy$ is on the right hand side of the line $x = q$	$< 1/2$	$> 1/2$

We know that $w = (w_1, w_2) = (a_1 + ib_1, a_2 + ib_2)$ and in order to show that $w \in \Omega_w$, it suffices to show that $h(w) < 1$. Observe that $a_1 > a_2$. Using (4.5)–(4.4) and Remark 4.1, we get

$$h_1(a_1 + ib_1) < 1/2 \quad \text{and} \quad h_2(a_2 + ib_2) < 1/2.$$

Thus, $h(w) = h_1(w_1) + h_2(w_2) < 1$, i.e. $w \in \Omega_w$.

4.1.2. *The case* $\operatorname{Re} w \in \triangle_{\mathbb{R}}$ *and* $\operatorname{Im} w \notin \triangle_{\mathbb{R}}$. The points in Ω $(= \mathbb{C}^2 \setminus \triangle_{\mathbb{C}})$ that remain to be discussed are those where $a_1 = a_2$ and $b_1 \neq b_2$. The solution of the associated Dirichlet problem is symmetric w.r.t. the real axis. This can be seen by observing the solutions directly (see (4.4)–(4.5)). Thus, with this method, we cannot distinguish between points on $\triangle_{\mathbb{C}}$ and points on $\triangle'_{\mathbb{C}}$ where

$$\triangle'_{\mathbb{C}} := \{w \in \mathbb{C}^2 : w_1 = \overline{w}_2\}.$$

Hence, we begin with studying points $w \in \Omega \setminus \triangle'_{\mathbb{C}}$, i.e. $a \in \triangle_{\mathbb{R}}$ and $|b_1| \neq |b_2|$. Without loss of generality, we can assume that $|b_1| > |b_2|$. Again, we have to choose $q \in \mathbb{R}$ and this time we know that $a := a_1 = a_2$ and $|b_1| > |b_2|$, so we replace the choice of q in (4.2) with $q > a_1$. *The choices of all other quantities are the same as before.* Now, we have to show that $h(w) < 1$. Let

$$\alpha_1 := |\arg(a - q + ib_1)|, \qquad \alpha_2 := |\arg(a - q + ib_2)|.$$

Using (4.4)–(4.5) gives us

$$h_1(a + ib_1) = \alpha_1/\pi \quad \text{and} \quad h_2(a + ib_2) = 1 - \alpha_2/\pi.$$

Thus,

$$h(w) = h_1(w_1) + h_2(w) = \frac{\alpha_1}{\pi} + 1 - \frac{\alpha_2}{\pi} = 1 - \frac{\alpha_2 - \alpha_1}{\pi}.$$

Since $|b_1| > |b_2|$, we know that $\alpha_2 > \alpha_1$, so $h(w) < 1$, i.e. $w \in \Omega_w$. Thus, we have proved that H extends analytically to a neighborhood Ω_w of $w \in \Omega'$ with $\Omega' := \Omega \setminus \triangle'_{\mathbb{C}}$. Since $w \in \Omega'$ is arbitrary, we conclude that H extends analytically to Ω'.

Finally, we study points $w \in \triangle'_{\mathbb{C}}$. The bianalytic transformation

$$(w_1, w_2) \mapsto \left(\frac{1}{2}(w_1 + w_2), \frac{i}{2}(w_1 - w_2)\right) =: (z_1, z_2)$$

maps points $w \in \triangle'_{\mathbb{C}}$ to points z where $\operatorname{Im} z = 0$. Thus, by Proposition 4.1, H extends analytically across $\triangle'_{\mathbb{C}}$ except at $(0,0)$ where the assumption in Proposition 4.1 does not hold. Hence, H extends analytically to Ω if it is analytic in Ω'. This concludes the proof of Theorem 4.2.

4.2. *A version of Theorem 4.2 with analyticity at infinity.* What we actually need is a version of Theorem 4.2 that includes analyticity at infinity; but first some definitions.

Definition 4.4. Define $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, i.e. $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{R}}$ are the usual real and complex projective spaces. We also embed $\widehat{\mathbb{R}}$ into $\widehat{\mathbb{C}}$ in the usual way, i.e. the ∞ in $\widehat{\mathbb{C}}$ is the same as in $\widehat{\mathbb{R}}$.

The topologies of $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{R}}$ are well known and it is also well known that $\widehat{\mathbb{C}}$ is a complex-analytic manifold. Moreover, functions defined on $\mathbb{C}$ or $\mathbb{R}$ that extend continuously to ∞ can be defined on $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{R}}$, respectively. Thus, we can talk about functions defined on subsets of $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{R}}$.

Definition 4.5. Let f be defined in $\widehat{\mathbb{C}}$. Then f is *analytic at* ∞ if the function $\zeta \mapsto f(1/\zeta)$ is analytic at $\zeta = 0$ in the usual sense.

