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On the necessary and sufficient conditions for the analytic function to be univalent or p-valent in the usual and in the generalized sense

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I. It is proved by G. M. Goluzin [1] and W. Wolibner [2] that the function $f(z) = 1/z + b_1 z + \ldots$ in the unit circle $|z| < 1$ is simple (schlicht) if and only if:

(W) *For every polynomial W_n of degree n*

$$\sum_{k=-n}^{\infty} k|c_k|^2 \leqslant 0, \tag{1}$$

where c_k are defined by the formula

$$W_n[f(z)] = \sum_{k=-n}^{\infty} c_k z^k. \tag{2}$$

Condition (W) is a condition for the coefficients b_n, $n = 1, 2, \ldots$, of $f(z)$. This condition includes coeffients of W_n as parameters. We shall show that it is possible to release inequality (1) from these parameters and in that way to receive a condition which includes the coefficients b_n only.

Let $W_n(z) = d_n z^n + \ldots + d_1 z$ and $[f(z)]^m = \sum_{k=-m}^{\infty} b_k^{(m)} z^k$. Then $W_n[f(z)]$ $= \sum_{m=1}^{n} d_m \sum_{k=-m}^{\infty} b_k^{(m)} z^k = \sum_{k=-n}^{\infty} \Big(\sum_{m=1}^{n} d_m b_k^{(m)}\Big) z^k$, and, by (2),

$$c_k = \sum_{m=1}^{n} d_m b_k^{(m)}.$$

Inequality (1) can be written as

$$\sum_{k=-n}^{\infty} k \Big|\sum_{m=1}^{n} d_m b_k^{(m)}\Big|^2 \leqslant 0 \tag{3}$$

or

(4) $$\sum_{i,j=1}^{n}\Big(\sum_{k=-h}^{\infty} kb_k^{(i)}\overline{b_k^{(j)}}\Big)d_i d_j \leqslant 0, \qquad h=\max(i,j).$$

Let us write

(5) $$A_{ij}=\sum_{k=-h}^{\infty} kb_k^{(i)}\overline{b_k^{(j)}}.$$

It is easy to see that $A_{ij}=\bar{A}_{ji}$. Hence the left side of (4) is a Hermitian quadratic form. When n is constant, then inequality (4) expresses that this form is non-negative. It is well known that a Hermitian quadratic form is non-negative if and only if the following inequalities are satisfied:

(6) $$(-1)^m\begin{vmatrix} A_{11} & \dots & A_{1m}\\ \dots & \dots & \dots\\ A_{m1} & \dots & A_{mm}\end{vmatrix}\geqslant 0, \qquad m=1,2,\dots,n.$$

Inequality (1) is equivalent to all the inequalities (6), where $n=1,2,\dots$

The first inequality of (6) is $A_{11}\leqslant 0$. By (5) and $b_k=0$ for $k<-1$, this inequality can be written as

$$\sum_{k=1}^{\infty} k|b_k|^2\leqslant 1.$$

This is the well-known area formula of Bieberbach.

The second inequality of (6) can be written as

$$A_{11}A_{22}-A_{12}A_{21}\geqslant 0.$$

Since (5) and $b_k^{(2)}=0$ for $k<-2$ and $b_k=0$ for $k<-1$ and $b_2^{(2)}=1$, it follows that

$$A_{11}A_{22}-A_{12}A_{21}=\Big(\sum_{k=-1}^{\infty}k|b_k|^2\Big)\Big(\sum_{k=-2}^{\infty}k|b_k^{(2)}|^2\Big)-\Big(\sum_{k=-1}^{\infty}kb_k\overline{b_k^{(2)}}\Big)\Big(\sum_{k=-1}^{\infty}kb_kb_k^{(2)}\Big)$$

$$=-2\sum_{k=-1}^{\infty}k|b_k|^2-\sum_{\substack{i,j=-1\\ i\neq j}}^{\infty} ijb_ib_j^{(2)}(\overline{b_jb_i^{(2)}-b_ib_j^{(2)}}),$$

and finally

$$2\sum_{k=-1}^{\infty}k|b_k|^2+\sum_{\substack{i,j=-1\\ i\neq j}}^{\infty} ijb_ib_j^{(2)}(\overline{b_jb_i^{(2)}-b_ib_j^{(2)}})\leqslant 0.$$

The form of the next inequalities of (6), if they were written as the former ones, would be still more complicated.

