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A microlocal version of Cartan–Grauert's theorem

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Abstract. Tuboids are tube-like domains which have a totally real edge and look asymptotically near the edge as a local tube over a convex cone. For such domains we state an analogue of Cartan's theorem on the holomorphic convexity of totally real domains in $\mathbb{R}^n \subset \mathbb{C}^n$.

1. Classical Cartan theorem. In his paper [3] H. Cartan proved the following theorem.

Theorem (Cartan). *Let ω be a domain in $\mathbb{R}^n = \mathbb{R}^n + i0 \subset \mathbb{C}^n$. Then ω has a fundamental system of neighborhoods Ω in $\mathbb{C}^n$ which are domains of holomorphy. In other words, for any domain Ω' in $\mathbb{C}^n$, containing ω, there exists a domain of holomorphy Ω such that $\omega \subset \Omega \subset \Omega'$.*

H. Grauert in [4] extended this theorem to domains lying on analytic totally real submanifolds in $\mathbb{C}^n$.

One way to prove Cartan's theorem is to construct the desired domain Ω as (the interior of) the intersection of a family of complex hyperboloids.

In more detail, denote by Ω'_x the section of Ω' by the "imaginary" plane $x + i\mathbb{R}^n$ through $x \in \Omega' \cup \mathbb{R}^n =: \omega'$:

$$\Omega'_x = \{z = x + iy \in \mathbb{C}^n : z \in \Omega'\}.$$

Denote by $r(x)$ the "radius" of this section, i.e. $r(x) = \mathrm{dist}(x, \partial\Omega'_x)$ for $x \in \omega'$ and 0 otherwise. It is a lower semicontinuous function on $\mathbb{R}^n$.

A *complex hyperboloid* or a *pseudoball* is a domain of holomorphy in $\mathbb{C}^n$ of the form

$$U(z_0, r) = \{z \in \mathbb{C}^n : \mathrm{Re}(z - z_0)^2 + r^2 > 0\}.$$

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The section $U_x(z_0, r)$ of the hyperboloid $U(z_0, r)$ by the "imaginary" plane $x + i\mathbb{R}^n$ through $x \in \mathbb{R}^n$ is the ball of radius $\sqrt{(x-x_0)^2 + r^2}$ with centre at y_0.

Now we define the domain of holomorphy Ω as the intersection of all complex hyperboloids with centres on ω':

$$\Omega = \operatorname{Int} \bigcap_{x \in \omega'} U(x, r(x))$$

where Int denotes interior.

By construction, $\Omega \subset \Omega'$. We show that $\Omega \supset \omega' \supset \omega$. For any $x_0 \in \omega'$ we consider a neighborhood $o(x_0) = \{x \in \omega' : |x - x_0| < d_0\}$ with compact closure in ω'. Set $r_0 = \inf\{r(x) : x \in \overline{o(x_0)}\}$ (this is positive due to the semicontinuity of $r(x)$). We assert that Ω contains a neighborhood of x_0 of the form

$$O(x_0) = \{z = x + iy : |x - x_0| < \varepsilon,\ |y| < \varepsilon\},$$

where $\varepsilon = \frac{1}{2}\min\{d_0, r_0\}$. Indeed, if the centre x' of a hyperboloid $U(x', r(x'))$ is "close" to x_0, namely, if $x' \in o(x_0)$, then its section $U_x(x', r')$ at any point x with $|x - x_0| < d_0$ is the ball of radius $r' \ge r(x') \ge r_0$ with centre at $y_0 = 0$, containing $O_x(x_0)$. If the distance from x' to x_0 is greater than d_0, then the radius of $U_x(x', r')$ at any point x with $|x - x_0| < d_0/2$ satisfies $r' \ge |x - x'| \ge d_0/2 \ge \varepsilon$ and the section $U_x(x', r(x'))$ again contains $O_x(x_0)$.

This argument shows that Ω is open and contains $\omega' \supset \omega$. Moreover, it is connected because ω' is connected and the sections Ω_x are convex. Finally, Ω is a domain of holomorphy, being the interior of the intersection of complex hyperboloids.