Definition 4.6. Let $\widehat{\mathbb{R}}^2 := \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ and $\widehat{\mathbb{C}}^2 := \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ with the product topology. Define the "projective" versions of $\triangle_{\mathbb{R}}$ and $\triangle_{\mathbb{C}}$ as

$$\widehat{\triangle}_{\mathbb{R}} := \{x \in \widehat{\mathbb{R}}^2 : x_1 = x_2\}$$

and

$$\widehat{\triangle}_{\mathbb{C}} := \{z \in \widehat{\mathbb{C}}^2 : z_1 = z_2\}.$$

Remark 4.2. It is important that we see elements in $\widehat{\mathbb{R}}^2$ and $\widehat{\mathbb{C}}^2$ as *pairs* of elements from $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{C}}$, respectively, since it would be incorrect and unnatural to try to introduce a group structure (such as addition or multiplication) *between* the first and second coordinates for points in $\widehat{\mathbb{R}}^2$ and $\widehat{\mathbb{C}}^2$. Thus, with our definition, $(\infty,\infty) \in \widehat{\triangle}_{\mathbb{R}}$.

There is also a *geometric reason* which motivates Definition 4.6 (see Remark 4.3).

We now state and prove the following version of Theorem 4.2 with regularity at infinity.

Theorem 4.3. *Assume that* $H : \widehat{\mathbb{R}}^2 \setminus \widehat{\triangle}_{\mathbb{R}} \to \mathbb{C}$ *has the following properties*:

(4.6) $\quad x_1 \mapsto H(x_1, x_2)$ *extends analytically to* $\widehat{\mathbb{C}} \setminus \{x_2\}$ *for all* $x_2 \in \widehat{\mathbb{R}}$, (4.7) $\quad x_2 \mapsto H(x_1, x_2)$ *extends analytically to* $\widehat{\mathbb{C}} \setminus \{x_1\}$ *for all* $x_1 \in \widehat{\mathbb{R}}$.

Then H *extends analytically to* $\widehat{\mathbb{C}}^2 \setminus \widehat{\triangle}_{\mathbb{C}}$.

P r o o f. By Theorem 4.2, H extends analytically to $\mathbb{C}^2 \setminus \triangle_{\mathbb{C}}$, and we need to prove that H extends analytically to $\widehat{\mathbb{C}}^2 \setminus \widehat{\triangle}_{\mathbb{C}}$. Thus, if we prove that H is analytic at (∞, z_2) and (z_1, ∞) for all $z_1, z_2 \neq \infty$, we are finished.

From (4.6) we see that for all fixed real $x_2 \neq \infty$,

$$z_1 \mapsto H(z_1, x_2) \text{ is analytic except at } z_1 = x_2, \tag{4.8}$$

$$z_1 \mapsto H(1/z_1, x_2) \text{ is analytic at } z_1 = 0, \tag{4.9}$$

and

$$z_1 \mapsto H(z_1, \infty) \text{ is analytic for } z_1 \neq \infty. \tag{4.10}$$

Now, define $\widetilde{H}(x_1, x_2) := H(1/x_1, 1/x_2)$ for all $x_1, x_2 \in \mathbb{R}$; since H is defined on $\widehat{\mathbb{R}}^2 \setminus \widehat{\triangle}_{\mathbb{R}}$, $\widetilde{H}$ is defined on $\mathbb{R}^2 \setminus \triangle_{\mathbb{R}}$. From (4.8)–(4.9) we see that for all $x_2 \in \mathbb{R}$,

$$z_1 \mapsto \widetilde{H}(z_1, x_2) \text{ is analytic except at } z_1 = x_2 \text{ for all } x_2 \neq 0,$$

and when $x_2 = 0$, (4.10) shows that

$$z_1 \mapsto \widetilde{H}(z_1, 0) \text{ is analytic except at } z_1 = 0.$$

Thus, for all fixed $x_2 \in \mathbb{R}$,

$$z_1 \mapsto \widetilde{H}(z_1, x_2) \text{ is analytic except at } z_1 = x_2.$$

Similarly, (4.7) shows that for all fixed $x_1 \in \mathbb{R}$,

$$z_2 \mapsto \widetilde{H}(x_1, z_2) \text{ is analytic except at } z_2 = x_1.$$

By applying Theorem 4.2, we extend $\widetilde{H}$ to an analytic function in $\mathbb{C}^2 \setminus \triangle_{\mathbb{C}}$. In particular, $\widetilde{H}$ is analytic as a function of two variables at $(z_1, 0)$ and $(0, z_2)$ for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Now, $H(z_1, z_2) = \widetilde{H}(1/z_1, 1/z_2)$, so H is analytic at (z_1, ∞) and (∞, z_2). Hence, H extends analytically to $\widehat{\mathbb{C}}^2 \setminus \widehat{\triangle}_{\mathbb{C}}$. ∎

Definition 4.7. Define the following two curves in $\mathbb{R}^2$ and $\mathbb{C}^2$:

$$\Gamma_{\mathbb{R}} := \{(\xi, \eta) \in \mathbb{R}^2 : \xi = -1/\eta\} \quad \text{and} \quad \Gamma_{\mathbb{C}} := \{(\xi, \eta) \in \mathbb{C}^2 : \xi = -1/\eta\}.$$

