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Some applications of a new integral formula for $\overline{\partial}_{\mathrm{b}}$

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Abstract. Let M be a smooth q-concave CR submanifold of codimension k in $\mathbb{C}^n$. We solve locally the $\overline{\partial}_{\mathrm{b}}$-equation on M for $(0,r)$-forms, $0 \le r \le q-1$ or $n-k-q+1 \le r \le n-k$, with sharp interior estimates in Hölder spaces. We prove the optimal regularity of the $\overline{\partial}_{\mathrm{b}}$-operator on $(0,q)$-forms in the same spaces. We also obtain L^p estimates at top degree. We get a jump theorem for $(0,r)$-forms ($r \le q-2$ or $r \ge n-k-q+1$) which are CR on a smooth hypersurface of M. We prove some generalizations of the Hartogs–Bochner–Henkin extension theorem on 1-concave CR manifolds.

In [7] we proved the following

THEOREM 0.1. *Let M be a $\mathcal{C}^{2+l}$-smooth q-concave CR generic submanifold of codimension k in $\mathbb{C}^n$. Let $z_0 \in M$ and $s \in \mathbb{N}$ with $s \le n$. Then there exist an open neighborhood $M_0 \subseteq M$ of z_0 and kernels $\mathcal{R}_{s,r}(\zeta, z)$ for $r = 0, \ldots, q-1, n-k-q, \ldots, n-k$ with the following properties*:

(i) *$\mathcal{R}_{s,r}(\zeta, z)$ is of class $\mathcal{C}^\infty$ in z (resp. ζ) and $\mathcal{C}^l$ in ζ (resp. z) with $\zeta \ne z$ for $r \ge n-k-q$ (resp. $r \le q-1$);*

(ii) *$\mathcal{R}_{s,r}(\zeta, z)$ is of bidegree (s,r) with respect to z and of bidegree $(n-s, n-k-r-1)$ with respect to ζ;*

(iii) *$\overline{\partial}_z \mathcal{R}_{s,r-1}(\zeta, z) = -\overline{\partial}_\zeta \mathcal{R}_{s,r}(\zeta, z)$ for $0 < r \le q-1$ or $n-k-q+1 \le r < n-k$ and $\overline{\partial}_\zeta \mathcal{R}_{s,0}(\zeta, z) = \overline{\partial}_z \mathcal{R}_{s,n-k}(\zeta, z) = 0$;*

(iv) *there is a constant $C > 0$ such that for every $\varepsilon > 0$, we have*

$$\int\limits_{\substack{\zeta \in M_0 \\ |\zeta - z| \le \varepsilon}} \|\mathcal{R}_{s,r}(\zeta, z)\| \, d\lambda(\zeta) \le C\varepsilon;$$

(v) *for every domain $\Omega \Subset M_0$ with piecewise $\mathcal{C}^1$ boundary, if f is a $\mathcal{C}^1$*

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(s,r)-form on $\overline{\Omega}$ ($0 \le r \le q-1$ or $n-k-q+1 \le r \le n-k$), then

$$f = \overline{\partial}_{\mathrm{b}} \int_{\Omega} f \wedge \mathcal{R}_{s,r-1} - \int_{\Omega} \overline{\partial}_{\mathrm{b}} f \wedge \mathcal{R}_{s,r} + \int_{\mathrm{b}\Omega} f \wedge \mathcal{R}_{s,r}$$

on Ω;

(vi) *for every open set $\Omega \Subset M_0$ the integral operator $\int_{\Omega} \cdot \wedge \mathcal{R}_{s,r}$ is a bounded linear operator from $L^{\infty}_{s,r+1}(\Omega)$ to $\mathcal{C}^{1/2}_{s,r}(\overline{\Omega})$ for any $r \le q-1$ (provided $l \ge 1$) and any $r \ge n-k-q+1$;*

(vii) *let $\Omega \Subset M_0$ be an open set; if $f \in L^{\infty}_{s,r+1}(\Omega)$ is of class $\mathcal{C}^l$ then $\int_{\Omega} f \wedge \mathcal{R}_{s,r}$ is of class $\mathcal{C}^{l+1/2}$ for $r \ge n-k-q$, and the same holds for $r \le q-1$ if M is supposed to be of class $\mathcal{C}^{3+l}$.*

By a different method, Polyakov [24] proved sharp estimates in Lipschitz–Stein spaces (cf. [28]) for global solutions of $\overline{\partial}_{\mathrm{b}}$ on $\mathcal{C}^4$ q-concave CR manifolds. Optimal Hölder estimates for solutions of $\overline{\partial}_{\mathrm{b}}$ on hypersurfaces were obtained in [12] and [27].

The aim of this paper is to give some applications of Theorem 0.1.

In Sections 2 and 3 respectively we construct local integral solution operators for $\overline{\partial}_{\mathrm{b}}$ on forms of low and high degrees. Estimates for these operators are a consequence of Theorem 0.1(vii). An example showing that our estimates are sharp is also given.

In Section 4 we obtain L^p estimates for $\overline{\partial}_{\mathrm{b}}$ at top degree on 1-concave CR manifolds. Such estimates were proven on hypersurfaces in [8].

It is known from [3] that on q-concave CR manifolds one cannot solve in general the $\overline{\partial}_{\mathrm{b}}$ equation for $(0,q)$-forms. A criterion for global solvability on such forms was given by Henkin in [14]. In Section 5 we prove the optimal regularity for the $\overline{\partial}_{\mathrm{b}}$-operator in this critical case.

In Section 7 we show a jump theorem for differential forms on q-concave CR manifolds.

In [17] Henkin stated an analogous result to the classical Hartogs–Bochner theorem on smooth 1-concave CR manifolds. In Section 8 we prove some generalizations of Henkin's result to CR manifolds and CR functions with less smoothness.

Theorem 0.1 and the applications given in this paper essentially improve the results of Airapetjan and Henkin [14], [1], [2] and also of the author in [5] where homotopy formulas for $\overline{\partial}_{\mathrm{b}}$ were obtained with less explicit kernels giving almost optimal but not optimal estimates.

The study of the tangential Cauchy–Riemann equations by means of explicit integral formulas with uniform estimates was initiated by Henkin [15] and further developed later on in [10], [14], [1], [21], [22], [27]. For further references and results on CR manifolds we refer the reader to the survey by Henkin [16], the memoir of Trèves [29] and the book by Boggess [9].

1. Preliminaries

1.1. *CR manifolds*. Let M be a real submanifold of class $\mathcal{C}^2$ in $\mathbb{C}^n$ defined by

$$M = \{z \in \Omega : \varrho_1(z) = \ldots = \varrho_k(z) = 0\}, \quad 1 \le k \le n, \tag{1.1}$$

where Ω is an open subset of $\mathbb{C}^n$ and the functions ϱ_ν, $1 \le \nu \le k$, are real-valued functions of class $\mathcal{C}^2$ on Ω with $d\varrho_1(z) \wedge \ldots \wedge d\varrho_k(z) \neq 0$ for each $z \in M$.

We denote by $T_z^{\mathbb{C}}(M)$ the complex tangent space to M at $z \in M$, i.e.,

$$T_z^{\mathbb{C}}(M) = \left\{ \zeta \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \varrho_\nu}{\partial z_j}(z)\zeta_j = 0,\ \nu = 1, \ldots, k \right\}.$$

We have $\dim_{\mathbb{C}} T_z^{\mathbb{C}}(M) \ge n - k$. The submanifold M is called a *Cauchy–Riemann manifold* (*CR-manifold*) if $\dim_{\mathbb{C}} T_z^{\mathbb{C}}(M)$ does not depend on $z \in M$. M is said to be *CR generic* if $\dim_{\mathbb{C}} T_z^{\mathbb{C}}(M) = n - k$ for every $z \in M$. If M is CR generic, then we call M *q-concave*, $0 \le q \le (n-k)/2$, if for each $z \in M$ and every $x \in \mathbb{R}^k \setminus \{0\}$ the hermitian form

$$\sum_{\alpha,\beta} \frac{\partial^2 \varrho_x}{\partial z_\alpha \partial \overline{z}_\beta}(z)\zeta_\alpha \overline{\zeta}_\beta,$$

where $\varrho_x = x_1\varrho_1 + \ldots + x_k\varrho_k$, has at least q negative eigenvalues on $T_z^{\mathbb{C}}(M)$.

If M is CR generic then we denote by $\mathcal{C}_{s,r}^l(M)$ the space of differential forms of type (s,r) on M which are of class $\mathcal{C}^l$. Here, two forms f and g in $\mathcal{C}_{s,r}^l(M)$ are considered to be equal if and only if for each form $\varphi \in \mathcal{C}_{n-s,n-k-r}^\infty(\Omega)$ with compact support, we have

$$\int_M f \wedge \varphi = \int_M g \wedge \varphi.$$

We denote by $[\mathcal{C}_{s,r}^l(M)]'$ the dual space to $\mathcal{C}_{s,r}^l(M)$. We define the tangential Cauchy–Riemann operator on forms in $[\mathcal{C}_{n-s,n-k-r}^l(M)]'$ as follows. If $u \in \mathcal{C}_{s,r}^l(M)$, $l \ge 1$, then u can be extended to a smooth form $\widetilde{u} \in \mathcal{C}_{s,r}^l(\Omega)$ and we may set

$$\overline{\partial}_{\mathrm{b}} u := \overline{\partial}\widetilde{u}|_M.$$

It follows from the condition for equality of forms on M that this definition does not depend on the choice of the extended form $\widetilde{u}$. In general, for forms $u \in [\mathcal{C}_{n-s,n-k-r+1}^l(M)]'$ and $f \in [\mathcal{C}_{n-s,n-k-r}^l(M)]'$, by definition

$$\overline{\partial}_{\mathrm{b}} u = f$$

will mean that for each form $\varphi \in \mathcal{C}_{n-s,n-k-r}^\infty(\Omega)$ with compact support we have

$$\int_M f \wedge \varphi = (-1)^{r+s} \int_M u \wedge \overline{\partial}\varphi.$$

We denote by $\mathcal{C}^{\alpha}_{s,r}(M)$ $(0 < \alpha < 1)$ the space of differential forms which are of type (s, r) and whose coefficients are α-Hölder continuous on each compact set in M.

Let l be a nonnegative integer and $0 < \alpha < 1$. Then we say that f is a $\mathcal{C}^{l+\alpha}$ form on M if f is of class $\mathcal{C}^l$ and all derivatives of order $\leq l$ of f are α-Hölder continuous on M.