II. In this part we shall prove the conditions which are sufficient and necessary for any regular analytic function to be p-valent in the usual and in the generalized sense of Biernacki [3]. These conditions have been proposed by W. Wolibner. They are conditions for the coefficients of these functions too, but their form is very complicated, and therefore it cannot be used in applications.

THEOREM 1. *The function $f(z)$ which is regular in the unit circle $|z|<1$ and for which $|f(z)|<1$ is p-valent (in the usual sense) if and only if for every non-negative polynomial $W(x,y)$ the following inequality is satisfied*

(7) $$\iint_{x^2+y^2<1} W\big(\operatorname{re}f(z),\operatorname{im}f(z)\big)|f'(z)|^2dxdy\leqslant p\iint_{x^2+y^2<1}W(x,y)\,dxdy.$$

Proof. Necessity. Since $f(z)$ is p-valent, we have

(8) $$\iint_R W(u,v)\,dudv\leqslant p\iint_{x^2+y^2<1}W(x,y)\,dxdy$$

where R indicates Riemann's surface for $f(z)$ in the unit circle $x^2+y^2<1$. After the change, in the first integral of (8), of variables u, v into x, y we receive inequality (7).

Sufficiency. Suppose, on the contrary, that $f(z)$ is not p-valent. Then there is a value w_0 for which $f^{-1}(w_0)$ possesses less than $p+1$ points. Then there is a circle K_0, the centre of which is w_0, such that if $w\in K_0$ then $f^{-1}(w)$ possesses also less than $p+1$ points.

Let us consider a non-negative polynomial $W(x,y)$ which possesses the property

(9) $$\iint_K W(x,y)\,dxdy>(p+1)\iint_{K-K_0}W(x,y)\,dxdy,$$

where K is the unit circle. That polynomial exists, because there exists a continuous function which possesses property (9). The function $f(z)$ induces a representation of the circle K_0 onto a subset of R' Riemann's surface R. It follows from (9), that

$$\iint_R W(u,v)\,dudv=\iint_{R'}W(u,v)\,dudv+\iint_{R-R'}W(u,v)\,dudv$$

$$\geqslant(p+1)\iint_{K_0}W(x,y)\,dxdy+\iint_{R-R'}W(u,v)\,dudv$$

$$\geqslant(p+1)\iint_{K_0}W(x,y)\,dxdy$$

$$=p\iint_K W(x,y)\,dxdy+\Big[\iint_K W(x,y)\,dxdy-(p+1)\iint_{K-K_0}W(x,y)\,dxdy\Big]$$

$$>p\iint_K W(x,y)\,dxdy,$$

contrary to (7).

THEOREM 2. *A function $f(z)$ which is regular in the unit circle $|z|<1$ and for which $|f(z)|<1$ is p-valent in mean for the centre 0 (in the sense of [3]) if and only if for every non-negative polynomial $W(t)$, where t is the real variable, the following inequality is satisfied:*

$$(10)\qquad \int_0^{2\pi}\int_0^1 W(|f(z)|)|f'(z)|^2 r\,dr\,d\varphi \leqslant p\int_0^{2\pi}\int_0^1 W(|z|)r\,dr\,d\varphi .$$

Proof. Necessity. $f(z)$ is p-valent in mean for the centre 0, i. e. for every R, $0<R\leqslant 1$,

$$(11)\qquad \int_0^{2\pi} N(R,\Phi)\,d\Phi \leqslant p\int_0^{2\pi} d\varphi ,$$

where $N(R,\Phi)$ is the number of roots of the equation

$$(12)\qquad f(z)=Re^{i\Phi},\quad \text{where}\quad |z|<1.$$

From (11) it follows that

$$\int_0^1\int_0^{2\pi} W(R)N(R,\Phi)R\,dR\,d\Phi \leqslant p\int_0^1\int_0^{2\pi} W(r)r\,dr\,d\varphi .$$

After the change of variables in the first integral, as in (12), we receive inequality (10).

