Holomorphic functions of fast growth on submanifolds of the domain

by Piotr Jakóbczak (Kraków)

Abstract. We construct a function $f$ holomorphic in a balanced domain $D$ in $\mathbb{C}^N$ such that for every positive-dimensional subspace $\Pi$ of $\mathbb{C}^N$, and for every $p$ with $1 \leq p < \infty$, $f|\Pi \cap D$ is not $L^p$-integrable on $\Pi \cap D$.

1. Introduction. Let $D$ be an open set in $\mathbb{C}^N$, and let $F$ be some class of complex-valued functions in $D$ which are holomorphic in $D$ and satisfy some other conditions there. Given an affine subspace $M$ of positive dimension in $\mathbb{C}^N$, the problem is to determine what further properties (besides being holomorphic) the functions from the class $F$ have when restricted to the slice $M \cap D$. This problem was studied in many situations by several authors; see e.g. [2], [5], [8], [9], [11].

In [4] we have shown that there exists a function $f$ holomorphic in the unit ball $B$ in $\mathbb{C}^N$ such that for every positive-dimensional subspace $\Pi$ of $\mathbb{C}^N$, $f|\Pi \cap B$ is not $L^2$-integrable in $\Pi \cap B$. The proof consists of construction of a function $f$ with sufficiently fast growth near the boundary of each set of the form $\Pi \cap B$, and the use of the well-known estimates relating the growth near the boundary and the $L^2$-norm of a holomorphic function. (See also [10] for a much more explicit proof of this result.)

In the present note we carry out the construction from [4] for the more general situation of domains which are balanced domains of holomorphy, i.e. domains of holomorphy such that for every $z = (z_1, \ldots, z_N) \in D$ and every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, the point $\lambda z = (\lambda z_1, \ldots, \lambda z_N)$ also belongs to $D$. We obtain holomorphic functions with prescribed fast growth near the boundary of such domains; then we apply our construction in order to obtain functions which are holomorphic and not integrable on linear slices of the domain, or which are not in $O(\delta)$ on any such slice, where $O(\delta)$ denotes the
space of functions of δ-tempered growth, and δ is a given weight function (see e.g. [1]; the precise definition of $O(\delta)$ will be recalled later).

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2. A holomorphic function with prescribed growth on slices.

Let $D$ be a balanced domain of holomorphy in $\mathbb{C}^N$. Then there exists a strictly plurisubharmonic smooth exhaustion function $\varphi$ in $D$, i.e. a smooth function $\varphi$ which is strictly plurisubharmonic in $D$ and for every real $c$ the set $\{z \in D \mid \varphi(z) < c\}$ is relatively compact in $D$. For further use we need the existence of a sequence $\{D_n\}_{n=1}^{\infty}$ of strictly pseudoconvex, smoothly bounded, balanced domains which exhaust $D$ and every straight line in $\mathbb{C}^N$ passing through zero intersects the boundary $\partial D_n$ of every domain $D_n$ transversally. It seems that the existence of such a sequence is well known; the proof of the following proposition was suggested to us by M. Jarnicki, Ch. Kiselman and P. Pflug.

**Proposition 1.** Let $D$ be a balanced domain of holomorphy in $\mathbb{C}^N$. Then there exists $\varepsilon_0 > 0$ and a family $\{D_{\varepsilon}\}_{0 < \varepsilon \leq \varepsilon_0}$ of strictly pseudoconvex, smoothly bounded, balanced domains such that $\bigcup_{0 < \varepsilon \leq \varepsilon_0} D_{\varepsilon} = D$, $\overline{D}_{\varepsilon} \subset D_{\varepsilon'}$ for $0 < \varepsilon' < \varepsilon \leq \varepsilon_0$, and for every $\varepsilon$, every (real) straight line passing through zero in $\mathbb{C}^N$ intersects $\partial D_{\varepsilon}$ transversally.