By $\mathcal{D}^l_{s,r}(M)$ we denote the space of all $f \in \mathcal{C}^l_{s,r}(M)$ with compact support and by $[\mathcal{D}^l_{s,r}(M)]'$ its dual.

We denote by $L^{\infty}_{s,r}(M)$ the Banach space of (s, r)-forms with bounded measurable coefficients on M endowed with the sup-norm.

1.2. *The generalized Koppelman lemma.* In this section we recall a formal identity (the generalized Koppelman lemma) which will be used in the definition of the kernels $\mathcal{R}_{s,r}$. The exterior calculus we use here was developed by Harvey and Polking in [13].

Let V be an open subset of $\mathbb{C}^n \times \mathbb{C}^n$. Suppose $G : V \to \mathbb{C}^n$ is a $\mathcal{C}^1$ map. We write

$$G(\zeta, z) = (g_1(\zeta, z), \ldots, g_n(\zeta, z))$$

and we use the following notations:

$$G(\zeta, z).(\zeta - z) = \sum_{j=1}^{n} g_j(\zeta, z)(\zeta_j - z_j),$$

$$G(\zeta, z).d(\zeta - z) = \sum_{j=1}^{n} g_j(\zeta, z)d(\zeta_j - z_j),$$

$$\overline{\partial}_{\zeta,z}G(\zeta, z).d(\zeta - z) = \sum_{j=1}^{n} \overline{\partial}_{\zeta,z}g_j(\zeta, z)d(\zeta_j - z_j),$$

where $\overline{\partial}_{\zeta,z} = \overline{\partial}_{\zeta} + \overline{\partial}_z$.

We define the *Cauchy–Fantappiè form* ω^G by

$$\omega^G = \frac{G(\zeta, z).d(\zeta - z)}{G(\zeta, z).(\zeta - z)}$$

on the set where $G(\zeta, z).(\zeta - z) \neq 0$.

Given m such maps, G^j, $1 \leq j \leq m$, we define the kernel

$$\Omega(G^1, \ldots, G^m) = \omega^{G^1} \wedge \ldots \wedge \omega^{G^m} \wedge \sum_{\alpha_1+\ldots+\alpha_m=n-m} (\overline{\partial}_{\zeta,z}\omega^{G^1})^{\alpha_1} \wedge \ldots \wedge (\overline{\partial}_{\zeta,z}\omega^{G^m})^{\alpha_m}$$

on the set where all the denominators are nonzero.

LEMMA 1.2 (The generalized Koppelmann lemma).

$$\overline{\partial}_{\zeta,z}\Omega(G^1,\dots,G^m)=\sum_{j=1}^m(-1)^j\Omega(G^1,\dots,\widehat{G}^j,\dots,G^m)$$

on the set where the denominators are nonzero; *the symbol* $\widehat{G}^j$ *means that the term* G^j *is deleted.*

For a proof of this lemma we refer the reader to [13] or [9].

1.3. *Barrier function.* In this section, we construct a barrier function for a hypersurface at a point where the Levi form has some positive eigenvalues. For a detailed proof of what will follow we refer the reader to Section 3 in [19].

Let H be an oriented real hypersurface of class $\mathcal{C}^2$ in $\mathbb{C}^n$ defined by

$$H=\{z\in\Omega:\varrho(z)=0\}$$

where Ω is an open subset of $\mathbb{C}^n$ and ϱ is a real-valued function of class $\mathcal{C}^2$ on Ω with $d\varrho(z)\neq 0$ for each $z\in H$.

Denote by $F(\zeta,\cdot)$ the *Levi polynomial* of ϱ at a point $\zeta\in\Omega$, i.e.

$$F(\zeta,z)=2\sum_{j=1}^n\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)-\sum_{j,k=1}^n\frac{\partial^2\varrho(\zeta)}{\partial\zeta_j\partial\zeta_k}(\zeta_j-z_j)(\zeta_k-z_k)$$

for $\zeta\in\Omega$ and $z\in\mathbb{C}^n$.

Let $z_0\in H$ and T be the largest vector subspace of $\mathbb{C}^n$ such that the Levi form of ϱ at z^0 is positive definite on T. Set $\dim T=d$ and suppose $d\geq 1$.

Denote by P the orthogonal projection from $\mathbb{C}^n$ onto T, and set $Q=I-P$. Then it follows from Taylor's theorem that there exist a number R and two positive constants A and α such that

$$\operatorname{Re}F(\zeta,z)\geq\varrho(\zeta)-\varrho(z)+\alpha|\zeta-z|^2-A|Q(\zeta-z)|^2 \tag{1.2}$$

for $|z_0-\zeta|\leq R$ and $|z_0-z|\leq R$. Since ϱ is of class $\mathcal{C}^2$ on Ω, we can find $\mathcal{C}^\infty$ functions a^{kj} $(k,j=1,\dots,n)$ on a neighborhood U of z_0 such that

$$\left|a^{kj}(\zeta)-\frac{\partial^2\varrho(\zeta)}{\partial\zeta_k\partial\zeta_j}\right|<\frac{\alpha}{2n^2}$$

for all $\zeta\in U$. Set

$$\widetilde{F}(\zeta,z)=2\sum_{j=1}^n\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)-\sum_{k,j=1}^n a^{kj}(\zeta)(\zeta_k-z_k)(\zeta_j-z_j)$$

for $(z,\zeta)\in\mathbb{C}^n\times U$. Denote by Q_{kj} the entries of the matrix Q, i.e. $Q=$

$(Q_{kj})_{k,j=1}^n$ (k = column index). For $(z,\zeta) \in \mathbb{C}^n \times U$ we set

$$\begin{aligned} g_j(\zeta,z) &= 2\frac{\partial\varrho(\zeta)}{\partial\zeta_j} - \sum_{k=1}^n a^{kj}(\zeta)(\zeta_k - z_k) + A\sum_{k=1}^n \overline{Q_{kj}(\zeta_k - z_k)},\\ G(\zeta,z) &= (g_1(\zeta,z),\dots,g_n(\zeta,z)),\\ \Phi(\zeta,z) &= G(\zeta,z).(\zeta - z). \end{aligned}$$

Since Q is an orthogonal projection, we have

$$\Phi(\zeta,z) = \widetilde{F}(\zeta,z) + A|Q(\zeta - z)|^2,$$

hence it follows from (1.2) that

$$\operatorname{Re}\Phi(\zeta,z) \geq \varrho(\zeta) - \varrho(z) + \frac{\alpha}{2}|\zeta - z|^2 \tag{1.3}$$

for $|z_0 - \zeta| \leq R$ and $|z_0 - z| \leq R$.

G is called a *Leray map* and Φ is called a *barrier function* of H (or ϱ) at z_0.

Definition 1.3. A map f defined on some complex manifold X will be called *k-holomorphic* if, for each point $\xi \in X$, there exist holomorphic coordinates $h_1,\dots,h_k$ in a neighborhood of ξ such that f is holomorphic with respect to $h_1,\dots,h_k$.

Lemma 1.4. *For every fixed $\zeta \in U$, the map $G(\zeta,z)$ and the function $\Phi(\zeta,z)$ defined above are d-holomorphic in $z \in \mathbb{C}^n$.*

1.4. *Some algebraic topology.* Here we state some elementary facts from algebraic topology which we need to define the kernels $\mathcal{R}_{s,r}$. Let N be a positive integer. Then a *p-simplex*, $1 \leq p \leq N$, will be every collection of p linearly independent vectors in $\mathbb{R}^N$. We define S_p as the set of all finite formal linear combinations, with integer coefficients, of p-simplices.

Let $\sigma = [a_1,\dots,a_p]$ be a collection of p vectors in $\mathbb{R}^N$. Then we set

$$\partial_j\sigma = [a_1,\dots,\widehat{a}_j,\dots,a_p]$$

for $1 \leq j \leq p$ and

$$\partial\sigma = \sum_{j=1}^p (-1)^j\partial_j\sigma.$$

If $1 \leq j_1 \leq p,\dots,1 \leq j_r \leq p - r$, we define

$$\partial^r_{j_r\dots j_1}\sigma = \partial_{j_r}(\partial^{r-1}_{j_{r-1}\dots j_1}\sigma)$$

where $\partial^1_j\sigma = \partial_j\sigma$. If σ is a p-simplex defined as above then we define the *barycenter* of σ by

$$b(\sigma) = \frac{1}{p}\sum_{j=1}^p a_j.$$

Now we define the *first barycentric subdivision* of σ by

$$\mathrm{sd}(\sigma) = (-1)^{p+1} \sum_{\substack{j_1,\ldots,j_{p-1} \\ 1\le j_i\le p-i+1}} (-1)^{j_1+\ldots+j_{p-1}} [b(\sigma), b(\partial_{j_1}\sigma), \ldots, b(\partial^{p-1}_{j_{p-1}\cdots j_1}\sigma)].$$

By linearity we can also define the first barycentric subdivision of any element of S_p. It is easy to see that

LEMMA 1.5. *If* σ *is an element of* S_p, *then* $\mathrm{sd}(\partial\sigma) = \partial\mathrm{sd}(\sigma)$.

The barycentric subdivision of higher order of an element σ of S_p is defined as follows: for $m \ge 2$ we set

$$\mathrm{sd}^m(\sigma) = \mathrm{sd}(\mathrm{sd}^{m-1}(\sigma)).$$

$\mathrm{sd}^0(\sigma)$ and $\mathrm{sd}^1(\sigma)$ are defined respectively as σ and $\mathrm{sd}(\sigma)$.

The following lemma is basic in algebraic topology ([23]).

LEMMA 1.6. *Given a simplex* σ *and* $\varepsilon > 0$, *there is an* m *such that each simplex of* $\mathrm{sd}^m\sigma$ *has diameter less than* ε.

