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Holomorphic functions of fast growth on submanifolds of the domain

by Piotr Jakóbczak (Kraków)

Abstract. We construct a function f holomorphic in a balanced domain D in $\mathbb{C}^N$ such that for every positive-dimensional subspace Π of $\mathbb{C}^N$, and for every p with $1 \le p < \infty$, $f|_{\Pi\cap D}$ is not L^p-integrable on $\Pi \cap D$.

1. Introduction. Let D be an open set in $\mathbb{C}^N$, and let F be some class of complex-valued functions in D which are holomorphic in D and satisfy some other conditions there. Given an affine subspace M of positive dimension in $\mathbb{C}^N$, the problem is to determine what further properties (besides being holomorphic) the functions from the class F have when restricted to the slice $M\cap D$. This problem was studied in many situations by several authors; see e.g. [2], [5], [8], [9], [11].

In [4] we have shown that there exists a function f holomorphic in the unit ball B in $\mathbb{C}^N$ such that for every positive-dimensional subspace Π of $\mathbb{C}^N$, $f|_{\Pi\cap B}$ is not L^2-integrable in $\Pi\cap B$. The proof consists of construction of a function f with sufficiently fast growth near the boundary of each set of the form $\Pi \cap B$, and the use of the well-known estimates relating the growth near the boundary and the L^2-norm of a holomorphic function. (See also [10] for a much more explicit proof of this result.)

In the present note we carry out the construction from [4] for the more general situation of domains which are balanced domains of holomorphy, i.e. domains of holomorphy such that for every $z = (z_1, \ldots, z_N) \in D$ and every $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$, the point $\lambda z = (\lambda z_1, \ldots, \lambda z_N)$ also belongs to D. We obtain holomorphic functions with prescribed fast growth near the boundary of such domains; then we apply our construction in order to obtain functions which are holomorphic and not integrable on linear slices of the domain, or which are not in $\mathcal{O}(\delta)$ on any such slice, where $\mathcal{O}(\delta)$ denotes the

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space of functions of δ-tempered growth, and δ is a given weight function (see e.g. [1]; the precise definiton of $\mathcal{O}(\delta)$ will be recalled later).

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2. A holomorphic function with prescribed growth on slices. Let D be a balanced domain of holomorphy in $\mathbb{C}^N$. Then there exists a strictly plurisubharmonic smooth exhaustion function ϱ in D, i.e. a smooth function ϱ which is strictly plurisubharmonic in D and for every real c, the set $\{z \in D \mid \varrho(z) < c\}$ is relatively compact in D. For further use we need the existence of a sequence $\{D_n\}_{n=1}^{\infty}$ of strictly pseudoconvex, smoothly bounded, balanced domains which exhaust D and every straight line in $\mathbb{C}^N$ passing through zero intersects the boundary ∂D_n of every domain D_n transversally. It seems that the existence of such a sequence is well known; the proof of the following proposition was suggested to us by M. Jarnicki, Ch. Kiselman and P. Pflug.

PROPOSITION 1. *Let D be a balanced domain of holomorphy in $\mathbb{C}^N$. Then there exists $\varepsilon_0 > 0$ and a family $\{D_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ of strictly pseudoconvex, smoothly bounded, balanced domains such that $\bigcup_{0<\varepsilon\le\varepsilon_0} D_\varepsilon = D$, $\overline{D}_\varepsilon \subset D_{\varepsilon'}$ for $0 < \varepsilon' < \varepsilon \le \varepsilon_0$, and for every ε, every (real) straight line passing through zero in $\mathbb{C}^N$ intersects ∂D_ε transversally.*