Also define their "projective" versions in $\widehat{\mathbb{R}}^2$ and $\widehat{\mathbb{C}}^2$:

$$\begin{aligned} \widehat{\Gamma}_{\mathbb{R}} &:= \{(\xi, \eta) \in \widehat{\mathbb{R}}^2 : \xi = -1/\eta\}, \\ \widehat{\Gamma}_{\mathbb{C}} &:= \{(\xi, \eta) \in \widehat{\mathbb{C}}^2 : \xi = -1/\eta\}. \end{aligned} \tag{4.11}$$

Remark 4.3. For the same reasons as explained in Remark 4.2, (4.11) is the natural definition of $\widehat{\Gamma}_{\mathbb{R}}$ and $\widehat{\Gamma}_{\mathbb{C}}$. There is also a geometric reason for choosing this definition. We will see later that the geometric meaning of ξ and η is that they parameterize points on a circle and that η and

$-1/\eta$ correspond to antipodal points. Thus, $\Gamma_{\mathbb{R}}$ is the set where ξ and η parameterize antipodal points on the circle, so it is geometrically natural to define either of them as a function of the other instead of defining them as a solution to a polynomial equation.

We now prove the result that we really need.

COROLLARY 4.1. *Assume that* $G : \widehat{\mathbb{R}}^2 \setminus \widehat{\Gamma}_{\mathbb{R}} \to \mathbb{C}$ *has the following properties*:

$$\xi \mapsto G(\xi,\eta) \text{ extends analytically to } \widehat{\mathbb{C}} \setminus \{-1/\eta\} \text{ for all } \eta \in \widehat{\mathbb{R}}, \tag{4.12}$$

$$\eta \mapsto G(\xi,\eta) \text{ extends analytically to } \widehat{\mathbb{C}} \setminus \{-1/\xi\} \text{ for all } \xi \in \widehat{\mathbb{R}}. \tag{4.13}$$

Then G *extends analytically to* $\widehat{\mathbb{C}}^2 \setminus \widehat{\Gamma}_{\mathbb{C}}$.

Proof. The mapping $(\xi,\eta) \mapsto (\xi, -1/\eta)$ maps the curve $\widehat{\Gamma}_{\mathbb{C}}$ in $\widehat{\mathbb{C}}^2$ to the curve $\widehat{\triangle}_{\mathbb{C}}$ in $\widehat{\mathbb{C}}^2$. Moreover, this mapping is bianalytic in $\widehat{\mathbb{C}}^2$, and it transforms the conditions stated in the corollary into the corresponding conditions of Theorem 4.3. Thus, the corollary follows directly from Theorem 4.3. ∎

5. Proof of Theorems 3.1 and 3.2. First of all, the existence of F follows immediately from (3.1). Thus, there exists a function $F : K_\mu^{\mathbb{C}} \to \mathbb{C}$ such that

$$f(\omega, it) = F(i(t\omega + \mu\omega^\perp)). \tag{5.1}$$

5.1. *Geometric observations.* We begin with some geometric observations. The main observation is a relationship between the set of lines in $\mathcal{L}$ and the set of points in K_μ. We have the following two situations:

1. Given $x \in K_\mu^\circ$, we can find $\omega, \sigma \in S^1$ and $t \in \mathbb{R}$ such that

$$x = \mu\omega^\perp + t\omega = \mu\sigma^\perp - t\sigma,$$

i.e. $\ell_\omega, \ell_\sigma \in \mathcal{L}$ and $x \in \ell_\omega \cap \ell_\sigma$. Then one can easily show that

$$\begin{aligned} \omega &= \frac{tx - \mu x^\perp}{|x|^2}, \\ \sigma &= \frac{-tx - \mu x^\perp}{|x|^2}, \\ t^2 &= |x|^2 - \mu^2. \end{aligned} \tag{5.2}$$

2. Given $\omega, \sigma \in S^1$ we find $x \in K_\mu^\circ$ such that $x \in \ell_\omega \cap \ell_\sigma$: using (5.2) we can show that

$$x = \mu \frac{(\omega + \sigma)^\perp}{1 + \omega \cdot \sigma}. \tag{5.3}$$

5.2. *Proof of Theorem 3.1.* Our strategy is to relate the present extension problem to the one considered in Corollary 4.1. This relation is established by choosing a new way to represent the lines in $\mathcal{L}$ that only involves rational functions.