2. The kernels $\mathcal{R}_{s,r}$. In this section, we recall the kernels $\mathcal{R}_{s,r}$. First we define some notations. Let k be an integer. Let $\mathcal{I}$ denote the set of all subsets $I \subseteq \{\pm1, \ldots, \pm k\}$ such that $|i| \ne |j|$ for all $i, j \in I$ with $i \ne j$. For $I \in \mathcal{I}$, $|I|$ denotes the number of elements in I. We set

$$\varDelta_{1\ldots|I|} = \Big\{(\lambda_1, , \ldots, , \lambda_{|I|}) \in (\mathbb{R}^+)^{|I|} : \sum_{j=1}^{|I|} \lambda_j = 1\Big\}.$$

We define $\mathcal{I}(l), 1 \le l \le k$, as the set of all $I \in \mathcal{I}$ with $|I| = l$; $\mathcal{I}'(l), 1 \le l \le k$, denotes the set of all $I \in \mathcal{I}(l)$ of the form $I = \{j_1, \ldots, j_l\}$ with $|j_\nu| = \nu$ for $\nu = 1, \ldots, l$. If $I \in \mathcal{I}$, then we set

$$\operatorname{sgn} I := \begin{cases} 1 & \text{if the number of negative elements in } I \text{ is even,} \\ -1 & \text{if the number of negative elements in } I \text{ is odd.} \end{cases}$$

Let now M be a $\mathcal{C}^2$-smooth CR q-concave manifold of codimension k in $\mathbb{C}^n$. Let $z_0 \in M$, $U \subseteq \mathbb{C}^n$ be a neighborhood of z_0 and $\widehat{\varrho}_1, \ldots, \widehat{\varrho}_k : U \to \mathbb{R}$ be functions of class $\mathcal{C}^2$ such that

$$M \cap U = \{\widehat{\varrho}_1 = \ldots = \widehat{\varrho}_k = 0\} \quad \text{and} \quad \partial\widehat{\varrho}_1(z_0) \wedge \ldots \wedge \partial\widehat{\varrho}_k(z_0) \ne 0.$$

Since M is q-concave, it follows from Lemma 3.1.1 of [1] that we can find a constant $C > 0$ such that the functions

$$\varrho_j := \begin{cases} \widehat{\varrho}_j + C\displaystyle\sum_{\nu=1}^{k} \widehat{\varrho}_\nu^2 & (j = 1, \ldots, k), \\ -\widehat{\varrho}_{-j} + C\displaystyle\sum_{\nu=1}^{k} \widehat{\varrho}_\nu^2 & (j = -1, \ldots, -k), \end{cases}$$

have the following property: for each $I \in \mathcal{I}$ and every $\lambda \in \Delta_{1\ldots|I|}$ the Levi form of $\lambda_1 \varrho_{I_1} + \ldots + \lambda_{|I|} \varrho_{I_{|I|}}$ at z_0 has at least $q+k$ positive eigenvalues.

Let $(e_1, \ldots, e_k)$ be the canonical basis of $\mathbb{R}^k$ and set $e_{-j} := -e_j$ for every $1 \leq j \leq k$. Let $I = (j_1, \ldots, j_k)$ be in $\mathcal{I}'(k)$; set

$$\widetilde{\Delta}_I = \Big\{ \sum_{i=1}^{k} \lambda_i e_{j_i} : \lambda_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \Big\},$$

and for each $a = \sum_{i=1}^{k} \lambda_i e_{j_i}$, let G_a and Φ_a be respectively the Leray map and the barrier function at z_0 corresponding to $\varrho_a = \lambda_1 \varrho_{j_1} + \ldots + \lambda_k \varrho_{j_k}$ (see Sect. 1.3). We call ϱ_a (resp. ϕ_a) the *defining function* (resp. *barrier function*) of M *in direction* a.

Let $\sigma = [a^1, \ldots, a^p]$, $p \geq 1$, be a collection of p vectors where $a^i \in \bigcup_{I \in \mathcal{I}'(k)} \Delta_I$ for every $1 \leq i \leq k$. Define

$$\widetilde{\Omega}[\sigma] := \Omega(G_{a^1}, \ldots, G_{a^p}).$$

We denote by S'_p the set of all finite formal linear combinations of such collections with integer coefficients and we extend $\widetilde{\Omega}$ by linearity to S'_p. For every $0 \leq s \leq n$, every $0 \leq r \leq n-p$ and any $\tau \in S'_p$, we define $\widetilde{\Omega}_{s,r}[\tau]$ as the piece of $\widetilde{\Omega}[\tau]$ which is of type (s, r) in z. We may rewrite Lemma 1.2 as follows:

Lemma 2.7. *For every $\tau \in S'_p$, we have $\overline{\partial}_{\zeta,z} \widetilde{\Omega}[\tau] = \widetilde{\Omega}[\partial\tau]$ outside the singularities.*

Let $I = (j_1, \ldots, j_l)$ be in $\mathcal{I}'(l)$, $1 \leq l \leq k$ and $\sigma_I = [e_{j_1}, \ldots, e_{j_l}]$. Then by continuity of the Levi form, by Lemma 1.4 and 1.6, we can find a positive integer m independent of I and l such that for every simplex $\tau = [a^1, \ldots, a^l]$ in $\mathrm{sd}^m(\sigma_I)$, the Leray maps of $G_{a^1}, \ldots, G_{a^l}$ are $q+k$-holomorphic in the same directions with respect to the variable $z \in \mathbb{C}^n$. Therefore we have

Lemma 2.8. *There is a positive integer m such that for every $I \in \mathcal{I}'(l)$, $1 \leq l \leq k$, any $s \geq 0$ and every $r \geq n-k-q+1$,*

(i) $\widetilde{\Omega}_{s,r}(\mathrm{sd}^m(\sigma_I)) = 0$,

(ii) $\overline{\partial}_z \widetilde{\Omega}_{s,r-1}(\mathrm{sd}^m(\sigma_I)) = 0$,

on the set where all the denominators are nonzero.

Let m be as in the previous lemma and $\nu^* \in \bigcup_{I \in \mathcal{I}'(k)} \widetilde{\Delta}_I$ be such that for any k-simplex τ in $\mathrm{sd}^m(\sigma_I)$, each collection of k elements in $[\nu^*, \tau]$ is a k-simplex. We adopt the following notation:

$$\Big[\nu^*, \sum_i c_i \sigma_i\Big] = \sum_i c_i [\nu^*, \sigma_i]$$

for any element $\sum_i c_i\sigma_i$ in S'_p. Set

$$R(\zeta,z)=\sum_{I\in\mathcal{I}'(k)}(\operatorname{sgn} I)\widetilde{\Omega}[\nu^*,\mathrm{sd}^m(\sigma_I)](\zeta,z). \tag{2.1}$$

DEFINITION 2.9. Let $s\in\mathbb{N}$ with $s\le n$. We define

$$\mathcal{R}_{s,r}(\zeta,z):=\begin{cases}(-1)^{r(k+1)}R_{s,r}(\zeta,z) & \text{if } n-k-q\le r\le n-k,\\ (-1)^{r(k+1)}R_{n-s,n-k-1-r}(z,\zeta) & \text{if } 0\le r\le q-1.\end{cases}$$

The coefficients of the kernel $\mathcal{R}_{s,r}(\zeta,z)$ have the following form (see [7]):

$$\frac{\mathcal{N}(\zeta,z)}{\prod_{i=1}^{k+1}(\Phi_{a^i}(\zeta,z))^{r_i}} \tag{2.2}$$

where $a^1,\dots,a^{k+1}$ are vectors in $\mathbb{R}^k$ such that every collection of k elements in $\{a^1,\dots,a^{k+1}\}$ is a family of linearly independent vectors, $r_i\ge1$ for all $1\le i\le k+1$, $\sum_{i=0}^{k+1}r_i=n$ and $|\mathcal{N}(\zeta,z)|\le C|\zeta-z|$.

3. Local solvability of $\overline{\partial}_{\mathrm{b}}$ in low degrees. In this section we are concerned with the local solvability of $\overline{\partial}_{\mathrm{b}}$ when the data is of bidegree $(0,r)$ where $r\le q-1$.

We construct a local homotopy formula for $r\le q-2$. Such a formula does not hold for $r=q-1$ (see [27]); instead we construct a solution operator in this case. We give an example showing that our estimates are sharp. We also derive a known result on the holomorphic extension of CR functions from 1-concave CR manifolds [14].

THEOREM 3.10. *Let M be a $\mathcal{C}^{3+l}$-smooth ($l\ge0$) q-concave CR generic submanifold of codimension k in $\mathbb{C}^n$ and z_0 a point in M. Then for every open neighborhood $U\subset M$ of z_0 and every r with $1\le r\le q-1$, there exist an open neighborhood $V\subset U$ of z_0 and linear integral operators*

$$T_r:\mathcal{C}^0_{0,r}(U)\to\mathcal{C}^0_{0,r-1}(V),\qquad S_r:\mathcal{C}^0_{0,r}(U)\to\mathcal{C}^0_{0,r-1}(V)$$

with the following properties:

(i) $f=\overline{\partial}_{\mathrm{b}}T_rf+S_{r+1}\overline{\partial}_{\mathrm{b}}f$ *for* $1\le r\le q-2$,
(ii) $f=\overline{\partial}_{\mathrm{b}}T_rf$ *if* $r=q-1$ *and* $\overline{\partial}_{\mathrm{b}}f=0$,
(iii) *if* $f\in\mathcal{C}^l_{0,r}(U)$ *then* $T_rf\in\mathcal{C}^{l+1/2}_{0,r}(V)$.

Proof. Without loss of generality we may assume that $U=M_0\cap B\Subset M_0$ where M_0 is defined as in Theorem 0.1 and B is a small ball centered at z_0. So we can use the integral formula from Theorem 0.1(v): for every $\mathcal{C}^1$ $(0,r)$-form f on $\overline{U}$ ($0\le r\le q-1$), we have

$$f(z)=\overline{\partial}_{\mathrm{b}}\mathcal{R}^U_{r-1}f(z)-\mathcal{R}^U_r\overline{\partial}_{\mathrm{b}}f(z)+\mathcal{R}^{\mathrm{b}U}_rf(z), \tag{3.1}$$

where

$$\mathcal{R}^U_{r-1} f(z) = \int\limits_{\zeta\in U} f(\zeta)\wedge \mathcal{R}_{0,r-1}(\zeta,z) \quad \text{and} \tag{3.2}$$

$$\mathcal{R}^{\mathrm{b}U}_{r} f(z) = \int\limits_{\zeta\in \mathrm{b}U} f(\zeta)\wedge \mathcal{R}_{0,r}(\zeta,z). \tag{3.3}$$

We must now analyze the boundary term. From the definition of the kernel $\mathcal{R}_{0,r}$ (see (2.2) and inequality (1.3)) it is clear that there is a small ball $B' \Subset B$ centered at z_0 such that the kernel $\mathcal{R}_{0,r}(\zeta,z)$ is nonsingular for $\zeta\in \mathrm{b}U$ and $z\in B'$ and therefore $\mathcal{R}^{\mathrm{b}U}_r f$ is of class $\mathcal{C}^{l+1}$ on B'.