**Proof.** Let $h$ be the Minkowski functional for $D$. Since $D$ is a domain of holomorphy and is balanced, $h$ is plurisubharmonic in $\mathbb{C}^N$, and $h(\lambda z) = |\lambda|h(z)$ for every $z \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$. For $\varepsilon > 0$, denote by $h_{\varepsilon}$ the regularization

$$h_{\varepsilon}(z) = \int_{\mathbb{C}^N} h(z - \varepsilon y) \phi(y) \, dm(y),$$

where $\phi$ is a smooth function in $\mathbb{C}^N$, supp $\phi$ is the unit ball, $\phi(y) = \phi(|y_1|, \ldots, |y_N|)$ for every $y = (y_1, \ldots, y_N) \in \mathbb{C}^N$, and $\int_{\mathbb{C}^N} \phi(y) \, dm(y) = 1$. (Here $m$ denotes the usual Lebesgue measure in $\mathbb{C}^N$.) It is well known that $h_{\varepsilon}$ is smooth and plurisubharmonic in $\mathbb{C}^N$, for each $z \in \mathbb{C}^N$, $h_{\varepsilon}(z)$ tends decreasingly to $h(z)$ as $\varepsilon$ decreases to zero, and $h_{\varepsilon}(e^{it} z) = h_{\varepsilon}(z)$, $z \in \mathbb{C}^N$, $t \in \mathbb{R}$. Since $h(0) = 0$, there exists $\varepsilon_0 > 0$ so small that $h_{\varepsilon_0}(0) < 1$ (and hence $h_{\varepsilon}(0) < 1$ for all $0 < \varepsilon \leq \varepsilon_0$). For $0 < \varepsilon \leq \varepsilon_0$, set

$$g_{\varepsilon}(z) = h_{\varepsilon}(z) + \varepsilon\|z\|^2.$$

Then $g_{\varepsilon}$ is a smooth and strictly plurisubharmonic function in $\mathbb{C}^N$. Let $D_{\varepsilon} = \{z \in \mathbb{C}^N \mid g_{\varepsilon}(z) < 1\}$. Then $0 \not\in \partial D_{\varepsilon}$ (because $g_{\varepsilon}(0) < 1$), $\overline{D}_{\varepsilon'} \subset D_{\varepsilon''}$ for $0 < \varepsilon'' < \varepsilon'$, and the domains $D_{\varepsilon}$ tend increasingly to $D$ as $\varepsilon$ decreases to zero; moreover, every domain $D_{\varepsilon}$ is pseudoconvex. Using the maximum
principle for subharmonic functions and the fact that $h_\varepsilon(e^{it}z) = h_\varepsilon(z)$ for $z \in \mathbb{C}^N$ and $t \in \mathbb{R}$, we have for $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ and $z \in \mathbb{C}^N$,

\begin{equation}
\label{eq:1}
h_\varepsilon(\lambda z) \leq \max_{t \in \mathbb{R}} h_\varepsilon(e^{it}z) = h_\varepsilon(z),
\end{equation}

and hence

$$g_\varepsilon(\lambda z) \leq g_\varepsilon(z), \quad z \in \mathbb{C}^N, \quad \lambda \in \mathbb{C}, \quad |\lambda| \leq 1.$$ 

Hence every domain $D_\varepsilon$ is balanced.

Now fix $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$, and $z \in \partial D_\varepsilon$. By (1) the function

$$\phi : [0, \infty) \ni t \mapsto h_\varepsilon(tz)$$

is non-decreasing. Denote by $\psi$ the function

$$\psi : [0, \infty) \ni t \mapsto g_\varepsilon(tz).$$

Then

$$\psi'(t) = \langle \text{grad} \ g_\varepsilon(tz), z \rangle_\mathbb{R} = \phi'(t) + 2\varepsilon t \|z\|^2,$$

where $\langle \cdot, \cdot \rangle_\mathbb{R}$ denotes the standard real scalar product in $\mathbb{C}^N = \mathbb{R}^{2N}$. Further,

\begin{equation}
\langle \text{grad} \ g_\varepsilon(z), z \rangle_\mathbb{R} = \phi'(1) + 2\varepsilon \|z\|^2 > 0
\end{equation}

(here we use the fact that $0 \notin \partial D_\varepsilon$). It follows form (2) that $\partial D_\varepsilon$ is smooth (and so $D_\varepsilon$ is strictly pseudoconvex), and that

$$\partial D_\varepsilon$$

is transversal to every (real) straight line passing through zero.