The geometric observation above shows that for each $x \in K_\mu$ there exist exactly two points on the circle corresponding to the lines in $\mathcal{L}$ that pass through x. Thus, by choosing a parameterization of the points on the circle, we have a parameterization of the elements in $\mathcal{L}$. The idea is to choose a rational parameterization of the points on the circle. Thus, define $\xi := \tan(\alpha/2)$ and

$$\omega := (\cos\alpha, \sin\alpha) = \left(\frac{1-\xi^2}{1+\xi^2}, \frac{2\xi}{1+\xi^2}\right). \tag{5.4}$$

Let $\ell(\xi)$ denote the line ℓ_ω where ξ corresponds to ω by (5.4). Then we can write $\mathcal{L} = \{\ell(\xi)\}_{\xi\in\mathbb{R}}$ where

$$\ell(\xi) : t \mapsto t\left(\frac{1-\xi^2}{1+\xi^2}, \frac{2\xi}{1+\xi^2}\right) + \mu\left(\frac{-2\xi}{1+\xi^2}, \frac{1-\xi^2}{1+\xi^2}\right). \tag{5.5}$$

Now, define the map $\varphi : \mathbb{R}^2 \to K_\mu^{\mathbb{C}}$ as follows:

$$\varphi(\xi,\eta) := ix \quad \text{where} \quad \{x\} := \ell(\xi) \cap \ell(\eta).$$

Thus, φ is the mapping that expresses the relationship between elements in $\mathcal{L}$, when parameterized by ξ, and points in $K_\mu^{\mathbb{C}}$. Let us explicitly write down φ in terms of ξ and η.

If $x \in K_\mu^\circ$, there are exactly two elements $\ell_\omega, \ell_\sigma \in \mathcal{L}$ such that $x \in \ell_\omega \cap \ell_\sigma$, and if ξ and η correspond to ω and σ as in (5.4), then by (5.3) and a little computation,

$$x = \frac{-\mu}{1+\xi\eta}(\xi+\eta, \xi\eta - 1). \tag{5.6}$$

Thus,

$$\varphi(\xi,\eta) = \frac{-i\mu}{1+\xi\eta}(\xi+\eta, \xi\eta - 1). \tag{5.7}$$

Observe that φ is naturally defined on $\widehat{\mathbb{C}}^2 \setminus \widehat{\Gamma}_{\mathbb{C}}$, so we will consider it as a map from $\widehat{\mathbb{C}}^2 \setminus \widehat{\Gamma}_{\mathbb{C}}$ to $\widehat{\mathbb{C}}^2$.

Let $G := F \circ \varphi$; we start by extending G. It is here that we relate our extension problem to the one treated in Corollary 4.1.

5.2.1. *Proving that G is analytic in $\widehat{\mathbb{C}}^2 \setminus \widehat{\Gamma}_{\mathbb{C}}$.* We know that G is defined in $\widehat{\mathbb{R}}^2 \setminus \widehat{\Gamma}_{\mathbb{R}}$ and we need to prove that G is separately analytic also at infinity, i.e. that (4.12) and (4.13) of Corollary 4.1 hold.

Let us show that (4.13) holds, i.e. $G_\xi(\eta) := G(\xi,\eta)$ extends analytically to $\widehat{\mathbb{C}} \setminus \{-1/\xi\}$ for all $\xi \in \widehat{\mathbb{R}}$. Fix $\xi \in \widehat{\mathbb{R}}$ and let η vary. Geometrically, this corresponds to fixing the line $\ell(\xi)$ and letting $\ell(\eta)$ vary. When η varies, we are moving along $\ell(\xi)$, which is the same as varying t, where t is the natural parameter on $\ell(\xi)$. We know that $\zeta \mapsto f(\omega, \zeta)$ is an entire function for all $\omega \in S^1$, in particular $t \mapsto f(\omega, it)$ extends to an entire function for all $\omega \in S^1$. To use this, we need to find a relationship between t and η (remember that ξ is fixed). But $x = t\omega + \mu\omega^\perp$, so taking the scalar product of this expression with ω gives $t = x \cdot \omega$. Now, by (5.3) and (5.4), we get

$$t = \mu \frac{\xi - \eta}{1 + \xi\eta}.$$

Thus, for fixed ξ, the relationship between t and η is expressed by a rational function. It follows that G_ξ can be extended to an analytic function on all of $\widehat{\mathbb{C}}$ except at the point $\eta = -1/\xi$. Note that $\eta = \infty$ corresponds to $t = -1/\xi$, thus G_ξ is also analytic at $\eta = \infty$ as long as $\xi \neq 0$. When $\xi = \infty$, the relationship between t and η takes the form $t = 1/\eta$. Hence, G_∞ is analytic in all of $\widehat{\mathbb{C}}$ except at $\eta = 0$.

Thus, G satisfies (4.13) and the proof of (4.12) is similar. By Corollary 4.1, G extends to an analytic function in $\widehat{\mathbb{C}}^2 \setminus \widehat{\mathit{\Gamma}}_{\mathbb{C}}$.

5.2.2. *Proving that F is entire in $\mathbb{C}^2$.* Since G is symmetric, F is well defined by the relation $G = F \circ \varphi$. To get information about F from G, one would like to invert φ. This can be done locally, and we will show that F is analytic at $z \in \mathbb{C}^2$ in each of the cases listed below.