Let H be Henkin's $\overline{\partial}$-homotopy operator on B'. Then on B' we have

$$\mathcal{R}^{\mathrm{b}U}_r f = H\overline{\partial}\mathcal{R}^{\mathrm{b}U}_r f + \overline{\partial}_z H\mathcal{R}^{\mathrm{b}U}_r f. \tag{3.4}$$

Lemma 1.2 implies that for $r\le q-1$, $\zeta\in \mathrm{b}U$ and $\xi\in B'$,

$$\begin{aligned}\overline{\partial}_\xi \mathcal{R}_{0,r}(\zeta,\xi) &\\ = -\overline{\partial}_\zeta \mathcal{R}_{0,r+1}(\zeta,\xi) &\pm \sum_{I\in\mathcal{I}'(k)} (\operatorname{sgn} I)\widetilde{\Omega}_{n,n-k-r-1}(\mathrm{sd}^m\sigma_I)(\xi,\zeta),\end{aligned} \tag{3.5}$$

because

$$\sum_{I\in\mathcal{I}'(k)} (\operatorname{sgn} I)\widetilde{\Omega}(\nu^*,\mathrm{sd}^m\partial\sigma_I)(\xi,\zeta) = 0.$$

Now for $r\le q-2$ the second term on the right-hand side of (3.5) vanishes by Lemma 2.8(i). Part (i) then follows from (3.1), (3.4), (3.5) and Stokes' theorem if we set $V = B'\cap M_0$, $T_r = H\mathcal{R}^{\mathrm{b}U}_r + \mathcal{R}^U_{r-1}$ and $S_{r+1} = (-1)^{k+r}H\mathcal{R}^{\mathrm{b}U}_{r+1} - \mathcal{R}^U_r$.

For $r=q-1$, suppose that $\overline{\partial}_{\mathrm{b}} f = 0$ on $\overline{U}$. First by Stokes' theorem for any $\xi\in B'$ we have

$$\int\limits_{\zeta\in \mathrm{b}U} f(\zeta)\wedge\overline{\partial}_\zeta \mathcal{R}_{0,q}(\zeta,\xi) = 0. \tag{3.6}$$

On the other hand, by Lemma 2.8(ii) we have, for every $I\in\mathcal{I}'(k)$,

$$\overline{\partial}_\zeta \widetilde{\Omega}_{n,n-k-q}(\mathrm{sd}^m\sigma_I)(\xi,\zeta) = 0$$

off the singularities. So after shrinking B' we can use similar arguments to [20] (see Lemma 5.4 and Lemma 5.5) to approach for every fixed $\xi\in B'$ the form $\widetilde{\Omega}_{n,n-k-q}(\mathrm{sd}^m\sigma_I)(\xi,\zeta)$ uniformly on $\mathrm{b}U$ by a sequence of $\overline{\partial}_\zeta$-closed forms on a neighborhood of $\overline{U}$. Thus by Stokes' theorem for every $I\in\mathcal{I}'(k)$ and any $\xi\in B'$ we obtain

$$\int\limits_{\zeta\in \mathrm{b}U} f(\zeta)\wedge \widetilde{\Omega}_{n,n-k-q}(\mathrm{sd}^m\sigma_I)(\xi,\zeta) = 0. \tag{3.7}$$

Now (3.5)–(3.7) imply that $\overline{\partial}\mathcal{R}^{\mathrm{b}U}_{q-1}f = 0$ on B'. Therefore by setting $T_{q-1} = H\mathcal{R}^{\mathrm{b}U}_{q-1} + \mathcal{R}^{U}_{q-2}$, we obtain (ii) from (3.1) and (3.4). (iii) is a consequence of the estimates from Theorem 0.1 and the regularity of the operator H. ■

We now exhibit an example showing that our estimates for the solution of $\overline{\partial}_{\mathrm{b}}$ are optimal.

Let $D = \{z \in \mathbb{C}^5 : |z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2 + |z_5|^2 < 1\}$ and $M = \mathrm{b}D \cap B$ where B is a small ball centered at $z_0 = (1, 0, 0, 0, 0)$. Then M is 2-concave near z_0. It is clear that for all $z \in \overline{D}$,

$$\mathrm{Re}(1 - z_1 + |z_2|^2 + |z_4|^2) \geq \tfrac{1}{2}(|z_1 - 1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2).$$

Let ln be the principal branch of logarithm in $\mathbb{C} \setminus \mathbb{R}^-$. We consider the function defined by $u(z_0) = 0$ and

$$u(z) = \frac{\overline{z}_2}{\ln(1 - z_1 + |z_2|^2 + |z_4|^2)} \quad \text{for } z \in \overline{D} \setminus \{z_0\}.$$

The function u is continuous on $\overline{D}$ and of class $\mathcal{C}^\infty$ on $\overline{D} \setminus \{z_0\}$. It is easy to see that $\overline{\partial}u$ extends to a continuous $(0,1)$-form on $\overline{D}$. Set $f = \overline{\partial}_{\mathrm{b}}u$.

Proposition 3.11. *There exists no function v on M with $\overline{\partial}_{\mathrm{b}}v = f$ such that $\|v\|_{\alpha,M} < \infty$ with $\alpha > 1/2$.*

Proof. See [4]. ■

Let M be a $\mathcal{C}^3$-smooth 1-concave CR submanifold of $\mathbb{C}^n$. Let z_0, M_0, $\mathcal{R}_{0,0}$ be as in Theorem 0.1. Let $U = M_0 \cap B \Subset M_0$, where B is a small ball centered at z_0. It follows from the proof of Theorem 3.10 that if f is a $\mathcal{C}^1$ function with $\overline{\partial}_{\mathrm{b}}f = 0$ on $\overline{U}$ then $\overline{\partial}\mathcal{R}^{\mathrm{b}U}_0 f = 0$ on a $\mathbb{C}^n$-neighborhood of z_0. By using the fact that CR generic manifolds are uniqueness sets for holomorphic functions (see [9]), this yields a proof of the following known extension theorem (see [14]).

Proposition 3.12. *Let M be a 1-concave CR submanifold of class $\mathcal{C}^3$. Then any $\mathcal{C}^1$ CR function defined on an open set $U \subseteq M$ extends to a holomorphic function on some $\mathbb{C}^n$-neighborhood of U.*

4. Local solvability of $\overline{\partial}_{\mathrm{b}}$ in high degrees. This section is devoted to the construction of a local $\overline{\partial}_{\mathrm{b}}$ homotopy formula for forms of bidegree $(0, r)$ with $r \geq n - k - q + 1$. In contrast to low degree forms the homotopy formula here needs no shrinking of the domains.

Theorem 4.13. *Let M be a q-concave CR generic submanifold of codimension k and of class $\mathcal{C}^{l+2}$ in $\mathbb{C}^n$ ($l \geq 0$), z_0 a point in M, and M_0 an open neighborhood of z_0 as in Theorem 0.1. Let V be a convex domain with $\mathcal{C}^2$ boundary such that $U = V \cap M_0 \Subset M_0$ and $\mathrm{b}U$ is of class $\mathcal{C}^1$. Then for* *every r with $n-k-q+1 \le r \le n-k$, there exist linear integral operators*

$$T_r : \mathcal{C}^0_{0,r}(\overline{U}) \to \mathcal{C}^0_{0,r-1}(U), \qquad S_r : \mathcal{C}^0_{0,r}(\overline{U}) \to \mathcal{C}^0_{0,r-1}(U)$$

with the following properties:

(i) $f = \overline{\partial}_{\mathrm{b}} T_r f + S_{r+1} \overline{\partial}_{\mathrm{b}} f$,

(ii) *if* $f \in \mathcal{C}^0_{0,r}(\overline{U}) \cap \mathcal{C}^l_{0,r}(U)$ *then* $T_r f \in \mathcal{C}^{l+1/2}_{0,r}(U)$.

Proof. By the integral formula from Theorem 0.1, for every $\mathcal{C}^1$ $(0,r)$-form f on $\overline{U}$ $(n-k-q+1 \le r \le n-k)$ we have

$$f(z) = \overline{\partial}_{\mathrm{b}} \mathcal{R}^U_{r-1} f(z) - \mathcal{R}^U_r \overline{\partial}_{\mathrm{b}} f(z) + \mathcal{R}^{\mathrm{b}U}_r f(z),$$

where $\mathcal{R}^U_{r-1} f$ and $\mathcal{R}^{\mathrm{b}U}_r f$ are defined respectively by (3.2) and (3.3).

Let $G_0(\cdot, z)$ be the Leray map of $\mathrm{b}V$ defined for $z \in \mathbb{C}^n$. It is known that $G_0(\cdot, z)$ is holomorphic with respect to z and the associated barrier function does not vanish for $z \in U$ and $\zeta \in \mathrm{b}U$. Define

$$\widetilde{\Omega}^0[\tau] := \Omega(G_0, G_{\nu^1}, \ldots, G_{\nu^k})$$

for any simplex $\tau = [\nu^1, \ldots, \nu^k]$ in S'_k. Extend this operation, by linearity, to all elements of S'_k (see Section 2 for notations). Set

$$F_{0,r} := (-1)^{r(k+1)} \sum_{I \in \mathcal{I}'(k)} (\operatorname{sgn} I) \widetilde{\Omega}^0_{0,r}[\nu^*, \mathrm{sd}^m(\sigma_I)].$$

For each $I \in \mathcal{I}'(k)$ and every component $\tau = [\nu^1, \ldots, \nu^k]$ in $\mathrm{sd}^m(\sigma_I)$,

$$\widetilde{\Omega}^0_{0,r}[\tau] = 0 \tag{4.1}$$

for $r \ge n-k-q+1$, because the maps $G_0, G_{\nu^1}, \ldots, G_{\nu^k}$ are $q+k$-holomorphic with respect to the variable z in the same directions. Since

$$\sum_{I \in \mathcal{I}'(k)} (\operatorname{sgn} I) \widetilde{\Omega}^0(\nu^*, \mathrm{sd}^m \partial \sigma_I)(\zeta, z) = 0,$$

it follows from Lemma 2.7, Lemma 1.5 and (4.1) that

$$\overline{\partial}_\zeta F_{0,r}(\zeta, z) + \overline{\partial}_z F_{0,r-1}(\zeta, z) = -\mathcal{R}_{0,r}(\zeta, z)$$

for $\zeta, z \in M_0$ with $\zeta \ne z$ and $r \ge n-k-q+1$. Part (i) then follows by setting

$$T_r f = (-1)^k \int_{\mathrm{b}U} f \wedge F_{0,r-1} + \mathcal{R}^U_{r-1} f,$$

$$S_{r+1} \overline{\partial}_{\mathrm{b}} f = (-1)^{r+1} \int_{\mathrm{b}U} \overline{\partial}_{\mathrm{b}} f \wedge F_{0,r} + \mathcal{R}^U_r \overline{\partial}_{\mathrm{b}} f.$$

Part (ii) is a direct consequence of (i) and Theorem 0.1(vii). ∎

5. Hölder and L^p estimates for $\overline{\partial}_{\mathrm{b}}$ at top degree. Let M be a CR 1-concave manifold of class $\mathcal{C}^{2+l}$ $(l \ge 0)$ and of codimension k in $\mathbb{C}^n$.

