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On some new geometrical interpretations of the torsion of a skew curve

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Introduction. While the first curvature of a curve has a great number of geometrical meanings, for the second curvature (or torsion) we do not have many geometrical interpretations. The aim of this note is to give two new geometrical interpretations of torsion, or rather of its modulus. These meanings are contained in two theorems below. Both concern the case when the first as well as the second curvature at the point considered are distinct from zero. Although for defining the torsion analytically it is sufficient to suppose that the curve is of the class C_3^{**} (i. e., that the radius-vector r of the curve is thrice differentiable and moreover that vectors r' and r'' are linearly independent), both theorems below require assumptions of stronger regularity, namely in theorem 1 we shall suppose that the curve is of the class C_5 and in theorem 2 we shall even need the analyticity of the curve k. The idea of the geometrical meaning contained in theorem 2 has arisen from a tendency to generalize the geometrical meaning of the first curvature, which consists of a comparison of the length of an arc with that of a chord and which we find in the thesis of P. Finsler (and perhaps it occurs even earlier), and which has been used so beautifully by J. Haantjes for curves in general metric spaces.

A detailed proof of theorem 2 requires long and cumbersome calculations. We restrict ourselves merely to sketching particular steps in the proof, quoting no detailed calculations and giving only results.

§ 1. Suppose that we are given a curve k of the class C_5^{***}. This means that the radius-vector r of this curve, is five times differentiable with respect to the parameter and, moreover, the vectors r', r'' and r''' are linearly independent. Let us fix a point p on the curve and introduce the arc s, measured from the point p. Taking the origin of the coordinate system at the point p, we can write the expansion

$$r = \sum_{i=1}^{4} \frac{s^i}{i!}\left(\frac{d^i r}{ds^i}\right)_0 + \frac{s^5}{5!}\left\{\left(\frac{d^5 r}{ds^5}\right)_0 + \varepsilon(s)\right\}$$

where $\varepsilon(s)$ is an infinitely small vector. Denoting by (t, n, b) Frenet's trihedron at the point p and making use of Frenet's formulae, we can write

$$\left(\frac{dr}{ds}\right)_0 = t, \quad \left(\frac{d^2r}{ds^2}\right)_0 = \varkappa n, \quad \left(\frac{d^3r}{ds^3}\right)_0 = -\varkappa^2 t + \varkappa' n + \varkappa\tau b,$$

(1)
$$\left(\frac{d^4r}{ds^4}\right)_0 = -3\varkappa\varkappa' t + [\varkappa'' - \varkappa^3 - \varkappa\tau^2]n + [2\varkappa'\tau + \varkappa\tau']b,$$

$$\left(\frac{d^5r}{ds^5}\right)_0 = [-4\varkappa\varkappa'' - 3\varkappa'^2 + \varkappa^4 + \varkappa^2\tau^2]t + [\varkappa''' - 6\varkappa^2\varkappa' - 3\varkappa'\tau^2 - -3\varkappa\tau\tau']n + [3\varkappa''\tau + 3\varkappa'\tau' + \varkappa\tau'' - \varkappa^3\tau - \varkappa\tau^3]b.$$

In the above relations $\varkappa$, τ denote the first and second curvature of the curve k at the point p. According to our suppositions $\varkappa > 0$, $\tau \neq 0$. Now let us denote by K the plane curve whose natural equation is $\varkappa = \varkappa(s)$ where $\varkappa(s)$ is the first curvature of the curve k. The position of the curve K will be uniquely determined in space if at the point corresponding to the point p we suppose that the Frenet's trihedrons for both k and K coincide, and this we accordingly do. If we denote by R the radius-vector of the curve K, we can write

$$R = \sum_{i=1}^{4} \frac{s^i}{i!}\left(\frac{d^iR}{ds^i}\right)_0 + \frac{s^5}{5!}\left\{\left(\frac{d^5R}{ds^5}\right)_0 + \bar{\varepsilon}(s)\right\}$$

where $\bar{\varepsilon}(s)$ is again a vector-function infinitely small, but for $(d^iR/ds^i)_0$ we obtain the simpler formulae

$$\left(\frac{dR}{ds}\right)_0 = t, \quad \left(\frac{d^2R}{ds^2}\right)_0 = \varkappa n, \quad \left(\frac{d^3R}{ds^3}\right)_0 = -\varkappa^2 t + \varkappa' n,$$