Case 1: $z \in \varphi(\mathbb{C}^2 \setminus \triangle_{\mathbb{C}})$.
Case 2: $z \in \varphi(\triangle_{\mathbb{C}})$.
Case 3: $z \notin \varphi(\mathbb{C}^2)$.

Observe that $\mathbb{C}^2 \cap \varphi(\mathit{\Gamma}_{\mathbb{C}}) = \emptyset$, so it is enough to invert φ at points $(\xi, \eta) \in \widehat{\mathbb{C}}^2 \setminus \widehat{\mathit{\Gamma}}_{\mathbb{C}}$.

Proof that F is analytic at z in Case 1. The determinant of the Jacobian matrix of φ at (ξ, η) is given by

$$\frac{2\mu^2(\eta - \xi)}{(1 + \xi\eta)^3},$$

which is non-zero outside $\triangle_{\mathbb{C}}$. Thus, by the implicit function theorem and the analyticity of G in $(\xi, \eta) \in \mathbb{C}^2 \setminus \mathit{\Gamma}_{\mathbb{C}}$, we conclude that F is analytic at $z \in \varphi(\mathbb{C}^2 \setminus \triangle_{\mathbb{C}})$.

Proof that F is analytic at z in Case 2. We will show that φ is locally a fold near points of $\triangle_{\mathbb{C}}$. Let $\pi(p, q) := (p, q^2)$ denote the usual fold in the second variable. We also define $\phi(p, q) := (p + q, p - q)$. Then

$$\varphi \circ \phi(p, q) = \frac{-i\mu}{1 + p^2 - q^2}(2p, p^2 - q^2 - 1)$$

so $\varphi \circ \phi = \psi \circ \pi$ where

$$\psi(w) := \frac{-i\mu}{1 + w_1^2 - w_2}(2w_1, w_1^2 - w_2 - 1).$$

Now, let $(\xi, \eta) \in \triangle_{\mathbb{C}} \setminus \Gamma_{\mathbb{C}}$ be arbitrary. We know that G is analytic at (ξ, η), so if $\widetilde{G} := G \circ \phi$, then $\widetilde{G}$ is analytic at points $(p, q) \in \phi^{-1}(\triangle_{\mathbb{C}} \setminus \Gamma_{\mathbb{C}})$. G is also symmetric, which implies that $\widetilde{G}$ is an even function in the second variable q. Thus, there exists a function $\widetilde{F}$ such that

$$\widetilde{G} = \widetilde{F} \circ \pi,$$

and $\widetilde{F}$ is analytic at $(p, q) \in \phi^{-1}(\triangle_{\mathbb{C}} \setminus \Gamma_{\mathbb{C}})$. Since

$$G = F \circ \varphi, \quad \varphi \circ \phi = \psi \circ \pi, \quad \text{and} \quad \widetilde{G} = G \circ \phi,$$

we get $\widetilde{F} = F \circ \psi$. Points $(\xi, \eta) \in \triangle_{\mathbb{C}} \setminus \Gamma_{\mathbb{C}}$ correspond to points $w = (0, w_2) = (0, \eta)$, and by the implicit function theorem ψ is locally invertible near $(0, w_2)$. Thus, F is analytic at $z \in \varphi(\triangle_{\mathbb{C}})$.

Proof that F is analytic at z in Case 3. Here we use the analyticity of G at certain points at infinity, and the fact that φ is locally invertible there.

Observe that the points we are interested in are of the form $z = (z_1, -i\mu)$. Then we seek points $(\xi, \eta) \in \widehat{\mathbb{C}}^2 \setminus \mathbb{C}^2$ where φ is invertible and such that $(z_1, -i\mu) = \varphi(\xi, \eta)$. More precisely, if $\xi' = 1/\xi$, then the determinant of the Jacobian matrix of φ in the variables (ξ', η) is given by

$$\frac{2\mu^2(1 - \xi'\eta)}{(\xi' + \eta)^3}.$$

Since $z = (z_1, -i\mu)$ corresponds to $\xi' = 0$ and $\eta = -i\mu/z_1$, the determinant is $2\mu^2/\eta^3$ at $(0, \eta)$, which is non-zero when $\eta \neq \infty$. Thus, φ is locally invertible near $(\xi', \eta) = (0, \eta)$, which corresponds to $(\xi, \eta) = (\infty, \eta)$. Our assumption on the analyticity of G at points in $\widehat{\mathbb{C}}^2 \setminus \widehat{\Gamma}_{\mathbb{C}}$ implies that F is analytic at $z = (z_1, -i\mu)$ when $z_1 \neq 0$. Since $(0, -i\mu)$ is the only point where F might not be analytic, it is removable, i.e. F extends analytically to $(0, -i\mu)$. Thus F is analytic at all points z in Case 3.

We have shown that F is an entire function in $\mathbb{C}^2$. This concludes the proof of Theorem 3.1.