Let $z_0 \in M$ and $M_0 \Subset M$ be a neighborhood of z_0 as in Theorem 0.1. Let $\Omega \subset M_0$. Since the boundary term in the integral representation from Theorem 0.1(v) vanishes at top degree (i.e. for $r = n - k$), we can say more about the regularity of $\overline{\partial}_{\mathrm{b}}$ in this case; indeed, we obtain optimal Hölder estimates up to the boundary and L^p estimates. For $f \in L^\infty_{0,n-k}(\Omega)$, define

$$R^\Omega f(z) = \int_\Omega \mathcal{R}_{0,n-k-1}(\zeta, z) \wedge f(z).$$

THEOREM 5.14. *For* $f \in L^\infty_{0,n-k}(\Omega)$ *one has*

(i) $f = \overline{\partial}_{\mathrm{b}} R^\Omega f$,

(ii) *there is a constant* C *such that*

$$\frac{\|R^\Omega f(z^1) - R^\Omega f(z^2)\|}{|z^1 - z^2|^{1/2}} \le C\|f\|_\infty,$$

(iii) *if moreover* f *is of class* $\mathcal{C}^l(\overline{\Omega})$ *then* $\overline{\partial} R^\Omega f$ *is of class* $\mathcal{C}^{l+1/2}(\overline{\Omega})$,

(iv) *for* $1 \le p < 2n$ *and* $1 \le q < 2np/(2n-p)$, *one has*

$$\|R^\Omega f\|_{L^q} \le C\|f\|_{L^p},$$

(v) *if* $2n < p \le \infty$, *then*

$$\|R^\Omega f\|_{L^\infty} \le C\|f\|_{L^p}.$$

Proof. (i), (ii) and (iii) follow from Theorem 0.1.

To prove (iv) and (v) we need the following lemma:

LEMMA 5.15. *Let* $\mathcal{M}(\zeta, z)$ *denote any of the cofficients of the kernel* $\mathcal{R}_{0,n-k-1}(\zeta, z)$. *For each* s *with* $1 \le s < 2n/2n - 1$, *there is a constant* $C_s > 0$ *such that*

$$\text{(i)} \quad \int_{z\in\Omega} |\mathcal{M}(\zeta, z)|^s \, d\lambda(z) \le C_s,$$

$$\text{(ii)} \quad \int_{\zeta\in\Omega} |\mathcal{M}(\zeta, z)|^s \, d\lambda(\zeta) \le C_s.$$

Proof. It is easy to see from (1.3) and (2.2) that $|\mathcal{M}(\zeta, z)|^s$ is majorized by a finite number of terms of the type

$$\frac{C}{\prod_{i=1}^k |\Phi_{a^i}(\zeta, z)|^{s+s/k} |\zeta - z|^{(2n-2k-3)s}}$$

where $a^1, \ldots, a^k$ are linearly independent (cf. [7]).

Since M is CR generic, $\operatorname{Im}\Phi_{a^1}(\cdot,z),\dots,\operatorname{Im}\Phi_{a^k}(\cdot,z)$ can be taken as local coordinates on M_0 (cf. [5]). Taking into account (1.3) we obtain

$$\int\limits_{\zeta\in\Omega} |\mathcal{M}(\zeta,z)|^s\,d\lambda(\zeta) \le C \int\limits_{\substack{X\in\mathbb{R}^{2n-k}\\ |X|<A}} \frac{dX}{\prod_{j=1}^k(|X_j|+|X|^2)^{s+s/k}|X|^{(2n-2k-3)s}}$$
$$\le C \int\limits_{\substack{X\in\mathbb{R}^{2n-2k}\\ |X|<A}} \frac{dX}{|X|^{(2n-1)(s-1)}|X|^{2n-2k-1}}$$

where A is a positive number. The last integral is finite if $s<2n/(2n-1)$. (ii) is proved similarly.

The above lemma implies part (v) and also the following:

$$\|R^{\Omega}f\|_{L^q} \le C\|f\|_{L^1} \quad \text{for } 1\le q<2n/(2n-1)$$

(cf. [25], Appendix B). Interpolating this inequality with (v), we obtain (iv). ■

6. Regularity theorem for $\overline{\partial}_{\mathrm{b}}$. It is known from [3] that in general on q-concave CR manifolds one cannot solve locally the tangential Cauchy–Riemann equation for data of bidegree $(0,q)$. However, we shall prove the existence of a regular solution when the data is a regular $\overline{\partial}_{\mathrm{b}}$ exact $(0,q)$-form (see Theorem 6.19). First we need some preparation.

Let M be a $\mathcal{C}^{l+3}$-smooth CR generic q-concave submanifold of codimension k in $\mathbb{C}^n$. Let $z_0\in M$ and let $M_0\subset M$ be a neighborhood of z_0 as in Theorem 0.1.

Let $\Omega\subset M_0$ be a domain. Let $0\le r\le q-1$ or $n-k-q\le r\le n-k$ and for $f\in L^{\infty}_{n,r+1}(\Omega)$ set

$$\mathcal{R}^{\Omega}_r f(z) := \int\limits_{\Omega} f(\zeta)\wedge \mathcal{R}_{n,r}(\zeta,z).$$

By Theorem 0.1 we have

$$\mathcal{R}^{\Omega}_r : \mathcal{D}^l_{n,r+1}(\Omega)\to \mathcal{C}^l_{n,r}(\Omega).$$

Define

$$\widehat{\mathcal{R}}^{\Omega}_r : [\mathcal{C}^l_{n,r}(\Omega)]'\to[\mathcal{D}^l_{n,r+1}(\Omega)]'$$

by setting for $T\in[\mathcal{C}^l_{n,r}(\Omega)]'$ and $\varphi\in\mathcal{D}^l_{n,r+1}(\Omega)$,

$$\widehat{\mathcal{R}}^{\Omega}_r T(\varphi) = T(\mathcal{R}^{\Omega}_r\varphi).$$

By duality we obtain from Theorem 0.1

Proposition 6.16. *Let $\Omega\subset M_0$ be a domain. Then for any $T\in[\mathcal{C}^l_{n,r}(\Omega)]'$ with $\overline{\partial}_{\mathrm{b}}T\in[\mathcal{C}^l_{n,r-1}(\Omega)]'$ and $0\le r\le q$ or $n-k-q+1\le r\le n-k$,* *we have*

$$(-1)^k T = \widehat{\mathcal{R}}^{\Omega}_{r-1}\overline{\partial}_{\mathrm{b}}T + \overline{\partial}_{\mathrm{b}}\widehat{\mathcal{R}}^{\Omega}_{r}T.$$

By Fubini's theorem we obtain the following

LEMMA 6.17. *Let* $1 \le r \le q$ *or* $n-k-q+1 \le r \le n-k$, $f \in \mathcal{C}^l_{0,r}(\Omega)$ *and let* $\langle f\rangle$ *be the current associated with f. Then*

$$\widehat{\mathcal{R}}^{\Omega}_{n-k-r}\langle f\rangle = (-1)^k\Big\langle \int_{\zeta\in\Omega} f(\zeta)\wedge \mathcal{R}_{n,n-r-k}(\cdot,\zeta)\Big\rangle.$$

We also need the following

LEMMA 6.18. *If* $T \in [\mathcal{C}^l_{n,n-k-1}(\Omega)]'$ *then there exists* $g \in \mathcal{C}^{l+1}_{0,0}(\Omega\backslash\operatorname{supp} T)$ *such that*

$$(\widehat{\mathcal{R}}^{\Omega}_{n-k-1}T)\varphi = \langle g\rangle\varphi \quad \text{for all } \varphi \in \mathcal{D}^l_{n,n-k}(\Omega\setminus\operatorname{supp} T)\ .$$

Proof. It is sufficient to show that such a function g exists over each open set $U \Subset \Omega\backslash\operatorname{supp} T$. Fix such an open set U. Then choose a $\mathcal{C}^\infty$ function χ on $\mathbb{C}^n$ such that $\chi = 1$ in a neighborhood of $\operatorname{supp} T$ and $\chi = 0$ in some neighborhood of $\overline{U}$. Then for $\varphi \in \mathcal{D}^l_{n,n-k}(\Omega)$ one has

$$\begin{aligned}(\widehat{\mathcal{R}}^{\Omega}_{n-k-1}T)\varphi &= \Big\langle \chi(z)T, \int_{\zeta\in\Omega}(1-\chi)(\zeta)\mathcal{R}_{n,n-1-k}(\zeta,z)\wedge\varphi(\zeta)\Big\rangle\\ &= \Big\langle T, \int_{\zeta\in\Omega}\chi(z)(1-\chi)(\zeta)\mathcal{R}_{n,n-1-k}(\zeta,z)\wedge\varphi(\zeta)\Big\rangle\\ &= (-1)^k\int_{\zeta\in\Omega} T(\chi(z)(1-\chi)(\zeta)\mathcal{R}_{n,n-k-1}(\zeta,z))\wedge\varphi(\zeta)\end{aligned}$$

because $\chi(z)(1-\chi)(\zeta)\mathcal{R}_{n,n-k-1}(\zeta,z)$ is a differential form of class $\mathcal{C}^{l+1}$ for $\zeta\in\Omega$ and $z \in \operatorname{supp} T$. Set

$$g|_U := (-1)^k T(\chi(z)(1-\chi)(\cdot)\mathcal{R}_{n,n-k-1}(\cdot,z)). \ \blacksquare$$

THEOREM 6.19. *Assume M is a* $\mathcal{C}^{l+3}$*-smooth CR generic q-concave submanifold of codimension k in* $\mathbb{C}^n$.