Proof. Let h be the Minkowski functional for D. Since D is a domain of holomorphy and is balanced, h is plurisubharmonic in $\mathbb{C}^N$, and $h(\lambda z) = |\lambda| h(z)$ for every $z \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$. For $\varepsilon > 0$, denote by h_ε the regularization

$$h_\varepsilon(z) = \int_{\mathbb{C}^N} h(z - \varepsilon y)\phi(y)\, dm(y),$$

where ϕ is a smooth function in $\mathbb{C}^N$, $\operatorname{supp}\phi$ is the unit ball, $\phi(y) = \phi(|y_1|, \ldots \ldots, |y_N|)$ for every $y = (y_1, \ldots, y_N) \in \mathbb{C}^N$, and $\int_{\mathbb{C}^N} \phi(y)\, dm(y) = 1$. (Here m denotes the usual Lebesgue measure in $\mathbb{C}^N$.) It is well known that h_ε is smooth and plurisubharmonic in $\mathbb{C}^N$, for each $z \in \mathbb{C}^N$, $h_\varepsilon(z)$ tends decreasingly to $h(z)$ as ε decreases to zero, and $h_\varepsilon(e^{it}z) = h_\varepsilon(z)$, $z \in \mathbb{C}^N$, $t \in \mathbb{R}$. Since $h(0) = 0$, there exists $\varepsilon_0 > 0$ so small that $h_{\varepsilon_0}(0) < 1$ (and hence $h_\varepsilon(0) < 1$ for all $0 < \varepsilon \le \varepsilon_0$). For $0 < \varepsilon \le \varepsilon_0$, set

$$\varrho_\varepsilon(z) = h_\varepsilon(z) + \varepsilon\|z\|^2.$$

Then ϱ_ε is a smooth and strictly plurisubharmonic function in $\mathbb{C}^N$. Let $D_\varepsilon = \{z \in \mathbb{C}^N \mid \varrho_\varepsilon(z) < 1\}$. Then $0 \notin \partial D_\varepsilon$ (because $\varrho_\varepsilon(0) < 1$), $\overline{D}_{\varepsilon'} \subset D_{\varepsilon''}$ for $0 < \varepsilon'' < \varepsilon'$, and the domains D_ε tend increasingly to D as ε decreases to zero; moreover, every domain D_ε is pseudoconvex. Using the maximum principle for subharmonic functions and the fact that $h_\varepsilon(e^{it}z) = h_\varepsilon(z)$ for $z \in \mathbb{C}^N$ and $t \in \mathbb{R}$, we have for $\lambda \in \mathbb{C}$, $|\lambda| \le 1$ and $z \in \mathbb{C}^N$,

$$h_\varepsilon(\lambda z) \le \max_{t\in\mathbb{R}} h_\varepsilon(e^{it}z) = h_\varepsilon(z), \tag{1}$$

and hence

$$\varrho_\varepsilon(\lambda z) \le \varrho_\varepsilon(z), \quad z \in \mathbb{C}^N,\ \lambda \in \mathbb{C},\ |\lambda| \le 1.$$

Hence every domain D_ε is balanced.

Now fix ε with $0 < \varepsilon \le \varepsilon_0$, and $z \in \partial D_\varepsilon$. By (1) the function

$$\phi : [0,\infty) \ni t \mapsto h_\varepsilon(tz)$$

is non-decreasing. Denote by ψ the function

$$\psi : [0,\infty) \ni t \mapsto \varrho_\varepsilon(tz).$$

Then

$$\psi'(t) = \langle \operatorname{grad} \varrho_\varepsilon(tz), z\rangle_{\mathbb{R}} = \phi'(t) + 2\varepsilon t\|z\|^2,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ denotes the standard real scalar product in $\mathbb{C}^N = \mathbb{R}^{2N}$. Further,

$$\langle \operatorname{grad} \varrho_\varepsilon(z), z\rangle_{\mathbb{R}} = \phi'(1) + 2\varepsilon\|z\|^2 > 0 \tag{2}$$

(here we use the fact that $0 \notin \partial D_\varepsilon$). It follows form (2) that ∂D_ε is smooth (and so D_ε is strictly pseudoconvex), and that

(3) ∂D_ε is transversal to every (real) straight line passing through zero.