This ends the proof. ■

Fix $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$. It is well known that for a given compact subset $K$ of $\mathbb{C}^N$, and for $\varepsilon'$ sufficiently close to $\varepsilon$, the regularizations $h_{\varepsilon'}$ are arbitrarily close to $h_\varepsilon$ on $K$. Therefore the same is true for the functions $g_{\varepsilon'}$ and $g_\varepsilon$. Hence, given an arbitrary neighborhood $U$ of $\overline{D}_\varepsilon$, there exists $\varepsilon' < \varepsilon$ such that $D_{\varepsilon'} \subset U$. Suppose now that $f$ is a function holomorphic in some neighborhood $U$ of $\overline{D}_\varepsilon$, and fix $D_{\varepsilon'} \subset U$ as above. Then $f$ is holomorphic in $D_{\varepsilon'}$. Since $D_{\varepsilon'}$ is a balanced domain of holomorphy, there exists a series $\sum_{s=0}^{\infty} Q_s$ of homogeneous polynomials which converges to $f$ uniformly on compact subsets of $D_{\varepsilon'}$; in particular, the convergence is uniform on $\overline{D}_\varepsilon$.

This yields the following proposition:

**Proposition 2.** Let the domain $D$ and the family $\{D_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ be as in Proposition 1. Then given $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$, every function holomorphic in a neighborhood of $\overline{D}_\varepsilon$ can be uniformly approximated on $\overline{D}_\varepsilon$ by functions which are holomorphic in the whole domain $D$.

In the sequel, given $K \subset \mathbb{C}^N$ and $f \in \mathcal{C}(K)$, we denote by $\|f\|_K$ the usual supremum norm on $K$.

Suppose now that $\delta$ is a positive, bounded and continuous function in a domain $G$ in $\mathbb{C}^N$. Denote by $O(\delta)$ the space of all functions holomorphic in
$G$ such that there exists a positive integer $k$ with
$$\sup\{|\delta^k(z)f(z)| \mid z \in G\} < \infty.$$ If moreover $\delta$ satisfies the conditions:

(i) $|z|\delta$ is bounded on $\mathbb{C}^N$,

(ii) $|\delta(z) - \delta(z')| \leq |z - z'|$ for all $z, z' \in \mathbb{C}^N$,

then it is called a \textit{weight function} (see [1]). The theory of functions from the space $\mathcal{O}(\delta)$ was investigated by several authors (see e.g. [1]).

We will prove the following theorem on the existence of holomorphic functions with bad boundary behavior on submanifolds:

**Theorem 1.** Let $D$ be a balanced domain of holomorphy in $\mathbb{C}^N$, and $\delta$ a positive and continuous function in $D$. Then there exists a function $f$ holomorphic in $D$ such that for every positive-dimensional subspace $\Pi$ of $\mathbb{C}^N$, $f|_{\Pi \cap D} \notin \mathcal{O}(\delta|_{\Pi \cap D})$.

Let $\{D_{\varepsilon}\}_{0 < \varepsilon \leq \varepsilon_0}$ be the family of domains constructed in Proposition 1. Choose an arbitrary sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ with $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots$ and $\lim_{n \to \infty} \varepsilon_n = 0$. We have $\overline{D}_{\varepsilon_n} \subset D_{\varepsilon_{n+1}}$ for $n = 1, 2, \ldots$. For each $n$, choose a neighborhood $U_n$ of $\partial D_{\varepsilon_n}$ such that $\overline{U}_n \subset D$, and $\overline{U}_n \cap \overline{U}_k = \emptyset$ for $n \neq k$. It follows from the proof of Proposition 1 that for every $n$ and every $z \in \partial D_{\varepsilon_n}$, grad $\varrho_{\varepsilon_n}(z) \neq 0$ (where $\varrho_{\varepsilon_n}$ is a defining function for $D_{\varepsilon_n}$, obtained in the proof of Proposition 1). Shrinking the neighborhoods $U_n$ if necessary we may assume that

(4) for every $n$ and for every $z \in U_n$, grad $\varrho_{\varepsilon_n}(z) \neq 0$.