(i) *If* $q=1$ *and T is a distribution of order l on M such that* $\overline{\partial}_{\mathrm{b}}T$ *is defined by a* $\mathcal{C}^l$ 1*-form on M, then T is defined by a* $\mathcal{C}^{l+1/2}$ *function.*

(ii) *Let* $z_0 \in M$. *Then there is a neighborhood* $M_0 \subset M$ *of* z_0 *such that for each* $f \in \mathcal{C}^l_{0,q}(M_0)$, *if T is a compactly supported current of bidegree* $(0,q-1)$ *on* M_0 *satisfying* $\overline{\partial}_{\mathrm{b}}T = \langle f\rangle$ *then there exists a current S of order l and of bidegree* $(0,q-1)$ *such that* $T-\overline{\partial}_{\mathrm{b}}S$ *is defined by a* $\mathcal{C}^{l+1/2}$ *form.*

Proof. If $M_0\subset M$ is a neighborhood of z_0 as in Proposition 6.16 then we can write

$$(-1)^k T - \overline{\partial}_{\mathrm{b}}\widehat{\mathcal{R}}^{M_0}_{n-k-q+1}T = \widehat{\mathcal{R}}^{M_0}_{n-k-q}\langle f\rangle.$$

The result in (ii) now follows from Lemma 6.17 and Theorem 0.1(vii).

Let us prove (i): It is sufficient to prove the statement on a neighborhood of each point. Let $z_0 \in M$ and let $M_0 \subset M$ be a neighborhood of z_0 as in Proposition 6.16. Let $\Omega \Subset M_0$ be a domain and χ a compactly supported function on Ω with $\chi \equiv 1$ on a neighborhood of $\overline{\Omega}$. By using Lemma 6.17 one obtains

$$\begin{aligned}(-1)^k T = (-1)^k \chi T &= (\widehat{\mathcal{R}}^{M_0}_{n-k-1} \overline{\partial}_{\mathrm{b}}(\chi T)) \\ &= (\widehat{\mathcal{R}}^{M_0}_{n-k-1}(\overline{\partial}_{\mathrm{b}}\chi) \wedge T) \pm \Big\langle \int_{\zeta \in \Omega} \chi(\zeta) \wedge f(\zeta) \wedge \mathcal{R}_{n,n-k-1}(\cdot, \zeta) \Big\rangle.\end{aligned}$$

In this way (i) follows from Lemma 6.18 and Theorem 0.1(vii). ∎

For hypersurfaces Theorem 6.19 was proved in [12]. For $\mathcal{C}^2$ q-concave CR manifolds, nonoptimal versions of this theorem were given in [2] and [5].

REMARK 6.20. In Theorem 6.19, if M is supposed to be only of class $\mathcal{C}^{l+2}$ then in (i) (resp. (ii)) T (resp. $T - \overline{\partial}_{\mathrm{b}} S$) will be of class $\mathcal{C}^l$ (cf. [7]).

Remark 6.20 together with Proposition 3.12 imply the following result (see [14]):

COROLLARY 6.21. *Let M be a $\mathcal{C}^{l+2}$-smooth 1-concave CR manifold ($l \geq 1$). Then every CR distribution of order r on M with $0 \leq r \leq l$ is defined by a function of class $\mathcal{C}^{l+2}$.*

7. Jump theorem for CR forms. Let M be a $\mathcal{C}^{l+3}$-smooth CR q-concave submanifold of codimension k in $\mathbb{C}^n$. Let V be a $\mathcal{C}^{l+1}$ 1-codimensional submanifold of M such that $M \setminus V$ has exactly two connected components V^+ and V^-.

DEFINITION 7.22. Let $f \in \mathcal{C}^0_{0,r}(V)$. We say that f is *CR* on V if $\int_V f \wedge \overline{\partial}\varphi = 0$ for all forms $\varphi \in \mathcal{C}^\infty_{0,n-k-1-r}(\mathbb{C}^n)$ such that $\operatorname{supp} \varphi \cap V \Subset V$.

THEOREM 7.23. *Suppose M is of class $\mathcal{C}^{l+4}$ (resp. $\mathcal{C}^{l+3}$) and let $f \in \mathcal{C}^{l+1}_{0,r}(V)$ with $0 \leq r \leq q-2$ (resp. $n-k-q+1 \leq r \leq n-k$) be a CR form on V. Then, for every point $z_0 \in V$, there is a neighborhood U of z_0 in M and two forms $F^\pm \in \mathcal{C}^{l+1/2}_{0,r}(U \cap \overline{V}^\pm)$ such that $F^\pm$ is CR on $U \cap V^\pm$ and*

$$f_{|V\cap U} = F^+_{|V\cap U} - F^-_{|V\cap U}.$$

Proof. Let $z_0 \in V$ and $M_0 \subset M$ be a neighborhood of z_0 where Theorem 0.1 holds. Let χ be a smooth cutoff function on M with $\operatorname{supp}\chi \Subset M_0$ and $\chi \equiv 1$ on a neighborhood U of z_0. Let $\Omega \subset M_0 \cap V^+$ be a relatively compact domain with $\mathcal{C}^{l+1}$ boundary in M_0 such that $\operatorname{supp}\chi \cap V \Subset b\Omega$. Let $\widetilde{f}$ be a $\mathcal{C}^{l+1}$ extension of f to Ω. Suppose $0 \leq r \leq q-2$ or $n-k-q+1 \leq r \leq n-k$

and define

$$G^+(z) = \chi\widetilde{f}(z) + \int_{\Omega} \overline{\partial}_{\mathrm{b}}(\chi\widetilde{f})(\zeta) \wedge \mathcal{R}_{0,r}(\zeta, z) \quad \text{for } z \in \Omega,$$

$$G^-(z) = \int_{\Omega} \overline{\partial}_{\mathrm{b}}(\chi\widetilde{f})(\zeta) \wedge \mathcal{R}_{0,r}(\zeta, z) \quad \text{for } z \in M_0 \setminus \overline{\Omega}.$$

Theorem 0.1 yields

$$G^{\pm}(z) = \int_{\mathrm{b}\Omega} \chi(\zeta) f(\zeta) \wedge \mathcal{R}_{0,r}(\zeta, z) + \overline{\partial}_{\mathrm{b}} \int_{z\in\Omega} \chi\widetilde{f}(\zeta) \wedge \mathcal{R}_{0,r-1}(\zeta, z)$$

for $z \in U \cap V^{\pm}$, $G^{\pm} \in \mathcal{C}^{l+1/2}(U \cap \overline{V}^{\pm})$ and

$$f_{|V\cap U} = G^+_{|V\cap U} - G^-_{|V\cap U}.$$

Next we show that $\overline{\partial}_{\mathrm{b}} G^{\pm} = H$ where H is a smooth form on a neighborhood of z_0. First recall from Theorem 0.1 that

$$\overline{\partial}_z \mathcal{R}_{0,r}(\zeta, z) = -\overline{\partial}_\zeta \mathcal{R}_{0,r+1}(\zeta, z).$$

Then using Stokes' theorem and the fact that f is a CR form on $\mathrm{b}\Omega$, we get

$$\overline{\partial}_{\mathrm{b}} G^{\pm}(z) = (-1)^{k+r+1} \int_{\mathrm{b}\Omega} \overline{\partial}\chi(\zeta) \wedge f(\zeta) \wedge \mathcal{R}_{0,r+1}(\zeta, z) = H(z).$$

Since $\chi = 1$ on U, H is of class $\mathcal{C}^{l+1}$ on a neighborhood z_0 (see Theorem 0.1(i)). Now by Theorems 3.10 and 4.13 we can solve the equation $\overline{\partial}_{\mathrm{b}} F = H$ on a neighborhood of z_0 with a $\mathcal{C}^{l+1}$ differential form F. After shrinking U we may set $F^{\pm} = G^{\pm} + F$. ∎

8. The Hartogs–Bochner effect on CR manifolds. It is well known since Ehrenpreis [11] that the Hartogs–Bochner phenomenon is closely related to the solution of $\overline{\partial}$ with compact support. In [17] Henkin studied the solvability of $\overline{\partial}_{\mathrm{b}}$ with compact support in connection with the Hartogs–Bochner effect for smooth CR functions on 1-concave CR manifolds.

In this section we give some generalizations of Henkin's result to the case of CR manifolds and CR functions with less smoothness.

8.1. *Jump formulas.* Let M be a $\mathcal{C}^3$-smooth 1-concave CR submanifold of $\mathbb{C}^n$. Let z_0, M_0 and $\mathcal{R}_{0,0}$ be as in Theorem 0.1. In this subsection we prove some jump properties of the kernel $\mathcal{R}_{0,0}$ which are analogous to ones enjoyed by the Martinelli–Bochner kernel in $\mathbb{C}^n$. To do that we first establish some estimates for $\mathcal{R}_{0,0}$.

LEMMA 8.24. *Let* $\Omega \Subset M_0$ *be a domain with* $\mathcal{C}^2$ *boundary. Let* $0 < \gamma \le 1$ *and* $0 < \alpha \le 1$. *Then*

$$\int_{\substack{\zeta\in\mathrm{b}\Omega \\ |\zeta - z| < \gamma}} \|\mathcal{R}_{0,0}(\zeta, z)_{|\mathrm{b}\Omega}\| |\zeta - z|^\alpha \, d\lambda(\zeta) \le C_\alpha \gamma^\alpha, \tag{8.1}$$

$$\int\limits_{\substack{\zeta\in \mathrm{b}\Omega\\ |\zeta-z|>\gamma}} \|\mathcal{R}_{0,0}(\zeta,z)_{|\mathrm{b}\Omega}\|\,d\lambda(\zeta)\le C(1-\ln\gamma), \tag{8.2}$$

$$\int\limits_{\substack{\zeta\in \mathrm{b}\Omega\\ |\zeta-z^1|>2\gamma}} \|(\mathcal{R}_{0,0}(\zeta,z^1)-\mathcal{R}_{0,0}(\zeta,z^2))_{|\mathrm{b}\Omega}\|\,|\zeta-z^2|^{\alpha}\,d\lambda(\zeta)\le C\frac{|z^1-z^2|}{\gamma^{2-\alpha}} \tag{8.3}$$

for all z^1,z^2 in M_0 with $\gamma>|z^1-z^2|$.