This ends the proof. ■

Fix ε with $0<\varepsilon \le \varepsilon_0$. It is well known that for a given compact subset K of $\mathbb{C}^N$, and for ε' sufficiently close to ε, the regularizations $h_{\varepsilon'}$ are arbitrarily close to h_ε on K. Therefore the same is true for the functions $\varrho_{\varepsilon'}$ and ϱ_ε. Hence, given an arbitrary neighborhood U of $\overline{D}_\varepsilon$, there exists $\varepsilon' < \varepsilon$ such that $D_{\varepsilon'} \subset U$. Suppose now that f is a function holomorphic in some neighborhood U of $\overline{D}_\varepsilon$, and fix $D_{\varepsilon'} \subset U$ as above. Then f is holomorphic in $D_{\varepsilon'}$. Since $D_{\varepsilon'}$ is a balanced domain of holomorphy, there exists a series $\sum_{s=0}^\infty Q_s$ of homogeneous polynomials which converges to f uniformly on compact subsets of $D_{\varepsilon'}$; in particular, the convergence is uniform on $\overline{D}_\varepsilon$. This yields the following proposition:

PROPOSITION 2. *Let the domain D and the family $\{D_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be as in Proposition 1. Then given ε with $0 < \varepsilon \le \varepsilon_0$, every function holomorphic in a neighborhood of $\overline{D}_\varepsilon$ can be uniformly approximated on $\overline{D}_\varepsilon$ by functions which are holomorphic in the whole domain D.*

In the sequel, given $K \subset \mathbb{C}^N$ and $f \in \mathcal{C}(K)$, we denote by $\|f\|_K$ the usual supremum norm on K.

Suppose now that δ is a positive, bounded and continuous function in a domain G in $\mathbb{C}^N$. Denote by $\mathcal{O}(\delta)$ the space of all functions holomorphic in

G such that there exists a positive integer k with

$$\sup\{|\delta^k(z)f(z)| \mid z \in G\} < \infty.$$

If moreover δ satisfies the conditions:

(i) $|z|\delta$ is bounded on $\mathbb{C}^N$,
(ii) $|\delta(z) - \delta(z')| \le |z - z'|$ for all $z, z' \in \mathbb{C}^N$,

then it is called a *weight function* (see [1]). The theory of functions from the space $\mathcal{O}(\delta)$ was investigated by several authors (see e.g. [1]).

We will prove the following theorem on the existence of holomorphic functions with bad boundary behavior on submanifolds:

THEOREM 1. *Let D be a balanced domain of holomorphy in $\mathbb{C}^N$, and δ a positive and continuous function in D. Then there exists a function f holomorphic in D such that for every positive-dimensional subspace Π of $\mathbb{C}^N$, $f|_{\Pi \cap D} \notin \mathcal{O}(\delta|_{\Pi \cap D})$.*

Let $\{D_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be the family of domains constructed in Proposition 1. Choose an arbitrary sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots$ and $\lim_{n\to\infty} \varepsilon_n = 0$. We have $\overline{D}_{\varepsilon_n} \subset D_{\varepsilon_{n+1}}$ for $n = 1, 2, \ldots$ For each n, choose a neighborhood U_n of $\partial D_{\varepsilon_n}$ such that $\overline{U}_n \subset D$, and $\overline{U}_n \cap \overline{U}_k = \emptyset$ for $n \ne k$. It follows from the proof of Proposition 1 that for every n and every $z \in \partial D_{\varepsilon_n}$, $\operatorname{grad} \varrho_{\varepsilon_n}(z) \ne 0$ (where ϱ_{ε_n} is a defining function for D_{ε_n}, obtained in the proof of Proposition 1). Shrinking the neighborhoods U_n if necessary we may assume that

$$\text{for every } n \text{ and for every } z \in U_n, \quad \operatorname{grad} \varrho_{\varepsilon_n}(z) \ne 0. \tag{4}$$

Moreover, according to the proof of Proposition 1, we have

$$D_{\varepsilon_n} = \{z \in \mathbb{C}^N \mid \varrho_{\varepsilon_n}(z) < 1\}, \tag{5}$$

and ϱ_{ε_n} is smooth and strictly plurisubharmonic in $\mathbb{C}^N$, and satisfies the condition

$$\varrho_{\varepsilon_n}(\lambda z) \le \varrho_{\varepsilon_n}(z), \quad z \in \mathbb{C}^N,\ \lambda \in \mathbb{C},\ |\lambda| \le 1. \tag{6}$$

