Thus we have proved that D is infinitely small of the fifth order. Moreover, we are able to formulate the following

THEOREM 2. *If the curve k is analytic and possesses at the point p a positive first curvature $\varkappa$ and a second curvature τ different from zero, and if the plane curve K has the same first curvature as k has, and if we denote by f and F the areas of the rectilinear pieces of surfaces, spread respectively on the chord c and arc $\breve{pq}$ or on the the chord C and arc $\breve{pQ}$ ($c = \overline{pq}$, $C = \overline{pQ}$), and if we denote by D the difference $F-f$, then we have*

(16) $$\lim_{s\to 0}\sqrt{\frac{540D}{\varkappa s^5}} = |\tau|.$$

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Differential equations for the extremal starlike functions

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In his paper [1] J. A. Hummel has proved that the function $f(z) = z+\ldots$, starlike for $|z| < 1$, for which the functional $\mathrm{re}\{E(a_2, \ldots, a_n)\}$ (E is a regular function) has its extremal value, satisfies the equation

$$\frac{zf'(z)}{f(z)}R(z) = Q(z)$$

where

$$R(z) = \sum_{\nu=2}^{n}\left[\lambda_\nu\sum_{\mu=1}^{\nu-1}\frac{a_\mu}{z^{\nu-\mu}} - \bar{\lambda}_\nu\sum_{\mu=1}^{\nu-1}\bar{a}_\mu z^{\nu-\mu}\right],$$
$$Q(z) = \sum_{\nu=2}^{n}\left[\lambda_\nu\sum_{\mu=1}^{\nu-1}\frac{\mu a_\mu}{z^{\nu-\mu}} + (\nu-1)\lambda_\nu a_\nu + \bar{\lambda}_\nu\sum_{\mu=1}^{\nu-1}\mu\bar{a}_\mu z^{\nu-\mu}\right].\qquad \lambda_\nu = \frac{\partial E}{\partial a_\nu},$$

It is easily seen that this equation concerns also the starlike functions in the ring $0 < |z| < 1$, because, if $f(z) = z+a_2z^2+\ldots$ is a starlike function, then $F(z) = 1/f(z)$ is also a starlike one in $0 < |z| < 1$ and vice versa. The coefficients of the function $F(z)$ are expressed with the coefficients of the functions $f(z)$ in the form of polynomials and vice versa.

Now let us study the class $\mathcal{G}$ of the starlike functions

(1) $$F(z) = \frac{1}{z}+b_0+b_1z+\ldots,\qquad 0 < |z| < 1.$$

Using the results of another paper of mine [3] we can strengthen the above result, at the same time simplifying the proof. Thus we have the following

THEOREM. *Let $E\big(F(z)\big) = E(b_1, b_2, \ldots, b_n) = E(x_1, \ldots, x_n; y_1, \ldots, y_n)$, $b_k = x_k+iy_k$, be a real function, differentiable with respect to each variable such as at every of the space of variability of the coefficients of the class $\mathcal{G}$ the function $\sum_{k=1}^{n}[(\partial E/\partial x_k)^2+(\partial E/\partial y_k)^2] \neq 0$. Then the function $F(z)$ for* *which the functional* $E(F)$ *has an extremal value satisfies the following differential equations* (I) *and* (II):

$$\text{(I)}\qquad \frac{zF'}{F}R_1(z) = Q_1(z)$$

where

$$R_1(z) = \sum_{p=1}^{n+1}\frac{1}{z^p}\sum_{k=-1}^{n-p} b_k\gamma_{p+k} - \sum_{p=1}^{n+1} z^p \sum_{k=-1}^{n-p}\bar{b}_k\bar{\gamma}_{p+k},$$

$$Q_1(z) = \sum_{p=1}^{n+1}\frac{1}{z^p}\sum_{k=-1}^{n-p} kb_k\gamma_{p+k} + \sum_{k=1}^{n+1} k\gamma_{k-1}b_{k-1} + \sum_{p=1}^{n+1} z^p\sum_{k=-1}^{n-p} k\bar{b}_k\gamma_{k+p},$$