Proof. Since $\mathcal{R}_{0,0}(\zeta,z)$ is of maximal holomorphic degree in ζ and M is CR generic then for every collection $a^1,\dots,a^k$ of linearly independent vectors in $\mathbb{R}^k$ there is a constant $C>0$ such that

$$\|\mathcal{R}_{0,0}(\zeta,z)_{|\mathrm{b}\Omega}\|\le C\|\mathcal{R}_{0,0}(\zeta,z)\|\cdot\|\partial\varrho_{a^1}(\zeta)\wedge\ldots\wedge\partial\varrho_{a^k}(\zeta)_{|\mathrm{b}\Omega}\|$$

for $\zeta\in\mathrm{b}\Omega$ and $z\in M_0$ with $z\ne\zeta$. For $1\le l\le k$ set

$$u_{a^l}(\zeta,z):=\operatorname{Im}\sum_{j=1}^{n}\frac{\partial\varrho_{a^l}(\zeta)}{\partial\zeta_j}(\zeta_j-z_j). \tag{8.4}$$

It is clear that (8.4) yields

$$\begin{aligned}&\|\partial\varrho_{a^1}(\zeta)\wedge\ldots\wedge\partial\varrho_{a^k}(\zeta)_{|\mathrm{b}\Omega}\|\\ &\le C\Big(|\zeta-z|^k+\sum_{(j_1,\dots,j_l)\in P'(k)}\|d_\zeta u_{a^{j_1}}(\zeta,z)\wedge\ldots\wedge d_\zeta u_{a^{j_l}}(\zeta,z)\|\cdot\|\zeta-z\|^{k-l}\Big)\end{aligned}$$

for $\zeta\in\mathrm{b}\Omega$ and $z\in M_0$. Then it is not difficult to see from (2.2) and (1.3) (cf. [7]) that the integral in (8.1) is bounded by

$$C\int\limits_{\substack{\zeta\in \mathrm{b}\Omega\\ |\zeta-z|<\gamma}}\frac{d\lambda(\zeta)}{|\zeta-z|^{2n-k-1-\alpha}}$$

and a finite sum of terms of the type

$$\int\limits_{\substack{\zeta\in \mathrm{b}\Omega\\ |\zeta-z|<\gamma}}\frac{\|d_\zeta u_{a^1}(\zeta,z)\wedge\ldots\wedge d_\zeta u_{a^{j_l}}(\zeta,z)_{|\mathrm{b}\Omega}\|\,d\lambda(\zeta)}{\prod_{s=1}^{l}(|u_{a^{j_s}}(z,\zeta)|+|\zeta-z|^2)^{1+1/k}|\zeta-z|^{2n-k-1-l-\alpha-2l/k}}$$

where we have used the following fact:

$$|u_{a^{j_s}}(\zeta,z)|+|\zeta-z|^2\le C(|u_{a^{j_s}}(z,\zeta)|+|\zeta-z|^2).$$

We obtain estimate (8.1) by using Range–Siu's trick (see the proof of Proposition 3.7 in [26]), which allows us to consider $u_{a^{j_1}}(\cdot,z),\dots,u_{a^{j_l}}(\cdot,z)$ as local coordinates on $\mathrm{b}\Omega$. (8.2) and (8.3) are shown in the same way. ∎

Now we can give a jump formula for functions defined on the boundary of a domain in M_0.

PROPOSITION 8.25. *Let* $\Omega \Subset M_0$ *be a domain with* $\mathcal{C}^2$ *boundary. Let* f *be a continuous Hölder function of order* α $(0 < \alpha \leq 1)$ *on* $\mathrm{b}\Omega$ *and* F *the function defined on* $M_0 \setminus \mathrm{b}\Omega$ *by*

$$F(z) := \int_{\zeta \in \mathrm{b}\Omega} f(\zeta)\mathcal{R}_{0,0}(\zeta, z).$$

Then $F_{|\Omega}$ (*resp.* $F_{|M_0 \setminus \overline{\Omega}}$) *has a Hölder continuous extension* F^+ (*resp.* F^-) *of order* $\alpha/2$ *up to* $\mathrm{b}\Omega$ *and* $F^+_{|\mathrm{b}\Omega} - F^-_{|\mathrm{b}\Omega} = f$.

Proof. Let $\widetilde{f}$ be an α-Hölder continuous extension of f to M_0. Set

$$G(z) = \int_{\zeta \in \mathrm{b}\Omega} (f(\zeta) - \widetilde{f}(z))\mathcal{R}_{0,0}(\zeta, z).$$

It follows from (8.1) that G is well defined for $z \in \mathrm{b}\Omega$. Let us now show that G is $\alpha/2$-Hölder continuous on W with $\Omega \Subset W \Subset M_0$.

Let $z^1, z^2 \in W$ and set $\gamma = |z^1 - z^2|^{1/2}$. Then we have

$$\begin{aligned} G(z^1) - G(z^2) = & \int_{\substack{\zeta \in \mathrm{b}\Omega \\ |\zeta - z^1| \leq 2\gamma}} (f(\zeta) - \widetilde{f}(z^1))\mathcal{R}_{0,0}(\zeta, z^1) \\ & - \int_{\substack{\zeta \in \mathrm{b}\Omega \\ |\zeta - z^1| \leq 2\gamma}} (f(\zeta) - \widetilde{f}(z^2))\mathcal{R}_{0,0}(\zeta, z^2) \\ & + \int_{\substack{\zeta \in \mathrm{b}\Omega \\ |\zeta - z^1| \geq 2\gamma}} (f(\zeta) - \widetilde{f}(z^2))(\mathcal{R}_{0,0}(\zeta, z^1) - \mathcal{R}_{0,0}(\zeta, z^2)) \\ & + (\widetilde{f}(z^2) - \widetilde{f}(z^1)) \int_{\substack{\zeta \in \mathrm{b}\Omega \\ |\zeta - z^1| \geq 2\gamma}} \mathcal{R}_{0,0}(\zeta, z^1). \end{aligned}$$

Then using Lemma 8.24 and the fact that $\widetilde{f}$ is α-Hölder, one obtains

$$|G(z^1) - G(z^2)| \leq C|z^1 - z^2|^{\alpha/2}.$$

Since $\overline{\partial}_\zeta \mathcal{R}_{0,0}(\zeta, z) = 0$ for $(\zeta, z) \in M_0 \times M_0$ with $z \neq \zeta$ (cf. Theorem 0.1(iii)) we have by Stokes' theorem

$$\int_{\zeta \in \mathrm{b}\Omega} \mathcal{R}_{0,0}(\zeta, z) = 0 \quad \text{for } z \in M_0 \setminus \overline{\Omega}. \tag{8.5}$$

On the other hand, Theorem 0.1(v) gives

$$\int_{\zeta \in \mathrm{b}\Omega} \mathcal{R}_{0,0}(\zeta, z) = 1 \quad \text{for } z \in \Omega. \tag{8.6}$$

(8.5) and (8.6) imply that $G(z) = F(z)$ for $z \in M_0 \setminus \overline{\Omega}$ and $G(z) = F(z) - \widetilde{f}(z)$ for $z \in \Omega$. Setting $F^+ = G + \widetilde{f}$ and $F^- = G$ completes the proof. ■

We now give a $\mathcal{C}^l$ version of the above jump theorem.

PROPOSITION 8.26. *Suppose that M is of class $\mathcal{C}^{l+4}$, $l \geq 0$. Let $\Omega \Subset M_0$ be a domain with $\mathcal{C}^{l+1}$ boundary. Let f be a $\mathcal{C}^{l+1}$ function on* $\mathrm{b}\Omega$ *and let F be the function defined on $M_0 \setminus \mathrm{b}\Omega$ by*

$$F(z) := \int_{\zeta \in \mathrm{b}\Omega} f(\zeta)\mathcal{R}_{0,0}(\zeta, z).$$

Then $F_{|\Omega}$ (resp. $F_{|M_0\setminus\overline{\Omega}}$) has a continuous extension F^+ (resp. F^-) which is of class $\mathcal{C}^{l+1/2}$ up to the boundary and $F^+_{|\mathrm{b}\Omega} - F^-_{|\mathrm{b}\Omega} = f$.

Proof. Let $\widetilde{f}$ be a $\mathcal{C}^{l+1}$ extension of f to M_0. Then it follows from Theorem 0.1(v) that

$$\int_{\mathrm{b}\Omega} f(\zeta) \wedge \mathcal{R}_{0,0}(\zeta, z) = \begin{cases} \widetilde{f}(z) + \int_{\Omega} \overline{\partial}_{\mathrm{b}}\widetilde{f}(\zeta) \wedge \mathcal{R}_{0,0}(\zeta, z) & \text{for } z \in \Omega, \\ \int_{\Omega} \overline{\partial}_{\mathrm{b}}\widetilde{f}(\zeta) \wedge \mathcal{R}_{0,0}(\zeta, z) & \text{for } z \in M_0 \setminus \overline{\Omega}. \end{cases}$$

The result follows from Theorem 0.1(vii) and the fact that $\overline{\partial}_{\mathrm{b}}\widetilde{f}$ is of class $\mathcal{C}^l$. ■

8.2. *Extension theorems.* We are now ready to prove some extension theorems of Hartogs–Bochner type for CR functions on 1-concave CR manifolds.