Moreover, according to the proof of Proposition 1, we have

(5) $D_{\varepsilon_n} = \{z \in \mathbb{C}^N \mid \varrho_{\varepsilon_n}(z) < 1\},$

and $\varrho_{\varepsilon_n}$ is smooth and strictly plurisubharmonic in $\mathbb{C}^N$, and satisfies the condition

(6) $\varrho_{\varepsilon_n}(\lambda z) \leq \varrho_{\varepsilon_n}(z), \quad z \in \mathbb{C}^N, \quad \lambda \in \mathbb{C}, \quad |\lambda| \leq 1.$

Therefore there exists a positive number $\omega_n$ such that for every $0 < \omega \leq \omega_n$, the domains

(7) $D_{\varepsilon_n, -\omega} = \{z \in \mathbb{C}^N \mid \varrho_{\varepsilon_n}(z) < 1 - \omega\}$

are strictly pseudoconvex, smoothly bounded, and balanced, $\overline{D}_{\varepsilon_n} \setminus \overline{U}_n \subset D_{\varepsilon_n, -\omega}$, and (as in (3)) $\partial D_{\varepsilon_n, -\omega}$ is transversal to every (real) straight line passing through zero.

Now fix $n \in \mathbb{N}$, and call $D_{\varepsilon_n} = G$, $\varrho_{\varepsilon_n} = \varrho$, $D_{\varepsilon_n, -\omega} = G_{-\omega}$, $U_n = U$.

It is well known that every strictly pseudoconvex domain is locally strictly convex with respect to convenient holomorphic coordinates in some neighborhood of a given point of its boundary. Examining the proof of this result
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(see e.g. [6], Lemma 3.2.3), and shrinking $U$ once more, we conclude that the following holds:

**Proposition 3.** For every $x \in \partial G$ there exist neighborhoods $Z_x, U_x, V_x$, and $W_x$ of $x$ with $Z_x \Subset U_x \Subset V_x \Subset W_x$, strictly convex domains $P_x, T_x, S_x$, and $R_x$ in $\mathbb{C}^N$ such that $P_x \Subset T_x \Subset S_x \Subset R_x$, and a biholomorphic mapping $\phi_x : W_x \to R_x$ such that

\[(8)\quad \varrho_n \circ \phi_x^{-1} \quad \text{is a strictly convex smooth function in } R_x,\]

\[\phi_x(Z_x) = P_x, \quad \phi_x(U_x) = T_x, \quad \phi_x(V_x) = S_x, \quad \text{and}\]

\[(9)\quad \overline{U} \subset \bigcup_{x \in \partial G} Z_x.\]

Now let $x \in \partial G$ be fixed. By a small perturbation of the function $\varrho_n$ we can obtain a strictly pseudoconvex domain $B \subset \mathbb{C}^N$ with smooth boundary such that $B \subset G$, $G \cap U_x \subset B$, $(\partial G \setminus V_x) \cap \overline{B} = \emptyset$, $\phi_x(B \cap W_x)$ is convex, there exists $\eta$ with $0 < \eta < \omega_n$ such that $G_{-\eta} \subset B$, and $B$ is star-shaped. (Note that since the deformation of $G$ is performed only near $x \in \partial G$, the domain $B$ need not be balanced (although $G$ is). Therefore $B$ is a star-shaped domain of holomorphy. It follows from [9] that every function holomorphic in $B$ can be approximated uniformly on compact subsets of $B$ by polynomials. In particular,

\[(10)\quad \text{every function holomorphic in } B \text{ can be approximated uniformly on compact subsets of } B \text{ by functions holomorphic in the whole domain } D.\]

Also, there exists $\theta$ with $0 < \theta < \eta$ such that $(G \setminus G_{-\theta}) \cap B = \emptyset$, and hence

\[(11)\quad B \cap (W_x \setminus V_x) \subset W_x \cap G_{-\theta}.\]