$$\gamma_k = \frac{\partial E}{\partial x_k} - i\frac{\partial E}{\partial y_k},$$

$$\text{(II)}\qquad \frac{zF'}{F}R_2(z) = Q_2(z)$$

where

$$R_2(z) = \sum_{p=1}^{n+1}\frac{1}{z^p}\cdot\frac{1}{p}\sum_{k=-1}^{n-p} b_k\gamma_{p+k} - \frac{1}{2}\sum_{p=1}^{n+1}\left[\frac{a_p}{p}\sum_{k=-1}^{n-p} b_k\gamma_{p+k} + \frac{\bar{a}_p}{p}\sum_{k=-1}^{n-p}\bar{b}_k\bar{\gamma}_{p+k}\right] + \sum_{p=1}^{n+1} z^p\frac{1}{p}\sum_{k=-1}^{n-p}\bar{b}_k\bar{\gamma}_{p+k},$$

$$Q_2(z) = -\sum_{p=1}^{n+1}\frac{1}{z^p}\sum_{k=0}^{n+1-p}\frac{a_k}{p+k}\sum_{l=-1}^{n-p-k} b_l\gamma_{l+p+k} - \frac{1}{2}\sum_{p=1}^{n+1}\left(\frac{a_p}{p}\sum_{k=-1}^{n-p} b_k\gamma_{p+k} - \frac{\bar{a}_p}{p}\sum_{k=-1}^{n-p}\bar{b}_k\bar{\gamma}_{p+k}\right) + \sum_{p=1}^{n+1} z^p\sum_{k=0}^{n+1-p}\frac{\bar{a}_k}{p+k}\sum_{l=-1}^{n-p-k}\bar{b}_l\bar{\gamma}_{l+p+k},$$

$$\text{(2)}\qquad a_k = (-1)^k\begin{vmatrix} b_0 & 1 & 0 & \dots & 0 \\ 2b_1 & b_0 & 1 & \dots & 0 \\ 3b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_{k-1} & b_{k-2} & b_{k-3} & \dots & b_0 \end{vmatrix}.$$

Note 1. If we assume additionally that $E(F)$ equates the real part of an analytical function of n variables and replace the function F by the functions $1/f$, then equation (I) acquires the form of Hummel's equation.

Note 2. In order to obtain an equation satisfied with a function for which the n-th coefficient has the extremal value, we have to put $E = \mathrm{re}\, b_n$, i. e. $\gamma_n = 1$, $\gamma_k = 0$ $(k \neq n)$.

Proof of the theorem. In paper [3] I have proved that the function for which the functional $E(F)$ has its extremum has to be of the form

$$\text{(3)}\qquad F(z) = \frac{1}{z}\prod_{k=1}^{m}(1-\sigma_k z)^{\beta_k}, \quad m \leqslant n+1, \qquad |\sigma_k| = 1, \quad \sigma_k \neq \sigma_j, \quad \sum_{k=1}^{m}\beta_k = 2, \quad \beta_k \geqslant 0$$

(see too [2]). Let us notice that from formula (3) it follows that the function

$$\text{(4)}\qquad \varphi(z) = \frac{zF'}{F} = -\sum_{k=0}^{\infty}\alpha_k z^k, \quad \alpha_0 = 1, \quad \alpha_k = \sum_{l=1}^{m}\beta_l\sigma_l^k$$

is a rational function with simple poles at the points $\bar{\sigma}_k$ only. Now let

$$P(z) = A_0 z^p + \ldots + A_p, \quad p \geqslant m$$

be a polynomial having zeros at the points $\bar{\sigma}_k$, and quite free outside them. We infer from formula (4) that

$$\text{(5)}\qquad \frac{zF'}{F}P(z) = S(z) = B_0 z^p + \ldots + B_p.$$

The polynomial $S(z)$ has m simple zeros at the points at which the function $\varphi(z)$ is equal to zero.

These points are situated on the circle $|z| = 1$ and correspond to the end points of the beams of the star on whose exterior domain the function $F(z)$ maps the ring $0 < |z| < 1$.