5.3. *Proof of Theorem 3.2.* Fix $N \in \mathbb{N}$. From (3.3) we know that there is a constant $C_N > 0$ (independent of ω) and $r > 0$ such that

$$|F(\zeta\omega + i\mu\omega^{\perp})| \le C_N(1 + |\zeta|)^{-N} e^{r|\operatorname{Im}\zeta|}.$$

But

$$\begin{aligned} &1 + |\zeta\omega + i\mu\omega^{\perp}| \le 1 + |\zeta| + \mu \le (1 + \mu)(1 + |\zeta|), \\ &|\operatorname{Im}(\zeta\omega + i\mu\omega^{\perp})| \ge |\operatorname{Im}\zeta|, \end{aligned}$$

so for $z \in M_\mu$ there exists $C_{1,N} > 0$ (depending only on N) such that

$$|F(z)| \le C_{1,N}(1+|z|)^{-N} e^{r|\operatorname{Im} z|}. \tag{5.8}$$

We want to prove that F satisfies (5.8) (with a different constant) on all of $\mathbb{C}^2$. By analyticity, it is enough to show that (5.8) holds outside a fixed bounded set in $\mathbb{C}^2$.

5.3.1. *Extending inequality* (*5.8*) *to a slightly larger set.* We begin with showing that (5.8) holds on a slightly larger set, $\widetilde{M_\mu}$ (to be defined below), which is easier to work with. For $|z| \ge \mu$, it is easy to show that the condition $z = x + iy \in M_\mu$ is equivalent to

$$|y \cdot x^\perp| = \mu |x|. \tag{5.9}$$

If we define $\widetilde{M_\mu}$ as the set of points $z \in \mathbb{C}^2$ satisfying (5.9), then $\widetilde{M_\mu}$ differs from M_μ by a bounded set ([3]), and it follows that F must satisfy (5.8) on $\widetilde{M_\mu}$.

5.3.2. *Showing that F is of exponential type on* $\mathbb{C}^2$. Our goal here is to show that F is of exponential type on $\mathbb{C}^2$. Let us begin with some notation. For $z = x + iy \in \mathbb{C}^2$, define

$$\varrho(z) := \operatorname{Im}(\overline{z}_1 z_2) = y \cdot x^\perp$$

and observe that the set $\{z \in \mathbb{C}^2 : \varrho(z) = 0\}$ equals

$$\mathbb{C}\mathbb{R}^2 := \{\zeta x \in \mathbb{C}^2 : x \in \mathbb{R}^2 \text{ and } \zeta \in \mathbb{C}\}.$$

$\mathbb{C}\mathbb{R}^2$ is sometimes called the *Beurling cone* and it divides $\mathbb{C}^2$ into two components.

To prove that F is of exponential type, we apply the maximum principle to the restriction of F to a suitably chosen complex line in $\mathbb{C}^2$ passing through an arbitrary point z_0. To be more precise, if $w \in \mathbb{C}^2$, define $\ell_{w,z} \subset \mathbb{C}^2$ as the complex line in $\mathbb{C}^2$ through z with complex direction w, i.e.

$$\ell_{w,z} := \{z + \zeta w \in \mathbb{C}^2 : \zeta \in \mathbb{C}\}.$$

Without loss of generality, we can assume that $\varrho(z_0) < 0$. As already noted, the idea is to choose ([4]) the line ℓ_{w,z_0}, i.e. choose $w \in \mathbb{C}^2$, that is *not* parallel to the "cone" $\widetilde{M_\mu}$. Then z_0 is contained in a component of $\ell_{w,z_0} \setminus \widetilde{M_\mu}$, and we can apply the maximum principle to F on this component to show that (5.14) holds at z_0. Since $z_0 \in \mathbb{C}^2 \setminus \widetilde{M_\mu}$ is arbitrary, and since (5.14) also holds at points in $\widetilde{M_\mu}$, we conclude that F is of exponential type in $\mathbb{C}^2$.

Observe that if $z_0, w \in \mathbb{C}^2$ where $w \ne 0$ and if z_0 and w lie in different components of $\mathbb{C}^2 \setminus \mathbb{C}\mathbb{R}^2$, then the component in $\ell_{w,z_0} \setminus \mathbb{C}\mathbb{R}^2$ containing z_0

([3]) One can show that $\widetilde{M_\mu} = M_\mu \cup \{x + iy \in \mathbb{C}^2 : x = 0 \text{ and } |y| < \mu\}$.

