ANNALES
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On quasianalytic classes of functions, expansible in series

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1. Denote by C_I^∞ the class of real functions $f(x)$, infinitely derivable in an interval I (closed or open, finite or infinite) and let $\{m_n\}$ be an arbitrary sequence of positive numbers. We say that $f\in C\{m_n\}\subset C_I^\infty$, if there exists a constant $A=A_f$ such that

$$\sup_{x\in I}|f^{(n)}(x)|\leqslant A^n m_n \quad \text{for} \quad n=0,1,2,\ldots$$

The class $C\{m_n\}$ will be said to be *quasianalytic* (*q-analytic*) if the conditions $f\in C\{m_n\}$ and $f^{(n)}(x_0)=0$, where $x_0\in I$, $n=0,1,2,\ldots$, imply $f(x)\equiv 0$. In the sequel we shall need two conditions of q-analicity:

1.1. *Write*

$$\beta_n=\inf_{k\geqslant n}\sqrt[k]{m_k}, \qquad T(r)=\sup_{n\geqslant 1}\frac{r^n}{m_n}, \qquad \textit{where} \quad r\geqslant 1;$$

the class $C\{m_n\}$ is q-analytic if and only if

$$\sum_{n=1}^{\infty}\frac{1}{\beta_n}=\infty \quad \textit{or} \quad \int_1^\infty \frac{\ln T(r)}{r^2}dr=\infty.$$

The first of these conditions is due to Carleman [1] and the second to Ostrowski [5]. *If the series or the integral in* 1.1 *are convergent, then there exists in $\langle a,b\rangle$ a non-negative function $f(x)\not\equiv 0$ (non-identically equal to zero) satisfying the following conditions*:

$$f^{(n)}(a)=f^{(n)}(b)=0, \quad |f^{(n)}(x)|<m_n \quad \textit{and} \quad f(a+x)=f(b-x)$$

for $n=0,1,2,\ldots$ and $x\in\langle a,b\rangle$.

Now let us assume $f(x)$ to be expanded in a series $\sum a_n\varphi_n(x)$. Clearly, if $\varphi_n\in C_I^\infty$ for $n=0,1,2,\ldots$ and the coefficients a_n satisfy suitable conditions, then $f(x)$ will belong to some q-analytic class. Such conditions given by suitable inequalities for the coefficients a_n were introduced by de la Vallée Poussin [6] and Mandelbrojt [4], $\varphi_n(x)$ being the trigonometrical system. The following definition will be applied in formulating the theorem of Mandelbrojt:

(M) *A function* $a(x)$, *defined for* $x>0$ *satisfies condition* (M) *if it is differentiable and if the function* $q(x) = xa'(x)$ *increases to infinity as* $x\to\infty$ [1].

Mandelbrojt proved the following theorem:

1.2. *Let* $a(x)$ *satisfy condition* (M).

I. *If* $|a_n| < e^{-a(n)}$, $|b_n| < e^{-a(n)}$ *and the integral*

$$(*)\qquad \int_1^\infty \frac{a(x)}{x^2}\,dx$$

is divergent, then the function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx),$$

$$\text{where}\quad f^{(n)}(0) = 0 \quad \text{for} \quad n = 0, 1, 2\ldots$$

identically equals zero.

II. *If the integral* $(*)$ *is convergent, then there exists a non-negative periodic function* $f(x)\not\equiv 0$ *with period* 2π, *even and satisfying the equalities*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad f^{(n)}(0) = 0$$

for every x *and* $n = 0, 1, 2, \ldots$, *with* $|a_n| < e^{-a(n)}$.

The theorem of Mandelbrojt I is a generalization of the theorem of de la Vallée Poussin. The latter author assumes that $xa'(x) > Ca(x)$ with a positive constant C and that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

In his proof de la Vallée Poussin applies, besides the condition of Carleman, only elementary methods. Mandelbrojt uses in his proof the condition of Ostrowski, but applies moreover a special lemma on entire functions. This lemma makes possible a further weakening of the assumptions on the coefficients a_n, but the modified assumptions are much more complicated.