The remaining $p-m$ zeros are common for the polynomials $S(z)$ and $P(z)$.

In the paper cited above I have proved that the figures σ_k and β_k defining the extremal function for the functional $E(F)$ satisfy the following system of algebraic equations:

$$\sum_{p=1}^{n+1}\frac{\sigma_k^p}{p}A_p+\lambda+\sum_{p=1}^{n+1}\frac{\bar\sigma_k^p}{p}\bar A_p=0,$$

$$\text{(6)}\qquad \sum_{p=1}^{n+1}\sigma_k^pA_p-\sum_{p=1}^{n+1}\bar\sigma_k^p\bar A_p=0,\qquad k=1,\ldots,m,$$

$$A_p=\sum_{l=-1}^{n-p}b_l\gamma_{p+l},\qquad \gamma_k=\frac{\partial E}{\partial x_k}-i\frac{\partial E}{\partial y_k},\qquad b_k=x_k+iy_k.$$

Now let us define the polynomial

$$R_1(z)=\sum_{p=1}^{n+1}\frac{A_p}{z^p}-\sum_{p=1}^{n+1}z^p\bar A_p.$$

Apparently $R_1(\bar\sigma_k)=0$, whence from (5) and considering that (6) gives

$$\sum_{k=1}^{n+1}A_ka_{p+k}-\sum_{k=1}^{p}\bar A_ka_{p-k}-\sum_{k=0}^{n+1-p}\bar a_k\bar A_{p+k}=0,\qquad p\leqslant n+1,$$

we have

$$-\frac{zF'}{F}R_1(z)=\sum_{p=1}^{n+1}\frac{1}{z^p}\sum_{k=0}^{n+1-p}a_kA_{k+p}+\sum_{k=1}^{n+1}a_kA_k+\sum_{p=1}^{n+1}z^p\sum_{k=0}^{n+1-p}\bar a_k\bar A_{k+p}$$
$$=-Q_1(z).$$

Taking into consideration the definition of the figure A_p and using the evident association (see [3])

$$a_{k-p+1}+b_0a_{k-p}+\ldots+a_1b_{k-p-1}=-(k-p+1)b_{k-p},$$

we have

$$\sum_{k=1}^{n+1}a_kA_k=-\sum_{k=1}^{n+1}k\gamma_{k-1}b_{k-1},\qquad \sum_{k=0}^{n+1-p}a_kA_{k+p}=-\sum_{k=-1}^{n-p}kb_k\gamma_{p+k},$$

which gives equation (I).

Now we define the polynomial

$$R_2(z)=\sum_{p=1}^{n+1}\frac{1}{z^p}\cdot\frac{A_p}{p}+\lambda+\sum_{p=1}^{n+1}z^p\frac{\bar A_p}{p}.$$

Of course $R_2(\bar\sigma_k)=0$. Analogically to the preceding case and considering that (6) gives

$$\sum_{k=1}^{n+1}\frac{A_k}{k}a_k+2\lambda+\sum_{k=1}^{n+1}\frac{\bar A_k}{k}\bar a_k=0,$$

$$\sum_{k=1}^{n+1}\frac{A_k}{k}a_{p+k}+\lambda a_p+\sum_{k=1}^{p}\frac{\bar A_k}{k}a_{p-k}+\sum_{k=0}^{n-p+k}\bar a_k\frac{\bar A_{p+k}}{p+k}=0,\qquad p\leqslant n+1,$$

we have

$$-\frac{zF'}{F}R_2(z)=\sum_{p=1}^{n+1}\frac{1}{z^p}\sum_{k=0}^{n+1-p}a_k\frac{A_{p+k}}{p+k}+\frac{1}{2}\sum_{k=1}^{n+1}\left(\frac{a_kA_k}{k}-\frac{\bar a_k\bar A_k}{k}\right)-$$
$$-\sum_{p=1}^{n+1}z^p\sum_{k=0}^{n+1-p}\bar a_k\frac{\bar A_{p+k}}{p+k}=-Q_2(z).$$

Using the definition of the figure A_p and taking into account the evident equality (2) we get equation (II).

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