2. Tuboids. Recall that a *local tube* over a domain $\omega \subset \mathbb{R}^n$ with profile V is a domain in $\mathbb{C}^n$ of the form

$$T_r(\omega, V) = \omega + iV_r = \{z = x + iy \in \mathbb{C}^n : x \in \omega,\ y \in V_r\}$$

where V is an open connected (non-empty) cone in $\mathbb{R}^n$, and $V_r := V \cap \{y \in \mathbb{R}^n : |y| < r\}$.

Tuboids are domains in $\mathbb{C}^n$ which look like local tubes with profile varying from point to point in $\mathbb{R}^n$. First we define more precisely what we mean by the varying profile.

Definition 1. Let ω be a domain in $\mathbb{R}^n$. A *profile* over ω is a domain Λ in $\mathbb{C}^n$ of the form

$$\Lambda = \Lambda(\omega) = \{z = x + iy : x \in \omega,\ y \in \Lambda_x\}$$

where Λ_x is an open connected (non-empty) cone in $\mathbb{R}^n$.

The set

$$\dot{\Lambda} = \{z = x + iy : x \in \omega,\ y \in \dot{\Lambda}_x\},$$

where $\dot{\Lambda}_x$ is the intersection of Λ_x with the unit sphere S^{n-1} in $\mathbb{R}^n$, will be called the *base* of the profile $\Lambda(\omega)$. (We always assume that the equality $\dot{\Lambda}_x = S^{n-1}$ for some $x \in \mathbb{R}^n$ implies $\Lambda_x = \mathbb{R}^n$.) A profile $\Lambda = \Lambda(\omega)$ is *compactly supported* in a profile $\Lambda' = \Lambda'(\omega)$, $\Lambda \Subset \Lambda'$, if the closure of the base $\dot{\Lambda}$ is contained in the base $\dot{\Lambda}'$.

For a profile $\Lambda = \Lambda(\omega)$ we define its *fiberwise convex hull* as a profile over ω of the form

$$\operatorname{ch}\Lambda = \{z = x + iy : x \in \omega,\ y \in \operatorname{ch}\Lambda_x\}$$

where $\operatorname{ch}\Lambda_x$ is the convex hull of the cone Λ_x. A profile is called *fiberwise convex* iff $\operatorname{ch}\Lambda = \Lambda$. Accordingly, the base $\dot{\Lambda}$ of a profile Λ is called *fiberwise convex* if Λ is fiberwise convex.

Definition 2. A *tuboid* in $\mathbb{C}^n$ with profile $\Lambda = \Lambda(\omega)$ is a domain Ω in $\mathbb{C}^n$ of the form

$$\Omega = \Omega(\Lambda) = \{z = x + iy : x \in \omega,\ y \in \Omega_x\},$$

where Ω_x is a domain in $\mathbb{R}^n$, such that the following condition is satisfied: for any $x \in \omega$ and any open connected cones V', V'' in $\mathbb{R}^n$ such that $V' \Subset \Lambda_x \Subset V''$ there exist a neighborhood $O(x)$ of x in $\mathbb{C}^n$ and a number $r > 0$ such that

$$T_r(o(x), V') \subset \Omega \cap O(x) \subset T_r(o(x), V'')$$

with $o(x) := O(x) \cap \mathbb{R}^n$. In other words, the intersection of Ω with a sufficiently small neighborhood $O(x)$ must, on the one hand, contain some local tube with profile V' and, on the other hand, be contained in some local tube with profile V''.

One can imagine a tuboid with profile $\Lambda(\omega)$ as a domain with the edge ω which approximates the cone Λ_x "asymptotically" near each $x \in \omega$. Local tubes and profiles can themselves serve as examples of tuboids. The notion of tuboid was introduced by J. Bros and D. Iagolnitzer in [1, 2].