(2)
$$\left(\frac{d^4R}{ds^4}\right)_0 = -3\varkappa\varkappa' t + [\varkappa'' - \varkappa^3]n,$$

$$\left(\frac{d^5R}{ds^5}\right)_0 = [-4\varkappa\varkappa'' - 3\varkappa'^2 + \varkappa^4]t + [\varkappa''' - 6\varkappa^2\varkappa']n.$$

Let us examine the order of smallness of

$$\omega(s) = |r| - |R|$$

where $|r|$ is the length of the chord $\overline{pq}$ for the curve k and $|R|$ that of $\overline{pQ}$ for the curve K such that

$$\breve{pq} = \breve{pQ} = s.$$

We have

$$\omega(s) = \sqrt{r\cdot r} - \sqrt{R\cdot R} = \frac{r\cdot r - R\cdot R}{\sqrt{r\cdot r} + \sqrt{R\cdot R}}$$

where a dot denotes a scalar product. We calculate the scalar product $r\cdot r$ making use of the relations (1) and developing it with respect to powers of the variable s. We obtain (taking into account that $t\cdot t = n\cdot n = b\cdot b = 1$ and $t\cdot n = t\cdot b = n\cdot b = 0$)

$$r\cdot r = s^2 + s^4\left(-\frac{2}{3!}\varkappa^2 + \frac{1}{(2!)^2}\varkappa^2\right) + s^5\left\{-\frac{6\varkappa\varkappa'}{4!} + \frac{\varkappa\varkappa'}{3!}\right\} + + s^6\left\{\frac{-8\varkappa\varkappa'' - 6\varkappa'^2 + 2\varkappa^4 + 2\varkappa^2\tau^2}{5!} + \frac{\varkappa\varkappa'' - \varkappa^4 - \varkappa^2\tau^2}{4!} + + \frac{1}{(3!)^2}(\varkappa^4 + \varkappa'^2 + \varkappa^2\tau^2) + \varepsilon^*(s)\right\}$$

where ε^* is a scalar function infinitely small. After reduction we obtain

$$r\cdot r = s^2 - s^4\frac{\varkappa^2}{12} - s^5\frac{\varkappa\varkappa'}{12} + s^6\left\{-\frac{\varkappa\varkappa''}{40} - \frac{\varkappa'^2}{75} + \frac{\varkappa^4}{360} + \frac{\varkappa^2\tau^2}{360} + \varepsilon^*\right\}.$$

Similarly

$$R\cdot R = s^2 - s^4\frac{\varkappa^2}{12} - s^5\frac{\varkappa\varkappa'}{12} + s^6\left\{-\frac{\varkappa\varkappa''}{40} - \frac{\varkappa'^2}{75} + \frac{\varkappa^4}{360} + \bar{\varepsilon}^*\right\}.$$

Hence we have

$$\omega(s) = \frac{s^6(\varkappa^2\tau^2/360 + \varepsilon^* - \bar{\varepsilon}^*)}{s(2 + \varepsilon^{**}(s))} = s^5\left\{\frac{\varkappa^2\tau^2}{720} + \eta(s)\right\}$$

where $\eta(s)$ is infinitely small.

It follows that $\omega(s)$ is infinitely small of the fifth order with respect to s. Thus we have the following

THEOREM 1. *If the curve k is of the class C_5^{***} and K is the plane curve with curvature coinciding with that of k, and if p and P are corresponding (fixed) points on both curves, and q and Q are variable points such that $\breve{pq} = \breve{PQ} = s$, and if c and C denote respectively the chords $c = \overline{pq}$, $C = \overline{PQ}$, and d the difference $d = c - C$, then*