[1] This implies continuity of $q(x)$, since $q(x) = (xa(x) - \int a(x)\,dx)'$.

2. We shall now show that the theorem of Mandelbrojt can be generalized by applying the following elementary lemma:

2.1. *Let us assume* $a(x)$ *to satisfy condition* (M). *Then the functions* $\Phi_k(x) = x^k e^{-a(x)}$ *have one and only one maximum in* $(0,\infty)$, *obtained for* $x = x_k$ *satisfying the equality*

$$(\mathrm{R})\qquad q(x_k) = k,$$

where $k = 1, 2, \ldots$ *Moreover, the sequence* $m_{nl} = \Phi_{n+l}(x_{n+l})$ [2] *satisfies the inequalities*

$$\frac{1}{r^{l+1}}e^{a(r)} \leqslant T(r) \leqslant \frac{1}{r^l}e^{a(r)}$$

for sufficiently large r *and for arbitrary fixed non-negative integer* l, $T(r)$ *being defined in* 1.1.

Proof. The properties of the function $q(x)$ imply that the equation $q(x) = k$ has for $k \geqslant n_0$ a unique solution $x = x_k$ and the function $\Phi_k(x)$ increases for $x < x_k$ and decreases for $x > x_k$, $k > n_0$. The definition of $T(r)$ yields

$$\ln T(r) = \max_{n\geqslant n_0}\{n\ln r - (n+1)\ln x_{n+l} + a(x_{n+l})\}.$$

Applying the equation (R) and the definition of $q(x)$ we obtain

$$\ln T(r) = \max_{n\geqslant n_0}\left\{q(x_{n+l})\int_{x_{n+l}}^{r}\frac{dx}{x} - \int_{x_{n+l}}^{r}\frac{q(x)}{x}\,dx + a(r) - l\ln r\right\}$$
$$= \max_{n\geqslant n_0}\{P(x_{n+l}, r) + a(r) - l\ln r\},$$

wherein

$$P(x_{n+l}, r) = q(x)\int_{x_{n+l}}^{r}\frac{dx}{x} - \int_{x_{n+l}}^{r}\frac{q(x)}{x}\,dx \leqslant 0$$

and $\ln T(r) \leqslant a(r) - l\ln r$ for every $r\geqslant 1$. Evidently,

$$P(x_{n+l}, r) \geqslant -\{q(r) - q(x_{n+l})\}(\ln r - \ln x_{n+l})$$

for $x_{n+l} < r$. Moreover, for sufficiently large r there exists an index n such that $q(x_{n+l}) = [q(r)]$. Then, for sufficiently large r and a certain n, $P(x_{n+l}, r) \geqslant -\ln r$. Hence, $\ln T(r) \geqslant a(r) - (l+1)\ln r$. Thus we have obtained

$$a(r) - (l+1)\ln r \leqslant \ln T(r) \leqslant a(r) - l\ln r.$$

[2] Since this equality does not define m_{nl} for $n < q(1) - l$, for such n we put $m_{nl} = m_{n_0 l}$, where $n_0 = [q(1) - l]$.

From lemma 2.1 and the condition of Ostrowski 1.1 we obtain the following corollary:

2.2. *Under the indications and assumptions of lemma* 2.1, *all classes* $C\{m_{nl}\}$ *are q-analytic (simultaneously for each* l) *if and only if the integral* $(*)$ *is divergent.*

Now we shall prove the following general theorem. Here I will denote an arbitrary interval, closed or open, finite or infinite.