The definitions of the profile and tuboid are easily extended to the case where the edge $\mathbb{R}^n$ is replaced by a totally real submanifold in $\mathbb{C}^n$. Namely, let M be a smooth totally real submanifold of dimension n in $\mathbb{C}^n = \mathbb{R}^n_{(x)} + i\mathbb{R}^n_{(y)}$ and let ω be a domain on M. Suppose that for any $z \in \omega$ the "imaginary" plane $z + i\mathbb{R}^n_{(y)}$ is transversal to M at z. Denote by $\{\Lambda_z\}$ a family of open connected (non-empty) cones in $\mathbb{R}^n_{(y)}$ parametrized by points $z \in \omega$. A *profile* over ω is a domain Λ in $\mathbb{C}^n$ of the form

$$\Lambda = \bigcup_{z \in \omega} \{z + \Lambda_z\}.$$

A profile Λ will be called a *tube profile* if for any $z \in \omega$ the cone Λ_z equals Λ_0, i.e. does not depend on z. Replacing the cone Λ_0 in this definition by its intersection with the ball $B(0, r)$ of radius r with centre at the origin, we get the definition of a *local tube profile*. A *tuboid* over ω is a domain of the form

$$\Omega = \bigcup_{z\in\omega} \{z + \Omega_z\}$$

where Ω_z is a domain in $\mathbb{R}^n_{(y)}$ which can be approximated at points of ω by local tube profiles in the same sense as in Definition 2 above.

3. Cartan–Grauert's theorem for tuboids. We now formulate a generalization of Cartan–Grauert's theorem to tuboids.

The standard complex metric $(z,z)_{\mathbb{C}} = z_1^2 + \ldots + z_n^2$ on $\mathbb{C}^n$ induces a real metric on $\mathbb{C}^n$, identified with $\mathbb{R}^{2n}$, of signature (n,n) given by

$$(z,z)_{\mathbb{R}} := \mathrm{Re}(z,z) = (x,x) - (y,y) \quad \text{for } z = x + iy \in \mathbb{C}^n.$$

Denote by E the vector space $\mathbb{R}^{2n}$ provided with this metric. A vector $z\in E$ is *real-like* if $(z,z)_{\mathbb{R}} > 0$, *imaginary-like* if $(z,z)_{\mathbb{R}} < 0$, and *null* if $(z,z)_{\mathbb{R}} = 0$. Accordingly, a linear subspace in E is called *real-like* (resp. *imaginary-like*, *null*) if every non-zero vector in this subspace is real-like (resp. imaginary-like, null).

We say that a smooth submanifold M in E is *real-like* if the tangent space $T_{z_0}M$ at any point $z_0 \in M$ is real-like. Evidently, such a submanifold is always totally real. A submanifold M is *strictly real-like* if it coincides with the graph of a smooth map

$$F : \mathbb{R}^n_{(x)} \to \mathbb{R}^n_{(y)}$$

with derivative satisfying the condition

$$\|F'(x)\| < 1 \quad \text{for any } x \in \mathbb{R}^n_{(x)}.$$

Theorem. *Let M be a C^2-smooth strictly real-like submanifold in $E \cong \mathbb{C}^n$, and ω be a domain on M. Suppose that Λ is a fiberwise convex profile over ω. Then for any tuboid Ω' with profile Λ there exists a tuboid $\Omega \subset \Omega'$ with the same profile which is a domain of holomorphy in $\mathbb{C}^n$.*

In order to obtain the classical Cartan theorem of Section 1 from the above theorem, take $M = \mathbb{R}^n_{(x)}$, $\Lambda = \mathbb{R}^n$. Our theorem may be considered as a "microlocal" variant of Cartan–Grauert's theorem because replacing the full profile $\Lambda = \mathbb{R}^n$ in the classical case by an arbitrary profile Λ corresponds to localization in "conormal directions". An extension of Cartan's theorem to tuboids was proved by J. Bros and D. Iagolnitzer in [1, 2] in the case when M is $\mathbb{R}^n$ or real-analytic.