Therefore there exists a positive number ω_n such that for every $0 < \omega \le \omega_n$, the domains

$$D_{\varepsilon_n,-\omega} = \{z \in \mathbb{C}^N \mid \varrho_{\varepsilon_n}(z) < 1 - \omega\} \tag{7}$$

are strictly pseudoconvex, smoothly bounded, and balanced, $\overline{D_{\varepsilon_n} \setminus U_n} \subset D_{\varepsilon_n,-\omega}$, and (as in (3)) $\partial D_{\varepsilon_n,-\omega}$ is transversal to every (real) straight line passing through zero.

Now fix $n \in \mathbb{N}$, and call $D_{\varepsilon_n} = G$, $\varrho_{\varepsilon_n} = \varrho$, $D_{\varepsilon_n,-\omega} = G_{-\omega}$, $U_n = U$. It is well known that every strictly pseudoconvex domain is locally strictly convex with respect to convenient holomorphic coordinates in some neighborhood of a given point of its boundary. Examining the proof of this result (see e.g. [6], Lemma 3.2.3), and shrinking U once more, we conclude that the following holds:

PROPOSITION 3. *For every $x \in \partial G$ there exist neighborhoods Z_x, U_x, V_x, and W_x of x with $Z_x \Subset U_x \Subset V_x \Subset W_x$, strictly convex domains P_x, T_x, S_x, and R_x in $\mathbb{C}^N$ such that $P_x \Subset T_x \Subset S_x \Subset R_x$, and a biholomorphic mapping $\phi_x : W_x \to R_x$ such that*

(8) $\varrho_n \circ \phi_x^{-1}$ *is a strictly convex smooth function in R_x*,

$\phi_x(Z_x) = P_x$, $\phi_x(U_x) = T_x$, $\phi_x(V_x) = S_x$, *and*

$$\overline{U} \subset \bigcup_{x\in\partial G} Z_x. \tag{9}$$

Now let $x \in \partial G$ be fixed. By a small perturbation of the function ϱ_n we can obtain a strictly pseudoconvex domain $B \subset \mathbb{C}^N$ with smooth boundary such that $B \subset G$, $G \cap U_x \subset B$, $(\partial G \setminus V_x) \cap \overline{B} = \emptyset$, $\phi_x(B \cap W_x)$ is convex, there exists η with $0 < \eta < \omega_n$ such that $\overline{G}_{-\eta} \subset B$, and B is star-shaped. (Note that since the deformation of G is performed only near $x \in \partial G$, the domain B need not be balanced (although G is). Therefore B is a star-shaped domain of holomorphy. It follows from [9] that every function holomorphic in B can be approximated uniformly on compact subsets of B by polynomials. In particular,

(10) every function holomorphic in B can be approximated uniformly on compact subsets of B by functions holomorphic in the whole domain D.

Also, there exists θ with $0 < \theta < \eta$ such that $((\overline{G \setminus G_{-\theta}}) \setminus V_x) \cap B = \emptyset$, and hence

$$B \cap (W_x \setminus V_x) \subset W_x \cap G_{-\theta}. \tag{11}$$

Assume now that K and L are compact subsets of $\phi_x((G \setminus G_{-\theta}) \cap U_x)$ such that

(12) K is a subset of a real $(2N-1)$-dimensional hyperplane Π of $\mathbb{C}^N$, and $\phi_x(G_{-\theta} \cap W_x)$ and L lie on one side of Π.

(This can happen, since by (8), $\phi_x(G_{-\theta} \cap W_x)$ is convex in $\mathbb{C}^N$.) The hyperplane Π has the form

$$\Pi = \{z \in \mathbb{C}^N \mid \mathrm{Re}\langle z - cz_0, z_0\rangle_{\mathbb{C}} = 0\}$$

with some $z_0 \in \mathbb{C}^N$, $\|z_0\| = 1$, and $c > 0$. (Here $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes the standard complex euclidean scalar product in $\mathbb{C}^N$.) The function

$$h(z) = b\exp(a\langle z - cz_0, z_0\rangle_{\mathbb{C}}), \quad z \in \mathbb{C}^N,\ a, b > 0,$$

is such that $|h|_\Pi| \equiv b$, and $|h(z)| < b$ for those $z \in \mathbb{C}^N$ which lie on the same side of the hyperplane Π as the point 0.