([4]) The choice of w must of course be independent of z_0. Actually, the direction of w may depend on z_0, the important fact is that w does not depend on the norm of z_0.

is bounded. This follows from the observation that

$$\begin{aligned}(5.10)\qquad \varrho(z_0+\zeta w) &= |\zeta|^2\varrho(w+\zeta^{-1}z_0)\\ &= |\zeta|^2(\varrho(w)+O(|\zeta|^{-1}))\quad \text{as } |\zeta|\to\infty,\end{aligned}$$

and the fact that $\varrho(z_0)$ and $\varrho(w)$ have different signs. Define (5)

$$\varrho_\mu(z) := \varrho(z) - \mu|x| \quad \text{where } z = x+iy, \text{ so } \{z\in\mathbb{C}^2 : \varrho_\mu(z)=0\}\subset \widetilde{M}_\mu.$$

Choose w so that $\varrho(w)>0$ (take e.g. $w:=(1,i)$). Noting that $\varrho_\mu(z)-\varrho(z) = O(|z|)$ as $|z|\to\infty$, and using (5.10) we see that the set

$$(5.11)\qquad \Omega^-_{w,z_0} := \{z_0+\zeta w\in\mathbb{C}^2 : \zeta\in\mathbb{C} \text{ and } \varrho_\mu(z_0+\zeta w)<0\}\subset \ell_{w,z_0}$$

is bounded. Thus, by the maximum principle,

$$|F(z_0)| \le \max_{z\in\partial\Omega^-_{w,z_0}} |F(z)|.$$

Since $\partial\Omega^-_{w,z_0}\subset\widetilde{M}_\mu$, we know that (5.8) holds at all points of $\partial\Omega^-_{w,z_0}$, so for $z_0\in\mathbb{C}^2\setminus\widetilde{M}_\mu$, there exists $z\in\partial\Omega^-_{w,z_0}$ such that

$$(5.12)\qquad |F(z_0)| \le C_{1,0}e^{r|z|}\quad \text{with } C_{1,0}>0 \text{ as in (5.8).}$$

The lemma below simply states the following. Let $z_0\notin\widetilde{M}_\mu$ and $z\in\ell_{w,z_0}\cap\widetilde{M}_\mu$ where we have chosen w as described before. Then $|z_0|$ is of the same order of magnitude as $|z|$.

This is rather easy to see for the case where we replace our "cone" $\widetilde{M}_\mu$ with the Beurling cone. The proof of the lemma can be found in [9, Lemma 5.1].

LEMMA 5.1. *Let $z = z_0+\zeta w$ where $|w|\ge 1$, $|z_0|\ge 1$, $\varrho(z_0)<0$, $\varrho(w)>0$ and $\varrho_\mu(z)=0$. Then there exist constants $C_1, C_2>0$, depending only on w, such that*

$$(5.13)\qquad C_1|z_0|\le |z|\le C_2|z_0|.$$

Using Lemma 5.1 in inequality (5.12) shows that

$$(5.14)\qquad |F(z_0)|\le C_{1,0}e^{C_2 r|z_0|}.$$

Thus, F is of exponential type.

5.3.3. *Estimating entire functions of exponential type.* We begin with stating the following version of the Phragmén–Lindelöf principle in $\mathbb{C}^n$.

THEOREM 5.1. *Let $u\in\mathcal{PSH}(\mathbb{C}^n,\mathbb{R})$ and assume that for some constants $C_3, C_4, C_5>0$:*

(5) When $\varrho(z_0)>0$, we define $\varrho_\mu(z):=\varrho(z)+\mu|x|$ and since $\{z\in\mathbb{C}^2 : \varrho_\mu(z)=0\}\subset\widetilde{M}_\mu$, we can argue similarly.

1. $u(x) \le C_3$ *for all* $x \in \mathbb{R}^n$,
2. $u(z) \le C_4 + C_5|z|$ *for all* $z \in \mathbb{C}^n$.

Then $u(z) \le C_3 + C_5|\operatorname{Im} z|$ *for all* $z \in \mathbb{C}^n$.

Proof. Let $\mathbb{H}^+ := \{\zeta \in \mathbb{C} : \operatorname{Im} z > 0\}$. The following 1-variable version of Theorem 5.1 is a Phragmén–Lindelöf principle in $\mathbb{C}$ (see e.g. [9, Lemma 5.3], or [4, exercise 18, p. 477]).

LEMMA 5.2. *Let* $u \in \mathcal{SH}(\mathbb{H}^+, \mathbb{R})$ *satisfy the following*:

1. $\limsup_{\zeta \to s} u(\zeta) \le C_3$ *for all* $s \in \mathbb{R}$,
2. $u(\zeta) \le C_4 + C_5|\zeta|$ *for* $\zeta \in \mathbb{H}^+$.

Then $u(\zeta) \le C_3 + C_5 \operatorname{Im} \zeta$ *for* $\zeta \in \mathbb{H}^+$.