2.3. *Let the sequence* $\varphi_n^{(k)}(x)$ *of k-th derivatives of the function* $\varphi_n(x)$ *satisfy the inequalities*

$$\max_{x\epsilon I}|\varphi_n^{(k)}(x)| \leqslant \sigma_{nk}$$

for $n = 1, 2, \ldots,\ k = 0, 1, 2, \ldots$ *Moreover, let us assume the following conditions to be satisfied:*

(a) *there exist a sequence* $\{\gamma_n\}$, $\inf\limits_n \gamma_n > 0$, *a non-negative integer* l *and a constant* A *such that* $\sigma_{nk} \leqslant A^k \gamma_n^{k+l}$;

(b) $f(x) = \sum\limits_{n=0}^{\infty} a_n \varphi_n(x)$ *in* I;

(c) *there exist positive functions* $\nu(x)$ *and* $\omega(x)$ *such that* $|a_n| < 1/\nu(\gamma_n)$, $\sum 1/\omega(\gamma_n) < \infty$ *and such that the function* $\alpha(x) = \ln\{\nu(x)/\omega(x)\}$ *satisfies the condition* (M) *and the integral* $(*)$ *is divergent.*

Then the function $f(x)$ *belongs to a q-analytic class; thus, if* $f^{(n)}(x_0) = 0$ *for* $n = 0, 1, 2, \ldots$ *and a point* $x_0 \epsilon I$, *then* $f(x) \equiv 0$ *in* I.

Proof. The assumptions of 2.3 imply

$$\sum_{n=0}^{\infty} |a_n \varphi_n^{(k)}(x)| \leqslant \sum_{n=0}^{\infty} \frac{\gamma_n^{k+l}\omega(\gamma_n)\sigma_{nk}}{\nu(\gamma_n)\gamma_n^{k+l}} \cdot \frac{1}{\omega(\gamma_n)}$$

$$\leqslant \sum_{n=0}^{\infty} A^k \Phi_{k+l}(\gamma_n) \frac{1}{\omega(\gamma_n)} \leqslant BA^k m_{kl},$$

B being a constant and $\Phi_k(x)$ and m_{kl} being defined in 2.1 Thus, $f \epsilon C\{m_{kl}\}$. It follows from (c) and 2.2 that the class $C\{m_{kl}\}$ is q-analytic.

It easily follows from the method of this proof that:

2.4. *If we replace in* 2.3 *the assumption* (a) *by*

(a′) $\dfrac{\omega(\gamma_n)\sigma_{nk}}{\gamma_n^{k+l}} \leqslant A^k$ *for* $n = 1, 2, \ldots,\ k = 0, 1, 2, \ldots,\ l > 0,\ A > 0$

then $\alpha(x)$ *in* (c) *may be replaced by* $\alpha(x) = \ln \nu(x)$.

3. The following theorem is a consequence of 2.4:

3.1. *Let* $\varphi(x)$ *belong to the class* $C\{1\}$ *in* $(-\infty, \infty)$ *and let* λ_n *be a sequence of numbers such that* $\inf\limits_n \lambda_n > 0$. *Put* $\varphi_n(x) = \varphi(\lambda_n x)$ *and assume* $f(x) = \sum a_n \varphi_n(x)$ *in an interval* I. *Moreover, let the function* $\alpha(x)$ *satisfy condition* (M) *and let the integral* $(*)$ *be divergent. If one of the following two conditions holds:*

1° $\{\lambda_n^{-1}\} \epsilon l^p$ *for a certain* p *and* $|a_n| < e^{-\alpha(\lambda_n)}$,

2° *there exist positive functions* $\nu(x)$ *and* $\omega(x)$ *such that* $|a_n| < 1/\nu(\lambda_n)$, $\sum 1/\omega(\lambda_n) < \infty$ *and* $\alpha(x) = \ln\{\nu(x)/\omega(x)\}$,

then $f(x)$ *belongs to the q-analytic class* $C\{m_{nl}\}$, *the sequence* $\{m_{nl}\}$ *being defined in* 2.1. *Thus, if* $f^{(n)}(x_0) = 0$ *for* $n = 0, 1, 2, \ldots$ *and a point* $x_0 \epsilon I$, *then* $f(x) \equiv 0$ *in* I.