Sufficiency. Suppose, on the contrary, that $f(z)$ is not p-valent in mean for centre 0. Then there exists an R_0, $0<R\leqslant 1$, such that

$$(13)\qquad \int_0^{2\pi} N(R_0,\Phi)\,d\Phi \geqslant 2\pi p(1+2\varepsilon),$$

where ε is positive. Let us denote by A an open set which contains the set $\underset{\Phi}{E}\{N(R_0,\Phi)=\infty\}$ and for which

$$\int_A N(R_0,\Phi)\,d\Phi \leqslant 2\pi p\varepsilon .$$

Then, by (13), we have

$$(14)\qquad \int_B N(R_0,\Phi)\,d\Phi \geqslant 2\pi p(1+\varepsilon),$$

where $B=\langle 0,2\pi\rangle - A$.

There exists a positive number δ such that for every R belonging to the interval $R_0-\delta\leqslant R\leqslant R_0+\delta$ inequality (14) holds. It is sufficient to prove that if $N(R_0,\Phi_0)$ is finite and equal to q, then there exists

a neighbourhood U of the point (R_0,Φ_0) such that if $(R,\Phi)\in U$ then $N(R,\Phi)\geqslant q$. Let us denote by $z_1,z_2,\ldots,z_q$ all points of the set $f^{-1}[(R_0,\Phi_0)]$. Let V_i, $i=1,2,\ldots,q$, be disjoint neighbourhoods of z_i. Then $U=\prod_{i=1}^{q} f(V_i)$ is the required neighbourhood of (R_0,Φ_0).

Let us consider a non-negative polynomial $W(R)$ for which

$$(15)\qquad \varepsilon\int_{R_0-\delta}^{R_0+\delta} RW(R)\,dR > \int_0^{R_0-\delta} RW(R)\,dR + \int_{R_0+\delta}^{1} RW(R)\,dR .$$

That polynomial exists, because there exists a continuous function which possesses the property (15).

From (14) it follows that

$$\int_0^1\int_0^{2\pi} N(R,\Phi)W(R)R\,dR\,d\Phi \geqslant \int_0^1\int_B N(R,\Phi)W(R)R\,dR\,d\Phi$$
$$\geqslant 2\pi p(1+\varepsilon)\int_0^1 W(R)R\,dR$$

and from (15) we have

$$2\pi p(1+\varepsilon)\int_0^1 W(R)R\,dR = 2\pi p(1+\varepsilon)\Big[\int_0^{R_0-\delta}+\int_{R_0-\delta}^{R_0+\delta}+\int_{R_0+\delta}^{1}\Big]$$
$$\geqslant 2\pi p(1+\varepsilon)\int_{R_0-\delta}^{R_0+\delta} RW(R)\,dR > 2\pi p\int_0^1 RW(R)\,dR,$$

contrary to (10).

THEOREM 3. *A function $f(z)$ which is regular in the unit circle $|z|<1$ and for which $|f(z)|<1$ is p-valent in area (in the sense of [3]) for the centre 0 if and only if for every decreasing polynomial $W(t)$, where t is the real variable, the following inequality is satisfied:*

$$(16)\qquad \int_0^{2\pi}\int_0^1 W(|f(z)|)|f'(z)|^2 r\,dr\,d\varphi \leqslant p\int_0^{2\pi}\int_0^1 W(|z|)r\,dr\,d\varphi .$$