(3)
$$\lim_{s\to 0}\frac{1}{\varkappa}\sqrt{\frac{720d}{s^5}} = |\tau| > 0.$$

§ 2. In order to obtain a second gemetrical meaning of the torsion τ, we again consider the curves k and K; on the basis of the hypothesis of the analyticity of the curves and of the preceding notation, we form two pieces of surfaces. Namely, denoting by t the current point of the arc $\widetilde{pq}$, we denote by t' its projection on the chord $\overline{pq}$. The set of segments tt', with t running through the whole arc $\widetilde{pq}$, will form a rectilinear piece of surface whose area will be denoted by f. In a similar manner we form the surface for the curve K and we denote its area by F; the surface F will evidently be a part of the plane. Since the chords c and C are infinitely small of the first order with respect to s, and since the curvature $\varkappa$ is supposed to be positive, one can foresee that the areas f and F are infinitely small of the third order with respect to s. It will be shown as before that the difference $D = F - f$ will be infinitely small of the fifth order and with the aid of this we shall obtain in the limit just the modulus of the torsion τ.

The current parameter (arc) will now be denoted by σ. Let number s denote for a time a fixed positive value corresponding to the points q and Q on the curves k and K respectively. After calculating f and F we shall let s tend to zero. Vectors $r(s)$ and $R(s)$ will be denoted shortly by a and A respectively.

The equations of the surfaces f and F, as is shown by easy calculation, are

$$(4)\quad r(\sigma, \varrho) = \frac{r(\sigma)\cdot a}{a\cdot a}\cdot a + \varrho\left\{r(\sigma) - \frac{r(\sigma)\cdot a}{a\cdot a}\cdot a\right\} \qquad (0 \leqslant \sigma \leqslant s,\ 0 \leqslant \varrho \leqslant 1)$$

and

$$(5)\quad R(\sigma, \varrho) = \frac{R(\sigma)\cdot A}{A\cdot A}\cdot A + \varrho\left\{R(\sigma) - \frac{R(\sigma)\cdot A}{A\cdot A}\cdot A\right\} \quad (0 \leqslant \sigma \leqslant s,\ 0 \leqslant \varrho \leqslant 1)$$

respectively. The area f is expressed — as we know — by the formula

$$f = \int_0^s d\sigma \int_0^1 \sqrt{g}\, d\varrho$$

where

$$g = g_{11}g_{22} - g_{12}^2, \qquad g_{11} = \left(\frac{\partial r}{\partial \sigma}\right)^2, \qquad g_{12} = \frac{\partial r}{\partial \sigma}\cdot\frac{\partial r}{\partial \varrho}, \qquad g_{22} = \left(\frac{\partial r}{\partial \varrho}\right)^2.$$

The differentiation with respect to parameter σ will be denoted in the sequel by r'. We shall write shortly r instead of $r(\sigma)$. Direct calculations give

$$g_{11} = \frac{(a\cdot r')^2}{a^2} + \varrho^2\left\{1 - \frac{(a\cdot r')^2}{a^2}\right\},$$

$$g_{12} = \varrho\left\{r\cdot r' - \frac{(a\cdot r)(a\cdot r')}{a^2}\right\},$$

$$g_{22} = r^2 - \frac{(a\cdot r)^2}{a^2}.$$

Hence we obtain

$$g = A(\sigma, s)\varrho^2 + B(\sigma, s)$$

where we put

$$(6)\quad \begin{aligned} A &= \frac{1}{a^2}\{a^2r^2 - (a\cdot r)^2 - r^2(a\cdot r')^2 - a^2(r\cdot r')^2 + 2(a\cdot r)(a\cdot r')(r\cdot r')\},\\ B &= \frac{1}{a^4}(a\cdot r')^2\{a^2r^2 - (a\cdot r)^2\}. \end{aligned}$$

In the integral

$$(7)\qquad f = \int_0^s d\sigma \int_0^1 \sqrt{A(\sigma, s)\varrho^2 + B(\sigma, s)}\, d\varrho$$

the integration with respect to ϱ may be realised and we obtain after simple transformations

$$f = \frac{1}{2}\int_0^s \sqrt{B}\left\{\sqrt{1+T} + \frac{1}{2\sqrt{T}}\log(1+T) + \frac{1}{\sqrt{T}}\log\left(1 + \sqrt{\frac{T}{1+T}}\right)\right\}d\sigma$$

where we put

$$(8)\qquad T = T(\sigma, s) = \frac{A(\sigma, s)}{B(\sigma, s)}.$$

To obtain an analogous (much simpler) formula for F it is sufficient to put in the above formula $A = 0$ and we obtain

$$F = \int_0^s \sqrt{B_0}\, d\sigma$$

where

$$B_0 = \frac{1}{A^4}(A\cdot R')^2\{A^2R^2 - (A\cdot R)^2\} \qquad (A = R(s)).$$

In what follows we denote for short

$$\Psi \stackrel{\text{df}}{=} \frac{1}{2}\left\{\sqrt{1+T} + \frac{1}{2\sqrt{T}}\log(1+T) + \frac{1}{\sqrt{T}}\log\left(1 + \sqrt{\frac{T}{1+T}}\right)\right\}.$$