Assume now that $K$ and $L$ are compact subsets of $\phi_x((G \setminus G_{-\theta}) \cap U_x)$ such that

\[(12)\quad \text{$K$ is a subset of a real $(2N - 1)$-dimensional hyperplane $\Pi$ of } \mathbb{C}^N, \text{ and } \phi_x(G_{-\theta} \cap W_x) \text{ and } L \text{ lie on one side of } \Pi.\]

(This can happen, since by (8), $\phi_x(G_{-\theta} \cap W_x)$ is convex in $\mathbb{C}^N$.) The hyperplane $\Pi$ has the form

\[\Pi = \{ z \in \mathbb{C}^N \mid \text{Re}(z - cz_0, z_0) = 0 \}\]

with some $z_0 \in \mathbb{C}^N$, $\|z_0\| = 1$, and $c > 0$. (Here $\langle \cdot, \cdot \rangle$ denotes the standard complex euclidean scalar product in $\mathbb{C}^N$.) The function

\[h(z) = b \exp(a(z - cz_0, z_0)) \quad z \in \mathbb{C}^N, \quad a, b > 0,\]

is such that $|h| \equiv b$, and $|h(z)| < b$ for those $z \in \mathbb{C}^N$ which lie on the same side of the hyperplane $\Pi$ as the point 0.
Choosing conveniently $a$ and $b$, and using (12), we may assume that
\begin{equation}
\|h\|_{\phi_x(G_x \cap W_x) \cup L} \leq m',
\end{equation}
and
\begin{equation}
\inf_K |h| \geq M',
\end{equation}
where $0 < m' < M'$ are given constants. Let $\psi$ be a smooth function in $\mathbb{C}^N$ with $0 \leq \psi \leq 1$, $\psi|_{W_x} \equiv 1$, and $\psi|_{\mathbb{C}^N \setminus W_x} \equiv 0$. Consider the function $g$ defined as $\psi(h \circ \phi_x)$ in $W_x$ and $0$ in $\mathbb{C}^N \setminus W_x$. Then $g$ is smooth in $\mathbb{C}^N$. The form $\overline{\partial} g$ is $\overline{\partial}$-closed in $\mathbb{C}^N$, and
\begin{equation}
\text{supp} \overline{\partial} g \subset W_x \setminus V_x.
\end{equation}
Moreover, by (11), (13), and (15),
\[\|\overline{\partial} g\|_B = \|(h \circ \phi_{n,x})\overline{\partial}\psi\|_B \leq \|h\|_{\phi_x(B \cap (W_x \setminus V_x))}\|\overline{\partial}\psi\|_{\mathbb{C}^N} \leq m'\|\overline{\partial}\psi\|_{\mathbb{C}^N}.\]
By [3] or [7] there exists $c > 0$ (depending only on $B$) and a function $v \in C^\infty(B)$ such that $\partial v = \overline{\partial} g$ in $B$, and
\[\|v\|_B \leq cm'\|\overline{\partial}\psi\|_{\mathbb{C}^N}.
\]
Then $f = g - v$ is holomorphic in $B$, and
\[\|f\|_{G_\eta \cup \phi_x^{-1}(L)} \leq \|h\|_{\phi_x(G_x \cap W_x) \cup L} + \|v\|_B \leq m' + cm'\|\overline{\partial}\psi\|_{\mathbb{C}^N}, \]
and by (14),
\[\inf_{\phi_x^{-1}(K)} |f| \geq \inf_K |h| - \|v\|_B \geq M' - cm'\|\overline{\partial}\psi\|_{\mathbb{C}^N}.
\]
Hence, by choosing $M'$ and $m'$ conveniently, we obtain
\begin{equation}
\inf_{\phi_x^{-1}(K)} |f| \geq M
\end{equation}
and
\begin{equation}
\|f\|_{G_\eta \cup \phi_x^{-1}(L)} < m,
\end{equation}
where $0 < m < M$ are given positive numbers.