Choosing conveniently a and b, and using (12), we may assume that

$$\|h\|_{\phi_x(G_{-\theta}\cap W_x)\cup L} \le m', \tag{13}$$

and

$$\inf_K |h| \ge M', \tag{14}$$

where $0 < m' < M'$ are given constants. Let ψ be a smooth function in $\mathbb{C}^N$ with $0 \le \psi \le 1$, $\psi|_{V_x} \equiv 1$, and $\psi|_{\mathbb{C}^N\setminus W_x} \equiv 0$. Consider the function g defined as $\psi(h\circ\phi_x)$ in W_x and 0 in $\mathbb{C}^N \setminus W_x$. Then g is smooth in $\mathbb{C}^N$. The form $\overline{\partial}g$ is $\overline{\partial}$-closed in $\mathbb{C}^N$, and

$$\operatorname{supp}\overline{\partial}g \subset W_x \setminus V_x. \tag{15}$$

Moreover, by (11), (13), and (15),

$$\begin{aligned}\|\overline{\partial}g\|_{\overline{B}} = \|(h\circ\phi_{n,x})\overline{\partial}\psi\|_{\overline{B}} &\le \|h\|_{\phi_x(B\cap(W_x\setminus V_x))}\|\overline{\partial}\psi\|_{\mathbb{C}^N}\\ &\le \|h\|_{\phi_x(W_x\cap G_{-\theta})}\|\overline{\partial}\psi\|_{\mathbb{C}^N} \le m'\|\overline{\partial}\psi\|_{\mathbb{C}^N}.\end{aligned}$$

By [3] or [7] there exists $c > 0$ (depending only on B) and a function $v \in \mathcal{C}^\infty(\overline{B})$ such that $\overline{\partial}v = \overline{\partial}g$ in $\overline{B}$, and

$$\|v\|_{\overline{B}} \le cm'\|\overline{\partial}\psi\|_{\mathbb{C}^N}.$$

Then $f = g - v$ is holomorphic in B, and

$$\|f\|_{G_{-\eta}\cup\phi_x^{-1}(L)} \le \|h\|_{\phi_x(G_{-\theta}\cap W_x)\cup L} + \|v\|_{\overline{B}} \le m' + cm'\|\overline{\partial}\psi\|_{\mathbb{C}^N},$$

and by (14),

$$\inf_{\phi_x^{-1}(K)} |f| \ge \inf_K |h| - \|v\|_B \ge M' - cm'\|\overline{\partial}\psi\|_{\mathbb{C}^N}.$$

Hence, by choosing M' and m' conveniently, we obtain

$$\inf_{\phi_x^{-1}(K)} |f| \ge M \tag{16}$$

and

$$\|f\|_{G_{-\eta}\cup\phi_x^{-1}(L)} < m, \tag{17}$$

where $0 < m < M$ are given positive numbers.

By (10) there exists a function k holomorphic in D such that

$$\inf\{|k(z)| \mid z \in \phi_x^{-1}(K)\} > M \tag{18}$$

and

$$\|k\|_{G_{-\eta}\cup\phi_x^{-1}(L)} < m. \tag{19}$$

We now return to the previous notations, i.e. we have the sequence $\{D_{\varepsilon_n}\}_{n=1}^\infty$ of balanced, strictly pseudoconvex, and smoothly bounded domains from (5), defined by the smooth and strictly plurisubharmonic functions ϱ_{ε_n} satisfying (6), and the numbers ω_n for which (7) holds. To simplify notations, we write $D_{\varepsilon_n}=D_n$ and $D_{\varepsilon_n,-\omega}=D_{n,-\omega}$. Let n be fixed. Since $\overline{U}_n$ is compact, by (9) there exist a finite number of points $x_{n,1},\ldots,x_{n,i_n}\in\partial D_n$ such that $\overline{U}_n\subset Z_{n,x_{n,1}}\cup\ldots\cup Z_{n,x_{n,i_n}}$. Let S be the unit sphere in $\mathbb{C}^N$, $S=\{w\in\mathbb{C}^N\mid \|w\|=1\}$. Note that for every $w\in S$, the half-line $I_w=\{tw\mid 0\le t<\infty\}$ intersects every $\partial D_{n,-\omega}$, $0<\omega\le\omega_n$. Hence