Proof. It follows from $\varphi \epsilon C\{1\}$ that we may take $\sigma_{nk} = A^k \lambda_n^k$. Assuming 1°, theorem 3.1 follows from 2.4 if we put $\gamma_n = \lambda_n$, $l = p$ and $\omega(x) = x^l$, since

$$\frac{\sigma_{nk}\omega(\gamma_n)}{\gamma_n^{k+l}} = \frac{A^k \lambda_n^k \lambda_n^l}{\lambda_n^{k+l}} = A^k \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\omega(\lambda_n)} < \infty.$$

The proof under assumption 2° is similar.

It is clear that theorem 3.1 remains true if $f(x) = \sum\{a_n \varphi_n(x) + b_n \psi_n(x)\}$ and if the assumptions of 3.1 are satisfied for $\varphi(x)$ and a_n as well as for $\psi(x)$ and b_n. Since the expansions of the form 2.3(b) are often applied we shall now consider some special cases. First let us note that if we put $\varphi(x) = \cos x$, $\psi(x) = \sin x$ and $\lambda_n = n$, then 3.1 yields theorem 1.2, I, of Mandelbrojt. The next application with $\varphi(x) = \cos x$ and $\psi(x) = \sin x$ follows for almost periodic functions:

3.2. *Let*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x)$$

be an almost periodic function, $\inf\limits_n \lambda_n > 0$. *Moreover, let* $\alpha(x)$ *satisfy condition* (M) *and let the integral* $(*)$ *be divergent. Then, if one of the assumptions* 1° *and* 2° *of* 3.1 *holds, then* $f(x)$ *belongs to a q-analytic class.*

Let us assume the sequence λ_n to be increasing. Then according to 3.2 it can be supposed that the slower λ_n increases, the weaker are the assumptions to be made about the coefficients a_n. Indeed, let us consider the following examples. If we take $\lambda_n = n^\varrho$, where $\varrho > 0$, we obtain

$|a_n| < e^{-\alpha(n^\varrho)}$, $|b_n| < e^{-\alpha(n^\varrho)}$. On the other hand, for $\lambda_n = \ln n$, it suffices to take $|a_n| < n^{-b}$, $|b_n| < n^{-b}$ with $b > 1$; to prove this we may choose $\nu(x) = e^{bx}$, $\omega(x) = e^{ax}$ and $\alpha(x) = (b-a)x$, where $b > a > 1$.

Theorem 3.2 may be applied to generalize a lemma which is of importance in some considerations of the theory of vector-valued analytic functions and is proved in [3], p. 34.

3.3. *Suppose that the function $\alpha(x)$ satisfies condition* (M), *the integral* (*) *is divergent and $|a_n| < e^{-\alpha(n^r)}$. Then the system of equations*

$$\sum_{n=1}^{\infty} a_n n^{rk+l} = 0, \qquad k = 1, 2, \ldots,$$

where l and $r > 0$ are fixed numbers has only the trivial solution $a_n = 0$ for $n = 1, 2, \ldots$

To prove 3.3, we put $\bar{a}_n = a_n n^l$, $\bar{\alpha}(x) = \alpha(x) + l\ln x$. Evidently, $\bar{\alpha}(x)$ satisfies condition (M) and $\int_1^\infty \bar{\alpha}(x)\,dx/x^2 = \infty$. Then the almost periodic function $f(x) = \sum_{n=1}^{\infty} \bar{a}_n \cos n^r x$ satisfies the assumptions of 3.2. Thus, it belongs to a q-analytic class and since the given system of equations yields $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots$, we obtain $f(x) \equiv 0$. This implies $a_n = 0$ for $n = 1, 2, \ldots$

We now give an application to Bessel series.

3.4. *Denote by $J_\nu(x)$ the ν-th Bessel function for a fixed positive integer ν and by $\{j_n\}$ the increasing sequence of all positive zeros of $J_\nu(x)$. Put $f(x) = \sum_{n=1}^{\infty} a_n J_\nu(j_n x)$. Let $\alpha(x)$ satisfy condition* (M) *and let the integral* (*) *be divergent. If we assume $|a_n| < e^{-\alpha(j_n)}$, then $f(x)$ belongs to a q-analytic class.*

This theorem follows from the fact that $\{j_n^{-1}\} \in l^2$.