THEOREM 8.27. *Let X be an n-dimensional complex analytic manifold. Let M be a $\mathcal{C}^3$ CR* 1*-concave submanifold of codimension k in X. Suppose that M has the following property*:

(∗) *For every $\overline{\partial}_{\mathrm{b}}$-closed and compactly supported $(0,1)$-current T of order 0 on M, there is a compactly supported measure S in M such that $\overline{\partial}_{\mathrm{b}}S = T$.*

Let D be a relatively compact domain with $\mathcal{C}^2$ boundary in M such that $M \setminus \overline{D}$ is connected and let f be a CR Hölder continuous function of order α, $0 < \alpha \leq 1$, on ∂D. Then there exists a unique Hölder continuous function F of order $\alpha/2$ on $\overline{D}$ which is CR on D and such that $F(z) = f(z)$ for all $z \in \partial D$. Moreover,

$$\max_{z\in\overline{D}} |F(z)| \leq \max_{z\in \mathrm{b}D} |f(z)|.$$

Proof. We consider the current $T \in [\mathcal{C}^0_{n,n-k-1}(M)]'$ defined by

$$T(\varphi) = \int_{\mathrm{b}D} f\varphi \quad \text{for } \varphi \in \mathcal{C}^0_{n,n-k-1}(M).$$

We have $\operatorname{supp} T = \mathrm{b}D$. Since f is CR on $\mathrm{b}D$ we have $\overline{\partial}_{\mathrm{b}}T = 0$. Now by the condition (∗) there exists a measure $S \in [\mathcal{C}^0_{n,n-k}(M)]'$ such that $\overline{\partial}_{\mathrm{b}}S = T$.

Since M is 1-concave, S is defined by a $\mathcal{C}^3$ CR function on each connected component of $M \setminus \mathrm{b}D$ (cf. Corollary 6.21).

Since $M \setminus \overline{D}$ is connected and S is a compactly supported CR function on $M\setminus\overline{D}$, by Proposition 3.12 and uniqueness of holomorphic functions (see also [1], Theorem 1) one has

$$S = 0 \quad \text{on } M \setminus \overline{D}. \tag{8.7}$$

Denote by F the CR function defining S on D and let us study the regularity of F up to the boundary. Since the problem is local we may work in an open subset of $\mathbb{C}^n$. Let $z_0 \in \mathrm{b}D$ and M_0 be a neighborhood of z_0 where the integral representation from Theorem 0.1 holds. Let χ be a smooth cutoff function on M such that $\operatorname{supp}\chi \Subset M_0$ and $\chi \equiv 1$ on a neighborhood U of z_0.

Set $T' = \overline{\partial}_{\mathrm{b}}(\chi S)$. Then $T' \in [\mathcal{C}^0_{n,n-k-1}(M_0)]'$. It follows from Proposition 6.16 that

$$\chi S = (-1)^k \widehat{\mathcal{R}}_{n-k-1} T'. \tag{8.8}$$

By Lemma 6.18, $\widehat{\mathcal{R}}_{n-k-1}T'$ is defined on $M_0 \setminus \operatorname{supp} T'$ and in particular on $(M_0 \setminus \mathrm{b}D) \cap U$ by the continuous function $(-1)^k T'(\mathcal{R}_{n,n-k-1}(z,\cdot))$.

Let $\Omega \Subset M_0 \cap \overline{D}$ be an open subset of M_0 such that $\operatorname{supp}\chi \cap \mathrm{b}D \Subset \mathrm{b}\Omega$. For $z \in M_0 \setminus \operatorname{supp} T'$ we have

$$T'(\mathcal{R}_{n,n-k-1}(z,\cdot)) = \langle(\overline{\partial}_{\mathrm{b}}\chi)S, \mathcal{R}_{0,0}(\cdot,z)\rangle + \int\limits_{\mathrm{b}\Omega} f(\zeta)\chi(\zeta)\wedge\mathcal{R}_{0,0}(\zeta,z). \tag{8.9}$$

Since $\overline{\partial}_{\mathrm{b}}\chi = 0$ on U, we see from Theorem 0.1(i) that

$$\langle(\overline{\partial}_{\mathrm{b}}\chi)S, \mathcal{R}_{0,0}(\cdot,z)\rangle \text{ is a } \mathcal{C}^2 \text{ function on } U. \tag{8.10}$$

Set $\Omega^+ := \Omega$ and $\Omega^- := M_0 \setminus \overline{\Omega}$. It follows from (8.7)–(8.10) and Proposition 8.25 that F has an $\alpha/2$-Hölder continuous extension to $\overline{\Omega^\pm} \cap U$, which we denote also by F, such that $F_{|\mathrm{b}\Omega\cap U} = f_{|\mathrm{b}\Omega\cap U}$.

Now if $w \in D$ is such that $\max_{z\in\mathrm{b}D}|F(z)| < |F(w)|$, it follows from what we have just proved that the function $1/F(z) - F(w)$ could be extended through w. But this is not possible, hence $\max_{z\in\overline{D}}|F(z)| \le \max_{z\in\mathrm{b}D}|f(z)|$. This implies in particular that the extension F is unique. ∎

On $\mathcal{C}^2$-smooth 1-concave CR manifolds, a weaker version of Theorem 8.27 was obtained in [6].

The proof of the following theorem is carried out exactly as above by using Proposition 8.26 instead of Proposition 8.25.

THEOREM 8.28. *Let X be an n-dimensional complex analytic manifold. Let M be a $\mathcal{C}^{l+4}$ CR 1-concave submanifold of codimension k in X ($l \ge 0$). Suppose that M satisfies condition $(*)$. Let D be a relatively compact domain with $\mathcal{C}^{l+1}$ boundary in M such that $M \setminus \overline{D}$ is connected and let f be a CR* *function of class* $\mathcal{C}^{l+1}$ *on* ∂D. *Then there exists a unique* $\mathcal{C}^{l+1/2}$ *function* F *on* $\overline{D}$ *which is CR on* D *and* $F(z) = f(z)$ *for all* $z \in \partial D$.

In conclusion, a few words on the condition $(*)$. In [17] Henkin gave some geometric conditions ensuring $(*)$. More general sufficient geometric conditions were given by Ch. Laurent in [18]. For $l = \infty$ Theorem 8.28 was obtained in both papers. Let us recall Henkin's conditions: Let M be a $\mathcal{C}^{l+2}$-smooth CR 1-concave submanifold of a complex manifold X defined by $M = \{z \in W : \varrho_1(z) = 0, \ldots, \varrho_k(z) = 0\}$, where W is a domain such that $W \Subset X$, and set $M^\varepsilon = \{z \in W : \varrho_1(z) = \varepsilon_1, \ldots, \varrho_k(z) = \varepsilon_k\}$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$. Let $\varepsilon_0 > 0$ be such that the manifold M^ε is 1-concave for all ε with $|\varepsilon| < \varepsilon_0$. We let W_0 be the union $\bigcup_{|\varepsilon|<\varepsilon_0} M^\varepsilon$. Henkin proved that for every sufficiently small strictly pseudoconvex θ in $\mathbb{C}^n$ with $\theta \Subset W_0$, $M \cap \theta$ satisfies the condition $(*)$. A small modification of Henkin's arguments yields the following general result:

THEOREM 8.29. *Let* $z_0 \in M$ *and* B *a small ball centered at* z_0 *with* $B \Subset W_0$ *and such that Theorem* 0.1 *holds on* $M_0 = M \cap B$. *Then for every* $\overline{\partial}_{\mathrm{b}}$*-closed and compactly supported* $(0,1)$*-current* T *of order* l *on* M_0, *there is a compactly supported current* S *of order* l *on* M_0 *such that* $\overline{\partial}_{\mathrm{b}} S = T$.

Proof. We use the notations of Section 6. By Theorem 8 of [1] for any $\varphi \in \mathcal{D}^{l+1}_{n,n-k-1}(M_0)$ the $\mathcal{C}^l$ form $\varphi - \mathcal{R}^{M_0}_{n,n-k-1}\overline{\partial}_{\mathrm{b}}\varphi$, which is CR by Theorem 0.1, can be approximated in the $\mathcal{C}^l$ topology by $\overline{\partial}$-exact $\mathcal{C}^\infty_{n,n-k-1}$ forms on B. Since $\overline{\partial}_{\mathrm{b}} T = 0$, we have $T(\varphi - \mathcal{R}^{M_0}_{n,n-k-1}\overline{\partial}_{\mathrm{b}}\varphi) = 0$ and therefore

$$-\overline{\partial}_{\mathrm{b}}\widehat{\mathcal{R}}^{M_0}_{n-k-1}T(\varphi) = T(\mathcal{R}^{M_0}_{n,n-k-1}\overline{\partial}_{\mathrm{b}}\varphi) = T(\varphi).$$

Since $\mathcal{D}^{l+1}_{n,n-k-1}(M_0)$ is dense in $\mathcal{D}^{l}_{n,n-k-1}(M_0)$, we obtain

$$-\overline{\partial}_{\mathrm{b}}\widehat{\mathcal{R}}^{M_0}_{n-k-1}T = T.$$

Now denote by ω_T the connected component of $M_0 \setminus \operatorname{supp} T$ whose boundary contains the boundary of M_0. From Lemma 6.18 it follows that on $M_0 \setminus \operatorname{supp} T$, $\widehat{\mathcal{R}}^{M_0}_{n-k-1}T$ is defined by a $\mathcal{C}^l$ CR function. If we choose a ball $B' \Subset B$ centered at z_0 and such that $\operatorname{supp} T \Subset B'$, then for each $\varphi \in \mathcal{D}^l_{n,n-k}(M_0 \setminus \overline{B'})$ the form $\mathcal{R}^{M_0}_{n-k-1}\varphi$ is CR on $M_0 \cap \overline{B'}$ (cf. Theorem 0.1(v)) and then can be approximated there in the $\mathcal{C}^l$ topology by $\overline{\partial}$-exact $\mathcal{C}^\infty_{n,n-k-1}$ forms on $\overline{B'}$ (see Theorem 8 of [1]). But since $\overline{\partial}_{\mathrm{b}} T = 0$ and $\operatorname{supp} T \subset B'$, this implies that $\widehat{\mathcal{R}}^{M_0}_{n-k-1}T(\varphi) = T(\mathcal{R}^{M_0}_{n,n-k-1}\varphi) = 0$ for all such φ. Hence $\widehat{\mathcal{R}}^{M_0}_{n-k-1}T = 0$ on $M_0 \setminus \overline{B'}$. By Proposition 3.12 and uniqueness of holomorphic functions, it follows that $\widehat{\mathcal{R}}^{M_0}_{n-k-1}T = 0$ on ω_T. We set $S = -\widehat{\mathcal{R}}^{M_0}_{n-k-1}T$. Then S is of order l since T is of order l and for any compact subset K of M_0, there is a positive constant C such that for every

$(n, n-k-1)$-form φ with support in K,

$$|\mathcal{R}^{M_0}_{n,n-k-1}\varphi|_l \le C|\varphi|_l,$$

where, $|\varphi|_l$ is the usual $\mathcal{C}^l$-norm of φ on M_0 (cf. [7]). ∎

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