We assert that with fixed $s > 0$ we have

$$(9)\qquad \lim_{\sigma\to 0} T = 0.$$

In fact, let us denote by α_0, $\alpha(\sigma)$, $\beta(\sigma)$, $\gamma(\sigma)$ respectively the angles

$$\alpha_0 = \sphericalangle(a, r'(0)), \quad \alpha = \sphericalangle(a, r), \quad \beta = \sphericalangle(a, r'), \quad \gamma = \sphericalangle(r, r').$$

We can write

$$A = r^2(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma),$$
$$B = r^2\sin^2\alpha\cos^2\beta.$$

Under our assumption concerning the curve k, we have $0 < \alpha_0 < \pi/2$ (at least for sufficiently small s); moreover

$$\lim_{\sigma\to 0}\alpha = \alpha_0, \quad \lim_{\sigma\to 0}\beta = \alpha_0, \quad \lim_{\sigma\to 0}\gamma = 0.$$

Hence we have

$$\lim_{\sigma\to 0}\frac{A}{B} = \frac{1-\cos^2\alpha_0-\cos^2\alpha_0-1+2\cos^2\alpha_0}{\sin^2\alpha_0\cos^2\alpha_0} = 0$$

and the relation (9) is proved.

Since T is infinitely small with respect to σ, the function Ψ expanded in a series gives

$$\Psi = 1+\tfrac{1}{6}T+\text{terms of higher order}.$$

Let us now write

$$D = F-f. \tag{10}$$

We have

$$D = \int_0^s \sqrt{B_0}\,d\sigma - \int_0^s \sqrt{B}\,\Psi\,d\sigma = \int_0^s \frac{B_0-B\Psi^2}{\sqrt{B_0}+\sqrt{B}\,\Psi}\,d\sigma \tag{11}$$
$$= \int_0^s \frac{B_0-B-\frac{1}{3}BT+\ldots}{\sqrt{B}+\sqrt{B_0}+\frac{1}{6}\sqrt{B}\,T+\ldots}\,d\sigma.$$

The next question is to expand the functions A, B, B_0 in series. For this purpose we assume that the coordinate system coincides with Frenet's trihedron common for both curves k and K at the point p, and making use of Frenet's formulae (and of their derivatives obtained by further differentiation) we write the expansion of the vector a with respect to s, and the expansions of the vectors $r(\sigma)$ and $r'(\sigma)$ with respect to σ, and after very cumbersome calculations of the scalar products $a\cdot a$, $r\cdot r$, $a\cdot r$, $a\cdot r'$, $r\cdot r'$ we obtain the expansion of the quantity A. We finally get

$$a^2 A = \sum_{i=10}^{\infty} W_i(\sigma, s)$$

where W_i are homogeneous polynomials of degree i with respect to the variables σ, s.

Similarly we obtain the relation

$$a^4 B = \sum_{i=8}^{\infty} V_i(\sigma, s) \tag{12}$$

where V_i are likewise homogeneous polynomials of degree i with respect to σ and s.

Finally we obtain

$$A^4 B_0 = \sum_{i=8}^{\infty} U_i(\sigma, s) \tag{13}$$

where U_i are likewise homogeneous polynomials of the variables σ, s.