By (10) there exists a function $k$ holomorphic in $D$ such that
\begin{equation}
\text{inf}\{|k(z)| \mid z \in \phi_x^{-1}(K)\} > M
\end{equation}
and
\begin{equation}
\|k\|_{G_\eta \cup \phi_x^{-1}(L)} < m.
\end{equation}

We now return to the previous notations, i.e. we have the sequence $\{D_{\varepsilon_n}\}_{n=1}^\infty$ of balanced, strictly pseudoconvex, and smoothly bounded domains from (5), defined by the smooth and strictly plurisubharmonic functions $\varrho_{\varepsilon_n}$ satisfying (6), and the numbers $\omega_n$ for which (7) holds. To simplify
notations, we write $D_{z_n} = D_n$ and $D_{z_n - \omega} = D_{n - \omega}$. Let $n$ be fixed. Since $\mathbb{U}_n$ is compact, by (9) there exist a finite number of points $x_{n,1}, \ldots, x_{n,i_n} \in \partial D_n$ such that $\mathbb{U}_n \subset Z_{n,x_{n,1}} \cup \ldots \cup Z_{n,x_{n,i_n}}$. Let $S$ be the unit sphere in $\mathbb{C}^N$, $S = \{ w \in \mathbb{C}^N \mid \|w\| = 1 \}$. Note that for every $w \in S$, the half-line $I_w = \{ tw \mid 0 \leq t < \infty \}$ intersects every $\partial D_{n - \omega}$, $0 < \omega \leq \omega_n$. Hence

$$\text{(20)}$$

\[ \text{every } I_w \text{ intersects some } Z_{n,x_{n,j}}. \]

Moreover, by Proposition 3, every such half-line $I_w$ intersects every $\partial D_{n - \omega}$, $0 < \omega \leq \omega_n$, transversally. By (8), for every $j = 1, \ldots, i_n$, the sets

$$\phi_n(x_{n,j})(D_{n - \omega} \cap W_{n,x_{n,j}})$$

are convex in $\mathbb{C}^N$ for every $0 < \omega \leq \omega_n$, and the lines $\phi_n(x_{n,j})(I_w)$ intersect $\phi_n(x_{n,j})(\partial D_{n - \omega} \cap W_{n,x_{n,j}})$ transversally (for those $w$ and $\omega$ for which the intersection is not empty). Hence it is rather easy to find for each $j = 1, \ldots, i_n$, a finite number of real $(2N - 1)$-dimensional hyperplanes $\Theta_n, \ldots, \Theta_n, s_{n,j}$ of $\mathbb{C}^N$, a family $K_n, 1, \ldots, K_n, s_{n,j}$ of compact subsets of $\mathbb{C}^N$, and a number $\omega_{n,j}$ with $0 < \omega_{n,j} < \omega_n$, as well as a number $\omega_{n,0}$, $0 < \omega_{n,0} < \omega_n$, such that:

- $K_n, l \subset \Theta_n, l \cap T_n, x_{n,j}$, $l = 1, \ldots, s_{n,j}$.

$$\text{(21)}$$

\[ \text{If for some } w \in S, \text{ the half-line } I_w \text{ intersects } Z_n, x_{n,j}, \text{ then } \phi_n(x_{n,j})(I_w) \]

(which is contained in $W_{n,x_{n,j}}$) intersects some $K_n, l$.

- For every $l = 1, \ldots, s_{n,j}$, the sets $\phi_n(x_{n,j})(W_{n,x_{n,j}} \cap D_{n - \omega_{n,j}})$ and $K_n, 1, \ldots, K_n, l - 1$ lie on the same side of $\Theta_n, l$ as the point zero, (we set $K_n, 0 = \emptyset$),

$$\omega_{n,0} > \omega_{n,1} > \ldots > \omega_{n,i_n}, \text{ so } D_{n - \omega_{n,i_n}} = \ldots \cap D_{n - \omega_{n,0}},$$