(20) every I_w intersects some $Z_{n,x_{n,j}}$.

Moreover, by Proposition 3, every such half-line I_w intersects every $\partial D_{n,-\omega}$, $0<\omega\le\omega_n$, transversally. By (8), for every $j=1,\ldots,i_n$, the sets

$$\phi_{n,x_{n,j}}(D_{n,-\omega}\cap W_{n,x_{n,j}})$$

are convex in $\mathbb{C}^N$ for every $0<\omega\le\omega_n$, and the lines $\phi_{n,x_{n,j}}(I_w)$ intersect $\phi_{n,x_{n,j}}(\partial D_{n,-\omega}\cap W_{n,x_{n,j}})$ transversally (for those w and ω for which the intersection is not empty). Hence it is rather easy to find for each $j=1,\ldots,i_n$ a finite number of real $(2N-1)$-dimensional hyperplanes $\Theta_{n,j,1},\ldots,\Theta_{n,j,s_{n,j}}$ of $\mathbb{C}^N$, a family $K_{n,j,1},\ldots,K_{n,j,s_{n,j}}$ of compact subsets of $\mathbb{C}^N$, and a number $\omega_{n,j}$ with $0<\omega_{n,j}<\omega_n$, as well as a number $\omega_{n,0}$, $0<\omega_{n,0}<\omega_n$, such that:

- $K_{n,j,l}\subset\Theta_{n,j,l}\cap T_{n,x_{n,j}}$, $\quad l=1,\ldots,s_{n,j}$.

(21) If for some $w\in S$, the half-line I_w intersects $Z_{n,x_{n,j}}$, then $\phi_{n,x_{n,j}}(I_w)$ (which is contained in $W_{n,x_{n,j}}$) intersects some $K_{n,j,l}$.

- For every $l=1,\ldots,s_{n,j}$, the sets $\overline{\phi_{n,x_{n,j}}(W_{n,x_{n,j}}\cap D_{n,-\omega_{n,j}})}$ and $K_{n,j,1},\ldots,K_{n,j,l-1}$ lie on the same side of $\Theta_{n,j,l}$ as the point zero,

(we set $K_{n,j,0}=\emptyset$),

$$\omega_{n,0}>\omega_{n,1}>\ldots>\omega_{n,i_n},\quad\text{so}\quad D_{n,-\omega_{n,1}}\Subset\ldots\Subset D_{n,-\omega_{n,i_n}},$$

and

$$K_{n,j,l}\subset\phi_{n,x_{n,j}}(W_{n,x_{n,j}}\cap(D_{n,-\omega_{n,j}}\setminus\overline{D}_{n,-\omega_{n,j-1}})),\qquad j=1,\ldots,i_n,\ l=1,\ldots,s_{n,j}.$$

Now we repeat essentially the construction from [4]. We order the sets $K_{n,j,l}$ into the sequence

(22) $$\{K_{1,1,1},K_{1,1,2},\ldots,K_{1,1,s_{1,1}},K_{1,2,1},\ldots,K_{1,2,s_{1,2}},\ldots,\\ K_{1,i_1,1},\ldots,K_{1,i_1,s_{1,i_1}},K_{2,1,1},\ldots,K_{2,1,s_{2,1}},\ldots\}=:\{K_1,K_2,\ldots\}.$$

Every subspace Π of $\mathbb{C}^N$ consists of real half-lines I_w, and, by (20) and (21),

(23) for every $w\in S$, the half-line I_w intersects infinitely many sets of the form $\phi^{-1}_{n,x_{n,j}}(K_{n,j,l})$.