The proof of Theorem 5.1 is now rather easy. Let $x, y \in \mathbb{R}^n$ and define $u_{x,y} : \mathbb{C} \to \mathbb{R}$ as

$$u_{x,y}(\zeta) := u(x + \zeta y).$$

Then $u_{x,y} \in \mathcal{SH}(\mathbb{C}, \mathbb{R})$, $u_{x,y}(s) \le C_3$ for $s \in \mathbb{R}$ and

$$u_{x,y}(\zeta) = u(x+\zeta y) \le C_4 + C_5|x+\zeta y| \le C_4 + C_5|y| \cdot |\zeta| + C_5|x| = C_7 + C_6|\zeta|,$$

where $C_6 := C_5|y|$ and $C_7 := C_4 + C_5|x|$. Lemma 5.2 now implies that

$$u_{x,y}(\zeta) \le C_3 + C_6|\operatorname{Im} \zeta| = C_3 + C_5|y| \cdot |\operatorname{Im} \zeta|.$$

Note that the constant C_7, which depends on x and can be very large, disappears. Let $\zeta = i$, which gives

$$u(x + iy) = u_{x,y}(i) \le C_3 + C_5|y|$$

and concludes the proof of Theorem 5.1. ■

For $N \in \mathbb{N}$, define $F_N : \mathbb{C}^2 \to \mathbb{C}$ as $F_N(z) := z_1^N F(z)$. Since $F \in \mathcal{O}(\mathbb{C}^2, \mathbb{C})$, we have $F_N \in \mathcal{O}(\mathbb{C}^2, \mathbb{C})$.

We know that (5.8) holds on $\widetilde{M}_\mu$ and that (5.14) holds in $\mathbb{C}^2$, so

$$|F_N(z)| \le C_{1,N} e^{r|\operatorname{Im} z|} \quad \text{for all } z \in \widetilde{M}_\mu, \tag{5.15}$$

and there exists $r' > 0$ (slightly larger than $C_2 r$) such that

$$|F_N(z_0)| \le C_{2,N} e^{r'|z_0|} \quad \text{for all } z_0 \in \mathbb{C}^2. \tag{5.16}$$

Now, define the 2-dimensional submanifold $M_{\mu,0} \subset M_\mu$ as

$$M_{\mu,0} := \{s\omega + i\mu\omega^\perp \in \mathbb{C}^2 : s \in \mathbb{R} \text{ and } \omega \in S^1\}.$$

Then (5.15) implies in particular that F_N is uniformly bounded on $M_{\mu,0}$, i.e.

$$|F_N(z)| \le C_{1,N} \quad \text{for all } z \in M_{\mu,0}. \tag{5.17}$$

LEMMA 5.3. *Let F_N be as above. Then*

$$|F_N(x)| \le C_{1,N} \quad \text{for all } x \in \mathbb{R}^2. \tag{5.18}$$

From Lemma 5.3 we deduce that inequality (5.17) above implies that F_N is uniformly bounded on $\mathbb{R}^2$. The idea of the proof is the following. Construct an analytic disc attached to $M_{\mu,0}$ for which the intersection with $\mathbb{R}^2$ is a circle centered at the origin with arbitrarily large radius. Thus, one can apply the maximum principle on the circle to estimate F_N at points in $\mathbb{R}^2$. The proof can be found in [9, Lemma 5.2].

Define $u_N(z) := \log|F_N(z)|$. Combining (5.16) with (5.18) allows us to apply Theorem 5.1 to u_N (6). Then

$$u_N(z) \le C_3 + C_5|\operatorname{Im} z| = \log C_{1,N} + r'|\operatorname{Im} z| \quad \text{for all } z \in \mathbb{C}^2,$$

i.e.

$$|F_N(z)| \le C_{1,N} e^{r'|\operatorname{Im} z|} \quad \text{for all } z \in \mathbb{C}^2.$$

Observe that for $z \in \mathbb{C}^2$ with $|z_1| \ge 1$ and $|z_2| \le |z_1|$,

$$1 + |z| \le 1 + |z_1| + |z_2| \le 3|z_1|$$

so

$$|z_1|^{-N} \le 3^N(1+|z|)^{-N}.$$

Thus, for such $z \in \mathbb{C}^2$,

$$|F(z)| = |z_1|^{-N}|F_N(z)| \le 3^N(1+|z|)^{-N}|F_N(z)|. \tag{5.19}$$

Now, defining $F_N(z) := z_2^N F(z)$ and repeating the above arguments, we find that the inequality in (5.19) also holds for $z \in \mathbb{C}^2$ where $|z_2| \ge 1$ and $|z_1| \le |z_2|$. Thus, for $z \in \mathbb{C}^2$ outside a fixed bounded set,

$$\begin{aligned} |F(z)| &\le 3^N C_{1,N}(1+|z|)^{-N} e^{r'|\operatorname{Im} z|} \\ &= C'_N(1+|z|)^{-N} e^{r'|\operatorname{Im} z|} \quad \text{for all } z \in \mathbb{C}^2 \end{aligned}$$

with $C'_N := 3^N C_{1,N}$. This concludes the proof of Theorem 3.2.

Acknowledgements. I would like to thank Jan Boman for his careful reading of the manuscript and extensive help. Most of the ideas in this article originate from him. Also, I would like to thank Andreas Nilsson for long and fruitful discussions.

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(6) We apply Theorem 5.1 with $n := 2$, $u := u_N$, $C_3 := \log C_{1,N}$, $C_4 := \log C_{2,N}$ and $C_5 := r'$.

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Reçu par la Rédaction le 29.11.1997
Révisé le 17.8.1998