4. Theorem 2.3 may also be applied to polynomial orthonormal systems in a finite interval. The result is the same as for trigonometrical Fourier series. The following theorem holds:

4.1. *Let $\varphi_n(x)$ be a system of polynomials ($\varphi_n(x)$ being a polynomial of degree n) orthonormal with weight-function $w(x)$ in $\langle a, b\rangle$, where $w(x) \geqslant m > 0$ in every subinterval $\langle a', b'\rangle \subset (a, b)$, m depending on a' and b'. Moreover, assume $f(x) = \sum a_n \varphi_n(x)$ in (a, b), where $|a_n| < e^{-\alpha(n)}$, $\alpha(x)$ satisfies condition* (M) *and the integral* (*) *is divergent. Then, if $f^{(k)}(x_0) = 0$ for $k = 0, 1, 2, \ldots$ and for a point $x_0 \in (a, b)$, we have $f(x) \equiv 0$ in (a, b).*

Proof. By a suitable linear transformation, the general case may be reduced to the case $a = -1$, $b = 1$. Let us fix an interval $\langle a', b'\rangle \subset (-1, 1)$. We apply the following inequality proved by G. Freud ([2], lemma 1, p. 222):

$$\sum_{\nu=0}^{n} \{\varphi_\nu^{(k)}(x)\}^2 \leqslant A_{a'b'} n^{2k+1} \quad \text{for} \quad n = 1, 2, \ldots, \quad k = 0, 1, 2, \ldots$$

We obtain $|\varphi_n^{(k)}(x)| \leqslant B_{a'b'} n^{k+1}$ for $n = 1, 2, \ldots$, $k = 0, 1, 2, \ldots$ Hence, we may put in 2.4 $\gamma_n = n$, $\sigma_{nk} = B_{a'b'} n^{k+1}$, $\omega(x) = x^2$, $l = 3$, $I = \langle a', b'\rangle$. We then obtain $f(x) \equiv 0$ in $\langle a', b'\rangle$. The interval $\langle a', b'\rangle$ being an arbitrary subinterval of $(-1, 1)$, we have $f(x) \equiv 0$ in $(-1, 1)$.

5. We now proceed to theorem 1.2, II. If $\varphi_n(x)$ is a trigonometrical system, the proof of 1.2, II, by means of lemma 2.1 should not essentially differ from that given by Mandelbrojt. In the case of almost periodic functions some complications occur, the system of functions $\cos\lambda_n x$ and $\sin\lambda_n x$ generally being non-complete. However, the following theorem holds:

5.1. *Let us suppose that the function $\alpha(x)$ satisfies condition* (M) *and that the integral* (*) *is convergent. Further, given a sequence $\{\lambda_n\}$ of positive numbers, let $\{\beta_n\}$ $(n = 1, 2, \ldots)$ denote the double sequence $m\lambda_n$ $(m = 1, 2, \ldots, n = 1, 2, \ldots)$, the arrangement of $\{\beta_n\}$ being arbitrary. Then there exists a non-negative, even, uniformly almost periodic function $f(x) \not\equiv 0$, expanding in the Fourier series $\sum A_n \cos\beta_n x$*(3) *and such that $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots$ and $|A_n| < e^{-\alpha(\beta_n)}$.*