and

$$K_n, l \subset \phi_n(x_{n,j})(W_{n,x_{n,j}} \cap (D_{n - \omega_{n,j}} \setminus D_{n - \omega_{n,j - 1}})), \quad j = 1, \ldots, i_n, l = 1, \ldots, s_{n,j}.$$
of $\delta$, there exists a function $f_1$ holomorphic in $D$ such that
\[
\inf\{|f_1(z)| \mid z \in \phi_{-1,1}^{-1}(K_1)\} \geq 1 \quad \text{and} \quad \|\delta f_1\|_{\overline{D},-\omega_{1,1}} \leq 2^{-1}.
\]
Suppose that the functions $f_1, \ldots, f_r$ are already chosen. Then we have
\[
K_{r+1} = K_{n_{r+1}, j_{r+1}, l_{r+1}}
\]
for uniquely determined $n_{r+1}, j_{r+1}$ with $1 \leq j_{r+1} \leq n_{r+1}$, and $l_{r+1}$ with $1 \leq l_{r+1} \leq s_{n_{r+1}, j_{r+1}}$. Moreover,
\[
\overline{D}_{n_{r+1}, -\omega_{n_{r+1}, j_{r+1}}} \subset D_{n_{r+1}, -\omega_{n_{r+1}, j_{r+1}}}, \quad j = 1, \ldots, j_{r+1} - 1, \quad \text{if } j_{r+1} > 1,
\]
and the set
\[
\phi_{n_{r+1}, x_{n_{r+1}, j_{r+1}}} (D_{n_{r+1}, -\epsilon_{n_{r+1}, j_{r+1}}} \cap W_{n_{r+1}, j_{r+1}}) \cup K_{n_{r+1}, j_{r+1}, 1} \cup \ldots \cup K_{n_{r+1}, j_{r+1}, l_{r+1}-1}
\]
lies on the same side of the hyperplane $\Theta_{n_{r+1}, j_{r+1}, l_{r+1}}$ as the point zero. By (17)–(19) and the fact that $\delta$ is positive, there exists a function $f_{r+1} = f_{n_{r+1}, j_{r+1}, l_{r+1}}$, holomorphic in $D$, such that
\[
(24) \quad \inf\{|\delta f_{r+1}(z)| \mid z \in \phi_{-1,1}^{-1}(K_{r+1})\} \geq (r + 1) + \sum_{p=1}^{r} \|\delta f_{r+1,p}\|_{K_{r+1}} + 1,
\]
and if we define
\[
L_r = D_{n_{r+1}, -\omega_{n_{r+1}, j_{r+1}}} \cup \phi_{-1,1}^{-1}(K_{n_{r+1}, j_{r+1}, 1} \cup \ldots \cup K_{n_{r+1}, j_{r+1}, l_{r+1}-1}),
\]
then
\[
(25) \quad \|f_{r+1}\|_{L_r} \left(= \|f_{n_{r+1}, j_{r+1}, l_{r+1}+1}\|_{L_r}\right) \leq 2^{-(r+1)},
\]
\[
(26) \quad \|\delta f_{r+1}\|_{L_p} \left(= \|\delta f_{n_{r+1}, j_{r+1}, l_{r+1}+1}\|_{L_p}\right) \leq 2^{-(r+1)}, \quad p = 1, \ldots, r.
\]
Set
\[
f(z) = \sum_{r=1}^{\infty} f_r(z), \quad z \in D.
\]
By (25), the function $f$ is well defined and holomorphic in $D$. By (20), (23), (24), and (26), for every $w \in S$ there exists a sequence $\{z_r\}_{r=1}^{\infty}$ of points of $I_w \cap D$ such that for infinitely many $r$,
\[
(27) \quad |\delta f(z_r)| \geq r.
\]
Therefore $f$ is not in $O(\delta|_{II \cap D})$ for any subspace $II$ of $\mathbb{C}^N$. This ends the proof of Theorem 1.
Given a domain \( G \) in \( \mathbb{C}^N \) and a number \( p \) with \( 1 \leq p < \infty \), we denote by \( L^p H(G) \) the space of all functions holomorphic in \( G \) such that
\[
\int |f(z)|^p \, dm(z) < \infty
\]
\((m\) denotes here the \( 2N \)-dimensional Lebesgue measure in \( \mathbb{C}^N \)). If \( G \) is a domain in a complex subspace \( M \) of \( \mathbb{C}^N \), the space \( L^p H(G) \) can be defined similarly, with \( m \) being the Lebesgue measure on \( M \).