In order to obtain the expansion of the function under the sign of integration in formula (11) we multiply the numerator and the denominator by a^4A^4 (because in the expressions for B and B_0, a^4 and A^4 respectively appear in the denominator). Then we obtain

$$D = \int_0^s \frac{a^4A^4B_0-a^4A^4B-\frac{1}{3}a^4A^4BT+\ldots}{a^4A^4(\sqrt{B}+\sqrt{B_0}+\frac{1}{6}\sqrt{B}\,T+\ldots)}\,d\sigma. \tag{14}$$

Since

$$a^4 = \sum_{i=4}^{\infty} u_i s^i \quad (u_4 = 1,\ u_5 = 0),$$
$$A^4 = \sum_{i=4}^{\infty} v_i s^i \quad (v_4 = 1,\ v_5 = 0)$$

we have according to (12), (13)

$$a^4A^4B_0-a^4A^4B = \sum_{i=4}^{\infty} u_i s^i \sum_{j=8}^{\infty} U_j(\sigma, s) - \sum_{i=4}^{\infty} v_i s^i \sum_{j=8}^{\infty} V_j(\sigma, s)$$
$$= s^4(U_8-V_8)+s^4(U_9-V_9)+s^6(u_6U_8-v_6V_8)+s^4(U_{10}-V_{10})+\ldots$$

But, as can be proved on account of the fact that the curves k and K have an identical first curvature,

$$U_8 = V_8 \quad\text{and}\quad U_9 = V_9, \quad u_6 = v_6,$$

so that finally

$$a^4A^4B_0-a^4A^4B = s^4(U_{10}-V_{10})+\ldots$$

Next we have

$$-\frac{1}{3}a^4A^4BT = -\frac{1}{3}A^4a^2a^2B\frac{A}{B} = -\frac{1}{3}A^4a^2a^2A$$

$$= -\frac{1}{3}(s^4+\ldots)(s^2+\ldots)(W_{10}+\ldots) = -\frac{1}{3}s^6W_{10}(\sigma, s)+\ldots$$

It follows that the numerator L of the function in the integral in formula (14) has the lowest term of the fourteenth degree

$$L = s^4(U_{10}-V_{10})+\ldots = \sum_{i=14}^{\infty} L_i(\sigma, s)$$

if $U_{10}-V_{10} \neq 0$.

Now let us consider the denominator M of the function under the sign of integral in (14). Expanding the function T in the infinite series (of homogeneous polynomials) we get

$$T = \sum_{i=0}^{\infty} T_i(\sigma, s).$$

Then from the identity

$$a^2a^2A = Ta^4B$$

we can write

$$(s^2+\ldots)\sum_{i=10}^{\infty} W_i = \sum_{i=0}^{\infty} T_i \sum_{j=8}^{\infty} V_j,$$

whence by comparing the two sides we obtain

$$T_0 = T_1 = T_2 = T_3 = 0, \quad T_4 = \frac{s^2W_{10}}{V}.$$

Further, expanding $\sqrt{a^4B}$ in the series

$$\sqrt{a^4B} = \sum_{i=0}^{\infty} K_i(\sigma, s)$$

and comparing

$$\sum_{i=8}^{\infty} V_i = \Big(\sum_{j=0}^{\infty} K_j\Big)^2$$

we get

$$K_0 = K_1 = K_2 = K_3 = 0, \quad K_4 = \sqrt{V_8}.$$

In an analogous way we have

$$\sqrt{A^4B_0} = \sum_{i=4}^{\infty} \overset{0}{K}_i(\sigma, s)$$

where $\overset{0}{K}_4 = K_4$. Thus we have

$$M = (s^4+\ldots)(s^2+\ldots)(K_4+\ldots)+(s^4+\ldots)(s^2+\ldots)(\overset{0}{K}_4+\ldots)+$$

$$+(s^4+\ldots)(s^2+\ldots)(K_4+\ldots)(T_4+\ldots)+\ldots = 2s^6K_4+\ldots = \sum_{i=10}^{\infty} M_i(\sigma, s).$$

At last we expand the whole expression under the sign of integral in (14)

$$\frac{L}{M} = R = \sum_{i=0}^{\infty} R_i(\sigma, s).$$

From the comparison of the series

$$\sum_{i=14}^{\infty} L_i = \sum_{i=10}^{\infty} M_i \sum_{j=0}^{\infty} R_j$$

we obtain

$$R_0 = R_1 = R_2 = R_3 = 0, \quad R_4 = L_{14}/M_{10}.$$

Let us remember that we have

$$M_{10} = 2s^6K_4 = 2s^6\sqrt{V_8}, \quad L_{14} = s^4(U_{10}-V_{10}).$$

Now an easy calculation gives

$$V_8 = \frac{\varkappa^2}{4}\sigma^2s^4(s-\sigma)^2.$$

A slightly longer calculation (which we omit) gives

$$L_{14} = \frac{\varkappa^2\tau^2}{72}\sigma^2s^8(s-\sigma)(s^3-2\sigma s+2\sigma^2s-\sigma^3).$$