Indeed, let us put $l_n = \pi/\lambda_n$, $I_n = \langle 0, 2l_n\rangle$ and $\bar{\alpha}(x) = \alpha(x) + \ln x$. If we denote by $\{m_n\}$ and $\{\overline{m}_n\}$ the sequences corresponding, by 2.1, to $\alpha(x)$ and $\bar{\alpha}(x)$ with $l = 0$, respectively, 2.2 implies that the classes $C\{m_n\}$ and $C\{\overline{m}_n\}$ are non-q-analytic. Thus, the theorem of Carleman-Mandelbrojt (see the remarks below 1.1) implies for each n the existence of a non-negative, even function $g_n(x) \not\equiv 0$ in I_n such that $g_n^{(k)}(0) = g_n^{(k)}(2l_n) = 0$ and $|g_n^{(k)}(x)| < \overline{m}_n$ for $k = 0, 1, 2, \ldots$ and $x \in I_n$. The function $g_n(x)$ may be defined periodically on the whole straight line by the equality $g_n(x \pm 2kl_n) = g_n(x)$ for $k = 1, 2, \ldots$ Let us expand $g_n(x)$ in a Fourier series

$$\frac{a_0^{(n)}}{2} + \sum_{m=1}^{\infty} (a_m^{(n)} \cos m\lambda_n x + b_m^{(n)} \sin m\lambda_n x)$$

and let a_n be an arbitrary sequence of positive numbers satisfying the inequality $\sum_{n=1}^{\infty} a_n \leqslant \sigma < 1$. We put $f(x) = \sum_{n=1}^{\infty} a_n g_n(x)$. Then $|f^{(k)}(x)| \leqslant \sum_{n=1}^{\infty} a_n |g_n^{(k)}(x)|$

(3) As we know, the coefficients A_n are defined by the formulas

$$A_n = M\{f(x)\cos\beta_n x\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\cos\beta_n x\,dx \quad \text{for} \quad n = 1, 2, \ldots$$

$\leqslant \sigma \overline{m}_k$ in $(-\infty, \infty)$ and $f(x)$ is a uniformly almost periodic function. Obviously, $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots$ and $f(x) \not\equiv 0$, and $\{\beta_n\}$ is the sequence of all Fourier exponents of $f(x)$. Moreover, partial integration yields the inequality

$$|A_n| \leqslant \frac{1}{\beta_n^p} \max_x |f^{(p)}(x)| \leqslant \frac{\sigma \overline{m}_p}{\beta_n^p} \quad \text{for} \quad p = 0, 1, 2, \ldots$$

Hence

$$|A_n| \leqslant \frac{\sigma}{\max\limits_p (\beta_n^p / \overline{m}_p)} = \frac{\sigma}{\overline{T}(\beta_n)}.$$

However, 2.1 with $l = 0$ yields $\overline{T}(r) \geqslant \frac{1}{r} e^{\bar{a}(r)} = e^{a(r)}$. Then $|A_n| \leqslant \sigma e^{-a(\beta_n)} < e^{-a(\beta_n)}$.

References

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Sur les solutions de classe (L^2) de l'équation différentielle $u'' + q(t)u = 0$

par Z. Opial (Kraków)

1. Envisageons l'équation différentielle

$$u'' + q(t)u = 0. \tag{1}$$

Nous dirons que la solution $u(t)$ de cette équation est *de classe* (L^2) si l'on a

$$\int_0^\infty u^2(t)\,dt < +\infty. \tag{2}$$

Désignons par $Q(t)$ la fonction

$$Q(t) = \max_{0 \leqslant s \leqslant t} (|q(s)|, 1).$$

La fonction $Q(t)$ est donc non décroissante dans l'intervalle $\langle 0, +\infty)$ et, par suite, à variation bornée dans tout intervalle fini. De plus $Q(t) \geqslant 1$ et

$$|q(t)| \leqslant Q(t) \tag{3}$$

dans tout l'intervalle $\langle 0, +\infty)$.

Théorème I. *Pour toute solution $u(t)$ de l'équation* (1) *de classe* (L^2) *on a*

$$\int_0^\infty \frac{u'^2(t)}{Q(t)}\,dt < +\infty. \tag{4}$$

Démonstration. En multipliant l'équation (1) par $u(t)$ on obtient

$$-u''(t)u(t) = q(t)u^2(t)$$

d'où, en vertu de (3):

$$-u''(t)u(t)/Q(t) \leqslant u^2(t). \tag{5}$$