Proof. Necessity. $f(z)$ is p-valent in area for centre 0, i. e. for every R, $0<R\leqslant 1$,

$$(17)\qquad \int_0^R\int_0^{2\pi} N(R,\Phi)R\,dR\,d\Phi \leqslant p\int_0^R\int_0^{2\pi} r\,dr\,d\varphi .$$

Let R_i, where $0<R_i\leqslant 1$ and $i=1,2,\ldots,n-1$, be a sequence of

positive numbers which is decreasing. Let $R_n = 0$. By (17) it follows that

$$\int_0^{R_i}\int_0^{2\pi} N(R,\Phi)R\,dR\,d\Phi \leqslant p\int_0^{R_i}\int_0^{2\pi} r\,dr\,d\varphi \tag{18}$$

for all $i = 1, 2, \ldots, n-1$, and hence

$$\sum_{i=1}^{n} C_i\int_0^{R_i}\int_0^{2\pi} N(R,\Phi)R\,dR\,d\Phi \leqslant p\sum_{i=1}^{n} C_i\int_0^{R_i}\int_0^{2\pi} r\,dr\,d\varphi, \tag{19}$$

where C_i are some numbers.

Now we consider the functions $S_n(R)$, whose value for $R_{m-1} < R < R_m$ is $\sum_{i=1}^{m} C_i$. Since R_i and C_i are arbitrary numbers, the function $S_n(R)$ is an arbitrary step-function which is never-increasing. Then inequality (19) may be written as

$$\int_0^{2\pi}\int_0^{1} S_n(R)N(R,\Phi)R\,dR\,d\Phi \leqslant p\int_0^{2\pi}\int_0^{1} S_n(r)r\,dr\,d\varphi. \tag{20}$$

For every decreasing polynomial $W(R)$ there is a sequence of functions $S_n(R)$ which is uniformly convergent to $W(R)$. Then in inequality (20) we may write $W(R)$ instead of $S_n(R)$. After the change of variables, as in the former theorems, we receive the required inequality.

Sufficiency. Let $f(z)$ not be p-valent in area for centre 0. Then there is such an R_0 that

$$\int_0^{2\pi}\int_0^{R_0} N(R,\Phi)R\,dR\,d\Phi > \pi R_0^2 p. \tag{21}$$

Let $S(R)$ be the function

$$S(R) = \begin{cases} 1 & \text{for} \quad R \leqslant R_0, \\ 0 & \text{for} \quad R > R_0. \end{cases}$$

From (21) it follows that

$$\int_0^{2\pi}\int_0^{1} S(R)N(R,\Phi)R\,dR\,d\Phi > p\int_0^{2\pi}\int_0^{1} S(r)r\,dr\,d\varphi. \tag{22}$$

Now we define a new function $S^*(R)$, which is linear in the following intervals

$$(0, R_0), \quad (R_0, R_0+\delta) \quad \text{and} \quad (R_0+\delta, 1)$$

and furthermore

$$S^*(c) = 1+\delta, \quad S^*(R_0) = 1, \quad S^*(R_0+\delta) = \delta \quad \text{and} \quad S^*(1) = 0.$$

If δ is sufficient small, then we may write S^* instead of S in (22): this is because $\int_0^{R_0}\int_0^{2\pi} N(R,\Phi)R\,dR\,d\Phi$ is a bounded function of the variable R_0 (which is easy to obtain from (16), if we write 1 instead of $W(R)$) and because $N(R,\Phi) \geqslant 0$.

There is a polynomial $W(R)$ which is decreasing and possesses the same values at points 0, R_0, $R_0+\delta$ and 1 as the function S^* and which is furthermore arbitrarily near S^* (by [4]).

Then in inequality (22) we may write W instead of S^*. Thus we receive

$$\int_0^{2\pi}\int_0^{1} W(R)N(R,\Phi)R\,dR\,d\Phi > p\int_0^{2\pi}\int_0^{1} W(r)r\,dr\,d\varphi,$$

contrary to (16).

Remark. In all these theorems the polynomials may be replaced by continuous functions which possess the same properties.

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