Hence we finally obtain

$$R_4 = \frac{\varkappa\tau^2}{72}\sigma(s^3-2\sigma s^2+2\sigma^2s-\sigma^3),$$

whence

$$D = \int_0^s (R_4+\ldots)d\sigma = \frac{\varkappa\tau^2}{72}\int_0^s [\sigma s^3-2\sigma^2s^2+2\sigma^3s-\sigma^4+\ldots]d\sigma \tag{15}$$

$$= \frac{\varkappa\tau^2}{72}s^5\left(\frac{1}{2}-\frac{2}{3}+\frac{2}{4}-\frac{1}{5}\right)+\ldots = \frac{\varkappa\tau^2}{540}s^5+\ldots$$

Thus we have proved that D is infinitely small of the fifth order. Moreover, we are able to formulate the following

THEOREM 2. *If the curve k is analytic and possesses at the point p a positive first curvature $\varkappa$ and a second curvature τ different from zero, and if the plane curve K has the same first curvature as k has, and if we denote by f and F the areas of the rectilinear pieces of surfaces, spread respectively on the chord c and arc $\breve{pq}$ or on the the chord C and arc $\breve{pQ}$ ($c = \overline{pq}$, $C = \overline{pQ}$), and if we denote by D the difference $F-f$, then we have*

$$\lim_{s\to 0}\sqrt{\frac{540D}{\varkappa s^5}} = |\tau|. \tag{16}$$

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Differential equations for the extremal starlike functions

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In his paper [1] J. A. Hummel has proved that the function $f(z) = z+\ldots$, starlike for $|z|<1$, for which the functional $\operatorname{re}\{E(a_2,\ldots,a_n)\}$ (E is a regular function) has its extremal value, satisfies the equation

$$\frac{zf'(z)}{f(z)}R(z) = Q(z)$$

where

$$R(z) = \sum_{\nu=2}^{n}\Big[\lambda_\nu\sum_{\mu=1}^{\nu-1}\frac{a_\mu}{z^{\nu-\mu}} - \bar\lambda_\nu\sum_{\mu=1}^{\nu-1}\bar a_\mu z^{\nu-\mu}\Big],$$

$$Q(z) = \sum_{\nu=2}^{n}\Big[\lambda_\nu\sum_{\mu=1}^{\nu-1}\frac{\mu a_\mu}{z^{\nu-\mu}} + (\nu-1)\lambda_\nu a_\nu + \bar\lambda_\nu\sum_{\mu=1}^{\nu-1}\mu\bar a_\mu z^{\nu-\mu}\Big].\qquad \lambda_\nu = \frac{\partial E}{\partial a_\nu},$$

It is easily seen that this equation concerns also the starlike functions in the ring $0<|z|<1$, because, if $f(z) = z+a_2z^2+\ldots$ is a starlike function, then $F(z) = 1/f(z)$ is also a starlike one in $0<|z|<1$ and vice versa. The coefficients of the function $F(z)$ are expressed with the coefficients of the functions $f(z)$ in the form of polynomials and vice versa.

Now let us study the class $\mathcal{G}$ of the starlike functions

$$F(z) = \frac{1}{z}+b_0+b_1z+\ldots,\qquad 0<|z|<1. \tag{1}$$

Using the results of another paper of mine [3] we can strengthen the above result, at the same time simplifying the proof. Thus we have the following

THEOREM. *Let $E\big(F(z)\big) = E(b_1, b_2,\ldots,b_n) = E(x_1,\ldots,x_n;y_1,\ldots,y_n)$, $b_k = x_k+iy_k$, be a real function, differentiable with respect to each variable such as at every of the space of variability of the coefficients of the class $\mathcal{G}$ the function $\sum_{k=1}^{n}[(\partial E/\partial x_k)^2+(\partial E/\partial y_k)^2]\neq 0$. Then the function $F(z)$ for*