To each $K_{n,j,l}=K_s$ we attach a function $f_{n,j,l}=f_s$ with the properties which we now describe inductively. By (16) and (17), and by the positivity of δ, there exists a function f_1 holomorphic in D such that

$$\inf\{|f_1(z)| \mid z \in \phi^{-1}_{1,x_{1,1}}(K_1)\} \geq 1 \quad \text{and} \quad \|\delta f_1\|_{\overline{D}_{1,-\omega_{1,1}}} \leq 2^{-1}.$$

Suppose that the functions $f_1, \ldots, f_r$ are already chosen. Then we have

$$K_{r+1} = K_{n_{r+1},j_{r+1},l_{r+1}}$$

for uniquely determined n_{r+1}, j_{r+1} with $1 \leq j_{r+1} \leq i_{n_{r+1}}$, and l_{r+1} with $1 \leq l_{r+1} \leq s_{n_{r+1},j_{r+1}}$. Moreover,

$$\overline{D}_{n_{r+1},-\omega_{n_{r+1},j}} \subset D_{n_{r+1},-\omega_{n_{r+1},j_{r+1}}}, \qquad j = 1, \ldots, j_{r+1} - 1, \qquad \text{if } j_{r+1} > 1,$$

or

$$\overline{D}_{n_r,-\omega_{n_r,i_{n_r}}} \subset D_{n_{r+1},-\omega_{n_{r+1},1}} \qquad \text{if } j_{r+1} = 1,$$

and the set

$$\phi_{n_{r+1},x_{n_{r+1},j_{r+1}}}(D_{n_{r+1},-\varepsilon_{n_{r+1},j_{r+1}}} \cap W_{n_{r+1},j_{r+1}}) \cup K_{n_{r+1},j_{r+1},1} \cup \ldots \cup K_{n_{r+1},j_{r+1},l_{r+1}-1}$$

lies on the same side of the hyperplane $\Theta_{n_{r+1},j_{r+1},l_{r+1}}$ as the point zero. By (17)–(19) and the fact that δ is positive, there exists a function $f_{r+1} = f_{n_{r+1},j_{r+1},l_{r+1}+1}$, holomorphic in D, such that

$$\inf\{|\delta^{r+1} f_{r+1}(z)| \mid z \in \phi^{-1}_{n_{r+1},x_{n_{r+1},j_{r+1}}}(K_{r+1})\} \geq (r+1) + \sum_{p=1}^{r} \|\delta^{r+1} f_p\|_{K_{r+1}} + 1, \tag{24}$$

and if we define

$$L_r = D_{n_{r+1},-\omega_{n_{r+1},j_{r+1}}} \cup \phi^{-1}_{n_{r+1},x_{n_{r+1},j_{r+1}}}(K_{n_{r+1},j_{r+1},1} \cup \ldots \cup K_{n_{r+1},j_{r+1},l_{r+1}-1}),$$

then

$$\|f_{r+1}\|_{L_r} \; (= \|f_{n_{r+1},j_{r+1},l_{r+1}+1}\|_{L_r}) \leq 2^{-(r+1)}, \tag{25}$$

$$\|\delta^p f_{r+1}\|_{L_r} \; (= \|\delta^p f_{n_{r+1},j_{r+1},l_{r+1}+1}\|_{L_r}) \leq 2^{-(r+1)}, \qquad p = 1, \ldots, r. \tag{26}$$

Set

$$f(z) = \sum_{r=1}^{\infty} f_r(z), \qquad z \in D.$$

By (25), the function f is well defined and holomorphic in D. By (20), (23), (24), and (26), for every $w \in S$ there exists a sequence $\{z_r\}_{r=1}^{\infty}$ of points of $I_w \cap D$ such that for infinitely many r,

$$|\delta^r(z_r) f(z_r)| \geq r. \tag{27}$$

Therefore f is not in $\mathcal{O}(\delta|_{\Pi \cap D})$ for any subspace Π of $\mathbb{C}^N$. This ends the proof of Theorem 1.