The condition of fiberwise convexity of the profile Λ is not very essential since for any tuboid Ω' over ω with profile Λ we can find a tuboid Ω'' over ω with profile $\mathrm{ch}\,\Lambda$ such that any function holomorphic in Ω extends holomorphically to Ω''. This is a microlocal version of Bochner's tube theorem.

It was proved for tuboids (in the case when M is $\mathbb{R}^n$ or real-analytic) in [1, 2].

We give here a sketch of the proof of the above theorem; the details are given in [5].

We shall construct the desired tuboid Ω using the same idea as in the proof of the classical Cartan theorem, as the intersection of a family of complex pseudoballs in E.

At every point $\xi \in M$ we have a decomposition of the tangent space

$$T_\xi E \cong E = \mathcal{R}_\xi \oplus \mathcal{J}_\xi,$$

where $\mathcal{R}_\xi = T_\xi M$ and $\mathcal{J}_\xi$ is its orthogonal complement in E. We introduce a curvature function of M determined by the norm of the second derivative $\|F''(\xi)\|$ at $\xi \in M$ (assuming that $F(\xi) = F'(\xi) = 0$). Having the curvature function one can define a "characteristic radius" $r(\xi) > 0$ of M at ξ which is a continuous function on M. We denote by B_ξ the ball of radius $r(\xi)$ in $\mathcal{J}_\xi \subset \xi + E$. The union of the balls B_ξ over all $\xi \in M$ forms a tube neighborhood B of M in E which can also be identified with the ball normal bundle of M in E. Points in B have "spherical coordinates" (ξ, s, r) where $\xi \in M$, s is a point of the unit sphere in $\mathcal{J}_\xi$, $0 \le r < r(\xi)$.

The desired tuboid Ω is constructed in several steps by "cutting off superfluous parts" from the original tuboid Ω'. We recall that Ω' has the fiberwise convex profile Λ. The first step is to construct a tuboid $\Omega_1 \subset B$ over ω with the same profile but having convex spherical sections. In terms of the spherical coordinates on B, introduced above, this means that all sections $\Omega_{1,\xi,r}$ of Ω_1 with fixed (ξ, r) are convex (i.e. the cones generated by $\Omega_{1,\xi,r}$ are convex). Moreover, $\Omega_1 \subset \Omega$.

The next step is to construct a tuboid $\Omega_2 \subset \Omega_1$ over ω with the same profile having fiberwise spherically convex fibres.

We recall the definition of spherical convexity. Any bounded convex domain D in $\mathbb{R}^n$, contained, for instance, in the ball of radius R, may be represented as the intersection of support hyperplanes containing D. Let $y \in \partial D$. Replace the hyperplane supporting D at y by the ball B_y of radius R which is tangent to the boundary of the support hyperplane at y. Taking the intersection of all such balls B_y over all $y \in \partial D$ we obtain a set

$$\breve{D} = \bigcap_{y \in \partial D} B_y$$

which is called the *spherical polar* of D. By construction, $\breve{D} \subset D$. The domain D is called *spherically convex* iff $\breve{D} = D$.

At the second step we construct a tuboid $\Omega_2 \subset \Omega_1 \subset B$ whose fibres $\Omega_{2,\xi}$ are spherically convex with respect to the ball B_ξ. We denote by $\breve{\Omega}_2$ the spherical polar of Ω_2, i.e. the set formed by the fiberwise spherical polars $\breve{\Omega}_{2,\xi}$ of $\Omega_{2,\xi}$ with respect to the balls B_ξ for $\xi \in \omega$.

The last step is the construction of the tuboid $\Omega \subset \Omega_2$ by taking the interior of the intersection of all pseudoballs $U((\xi, s, r), r(\xi))$ with centres at points $(\xi, s, r) \in \breve{\Omega}_2$ of "characterictic radius" $r(\xi)$ for $\xi \in \omega$. Then $\Omega \subset \Omega'$ is a tuboid over ω with the same profile Λ which is a domain of holomorphy.

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