In the same way as Theorem 1 we can prove the following theorem on functions from the space \( L^p H \) (for the case of the ball, see [4], Theorem 1):

**Theorem 2.** Let \( D \) be a balanced domain of holomorphy in \( \mathbb{C}^N \). Then there exists a function \( f \), holomorphic in \( D \), such that for every positive-dimensional subspace \( \Pi \) of \( \mathbb{C}^N \) and for every \( p \) with \( 1 \leq p < \infty \), \( f|_{D \cap \Pi} \notin L^p H(D \cap \Pi) \).

**Proof.** It is well known that if \( G \) is a domain in \( \mathbb{C}^M \), \( 1 \leq p < \infty \), and \( f \in L^p H(G) \) then for every \( z_0 \in G \),
\[
|f(z_0)| \leq \frac{M^{M/p}}{(\pi \text{dist}(z_0, \partial G)^2)^{M/p}} \|f\|_{G,p},
\]
where \( \|f\|_{G,p} \) denotes the \( L^p \)-norm of \( f \) in \( G \) and dist\((z_0, \partial G)\) is the Euclidean distance of \( z_0 \) to \( \partial G \). For \( z_0 \) sufficiently close to \( \partial G \), we have dist\((z_0, \partial G)\) \( < 1 \). Hence for \( 1 \leq p < \infty \),
\[
1 \leq \frac{1}{\text{dist}(z_0, \partial G)^{2M/p}} \leq \frac{1}{\text{dist}(z_0, \partial G)^{2M}}.
\]
Therefore, for all \( z_0 \in G \), and for every \( 1 \leq p < \infty \), we have
\[
\frac{1}{\text{dist}(z_0, \partial G)^{2M/p}} \leq 1 + \frac{1}{\text{dist}(z_0, \partial G)^{2M}}.
\]
Moreover, there exists \( c > 0 \) such that for all \( L = 1, \ldots, N \), and every \( 1 \leq p < \infty \),
\[
(L/\pi)^{L/p} \leq c.
\]

Consider the construction of the function \( f \) from the proof of Theorem 1. We now require that the function \( f \), constructed as before, satisfies the inequality
\[
|f(z)| \geq \frac{r}{\text{dist}(z_0, \partial D)^{2N}} + 1
\]
for all \( z \in K_r \) instead of (27). (Here the sets \( K_r \) are defined as in (22)). It follows from the above considerations and from (28) that the function \( f \) obtained in this way is holomorphic in \( D \), and for every subspace \( \Pi \) of \( \mathbb{C}^N \) and every \( 1 \leq p < \infty \), \( f \notin L^p (\Pi \cap D) \). This ends the proof. \[\blacksquare\]
Now let $D$ be a balanced domain of holomorphy in $\mathbb{C}^N$, as before. Then in particular Theorem 2 holds for $D$ and $p = 2$. Moreover, since $D$ is balanced, every function $f$ holomorphic in $D$ can be developed into a series of homogeneous polynomials,

$$f(z) = \sum_{s=0}^{\infty} Q_s(z),$$

where every $Q_s$ is a homogeneous polynomial of degree $s$, $s = 0, 1, \ldots$. In [10], Thm. 1, Wojtaszczyk constructed explicitly a sequence $\{p_n\}_{n=1}^{\infty}$ of homogeneous polynomials of degree $n$ in the unit ball $B$ in $\mathbb{C}^N$ such that the function

$$f(z) := \sum_{n} n^{\ln n} p_n(z)$$

is holomorphic in $B$, and for each hyperplane $\Pi \subset \mathbb{C}^N$ and any $p > 0$,

$$\int_{\Pi \cap B} |f(z)|^p \, dm_{\Pi}(z) = \infty$$

($m_\Pi$ is the Lebesgue measure on $\Pi$). It would be interesting to know whether the construction in the present note, given for an arbitrary balanced domain of holomorphy, can be done more explicitly, e.g. as in [10].

References


Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: jakobcza@im.uj.edu.pl

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