Given a domain G in $\mathbb{C}^N$ and a number p with $1 \le p < \infty$, we denote by $L^pH(G)$ the space of all functions holomorphic in G such that

$$\int_G |f(z)|^p \, dm(z) < \infty$$

(m denotes here the $2N$-dimensional Lebesgue measure in $\mathbb{C}^N$). If G is a domain in a complex subspace M of $\mathbb{C}^N$, the space $L^pH(G)$ can be defined similarly, with m being the Lebesgue measure on M.

In the same way as Theorem 1 we can prove the following theorem on functions from the space L^pH (for the case of the ball, see [4], Theorem 1):

THEOREM 2. *Let D be a balanced domain of holomorphy in $\mathbb{C}^N$. Then there exists a function f, holomorphic in D, such that for every positive-dimensional subspace Π of $\mathbb{C}^N$ and for every p with $1 \le p < \infty$, $f|_{D\cap\Pi} \notin L^pH(D \cap \Pi)$.*

Proof. It is well known that if G is a domain in $\mathbb{C}^M$, $1 \le p < \infty$, and $f \in L^pH(G)$ then for every $z_0 \in G$,

$$|f(z_0)| \le \frac{M^{M/p}}{(\pi \operatorname{dist}(z_0, \partial G)^2)^{M/p}} \|f\|_{G,p},$$

where $\|f\|_{G,p}$ denotes the L^p-norm of f in G and $\operatorname{dist}(z_0, \partial G)$ is the Euclidean distance of z_0 to ∂G. For z_0 sufficiently close to ∂G, we have $\operatorname{dist}(z_0, \partial G) < 1$. Hence for $1 \le p < \infty$,

$$1 \le \frac{1}{\operatorname{dist}(z_0, \partial G)^{2M/p}} \le \frac{1}{\operatorname{dist}(z_0, \partial G)^{2M}}.$$

Therefore, for all $z_0 \in G$, and for every $1 \le p < \infty$, we have

$$\frac{1}{\operatorname{dist}(z_0, \partial G)^{2M/p}} \le 1 + \frac{1}{\operatorname{dist}(z_0, \partial G)^{2M}}.$$

Moreover, there exists $c > 0$ such that for all $L = 1, \ldots, N$, and every $1 \le p < \infty$,

$$(L/\pi)^{L/p} \le c.$$

Consider the construction of the function f from the proof of Theorem 1. We now require that the function f, constructed as before, satisfies the inequality

$$|f(z)| \ge \frac{r}{\operatorname{dist}(z_0, \partial D)^{2N}} + 1 \tag{28}$$

for all $z \in K_r$ instead of (27). (Here the sets K_r are defined as in (22)). It follows from the above considerations and from (28) that the function f obtained in this way is holomorphic in D, and for every subspace Π of $\mathbb{C}^N$ and every $1 \le p < \infty$, $f \notin L^p(\Pi \cap D)$. This ends the proof. ∎

Now let D be a balanced domain of holomorphy in $\mathbb{C}^N$, as before. Then in particular Theorem 2 holds for D and $p = 2$. Moreover, since D is balanced, every function f holomorphic in D can be developed into a series of homogeneous polynomials,

$$f(z) = \sum_{s=0}^{\infty} Q_s(z),$$

where every Q_s is a homogeneous polynomial of degree s, $s = 0, 1, \ldots$ In [10], Thm. 1, Wojtaszczyk constructed explicitly a sequence $\{p_n\}_{n=1}^{\infty}$ of homogeneous polynomials of degree n in the unit ball B in $\mathbb{C}^N$ such that the function

$$f(z) := \sum_n n^{\ln n} p_n(z)$$

is holomorphic in B, and for each hyperplane $\Pi \subset \mathbb{C}^N$ and any $p > 0$,

$$\int_{\Pi \cap B} |f(z)|^p \, dm_{\Pi}(z) = \infty$$

(m_{Π} is the Lebesgue measure on Π). It would be interesting to know whether the construction in the present note, given for an arbitrary balanced domain of holomorphy, can be done more explicitly, e.g. as in [10].

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Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: jakobcza@im.uj.edu.pl

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