where $z_n(x) = g_n(x)/M_n$ $(n = 1, 2, \ldots)$. Moreover, in virtue of (*), (12), (13), (14) and (24), we have

$$\sup_{x \in I_0} |z_n(x)| = 1 \quad (n = 1, 2, \ldots), \tag{32}$$

$$\lim_{n\to\infty} \Delta_h^{(k)} z_n(x) = 0 \tag{33}$$

and for each rational w

$$\lim_{n\to\infty} z_n(w) = 0. \tag{34}$$

Further, if $x+jh \in I_0$ $(j = 0, 1, \ldots, k-1)$, then, according to (32),

$$|z_n(x+kh)| \leqslant |\Delta_h^{(k)} z_n(x)| + \sum_{j=0}^{k-1} \binom{k}{j} |z_n(x+jh)|$$
$$\leqslant |\Delta_h^{(k)} z_n(x)| + 2^k \quad (n = 1, 2, \ldots).$$

Hence and from (33) it follows immediately that

$$\limsup_{n\to\infty} |z_n(x+kh)| \leqslant 2^k.$$

By iterating of this procedure we finally obtain for every finite interval I the inequality

$$\sup_{x \in I} \limsup_{n\to\infty} |z_n(x)| < \infty.$$

Hence and from (33) and (34), applying lemma 2, we obtain the convergence $\lim_{n\to\infty} z_n(x) = 0$ for each x, which contradicts (31). The theorem is thus proved.

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Reçu par la Rédaction le 22. 5. 1958

ANNALES
POLONICI MATHEMATICI
VII (1959)

On a certain method of Toeplitz

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When considering a method of summability we come across a question of basic importance, namely that of the domain in which that method sums the analitical expansion $\sum a_n z^n$ of the function $f(z)$ to the function $f(z)$. The limitability of the geometrical sequence (a^n) plays a decisive part in considerations of this kind. The range of classical methods, as far as the limitability of a geometrical sequence is concerned, is rather restricted. The mean methods (the methods of Hölder and Cesàro), and the continuous methods of Abel-Poisson limit a geometrical sequence within the closed circle $|a| \leqslant 1$. The method of Euler (E, k) limits a geometrical sequence within an open circle $|a+k| < k+1$, adding the point $a = 1$ (see for instance [1], p. 178 below), whereas the classical method of Borel limits a geometrical sequence within the open half-plane $\mathrm{re}\, a < 1$, adding the point $a = 1$ (see [1], p. 183, th. 128).

In this paper we define a permanent method of Toeplitz which limits a geometrical sequence all over the complex plane, namely for $a = 1$ to one, for a real greater than one to ∞, and for any other complex a to zero. In this way the method in question sums the geometrical series $\sum z^n$ to the function $1/(1-z)$ all over the complex plane, with the exception of real numbers $z \geqslant 1$.

A sequence transformed by this method we define as follows:

$$\eta_m = 2^{-m} e^{-m} \sum_{n=0}^{\infty} \frac{m^{n\cdot 2^{-m}}}{\Gamma(n\cdot 2^{-m}+1)} \xi_n. \tag{1}$$

The construction of this method is connected with Borel's continuous method B_k ([4], p. 143) defined by the formula

$$B_k(t, x) = 2^k e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\cdot 2^k}}{\Gamma(n\cdot 2^k+1)} \xi_n.$$

Indeed, we have

$$\eta_m = B_{-m}(m, x); \tag{2}$$

this means that the m-th term of the transform by the method defined thus is "taken out" of the transform by the method B_{-m} (with $t = m$).

To begin our considerations we give the following lemma:

LEMMA 1. *For any complex a, non-negative t and q which is a positive integer the following formula is true:*

$$\sum_{n=0}^{\infty} \frac{t^{n\cdot 2^{-q}} a^n}{\Gamma(n\cdot 2^{-q}+1)} = e^{a^{2^q}\cdot t}\left[1+\sum_{r=1}^{2^q-1}\frac{\varepsilon^r}{\Gamma(r\cdot 2^{-q})}\int_0^{a^{2^q}\cdot t} e^{-u}u^{r\cdot 2^{-q}-1}\,du\right] \tag{3}$$

where ε is one of the roots $\sqrt[2^q]{1}$ chosen suitable for a. If, for $u = \varrho(\cos\varphi + i\sin\varphi)$, where $0 \leqslant \arg u = \varphi < 2\pi$, by u^α we mean $\varrho^\alpha(\cos\alpha\varphi + i\sin\alpha\varphi)$, then $\varepsilon = 1$ if and only if

$$0 \leqslant \arg a < \pi\cdot 2^{-q+1}. \tag{$*$}$$

Proof. In order to prove the above lemma we define a function

$$f(v) = \sum_{n=0}^{\infty}\frac{v^{n+2^{-q}}}{\Gamma(n+2^{-q}+1)} = v^{2^{-q}}\sum_{n=0}^{\infty}\frac{v^n}{\Gamma(n+2^{-q}+1)} \tag{4}$$

where v^α is understood in the above sense, and q is a fixed integer. Function (4) satisfies the differential equation

$$f'(v) - f(v) = \frac{v^{2^{-q}-1}}{\Gamma(2^{-q})} \tag{5}$$

where $f(0) = 0$. Hence the function is

$$f(v) = \frac{e^v}{\Gamma(2^{-q})}\int_0^v e^{-u}u^{2^{-q}-1}du; \tag{6}$$

we integrate along the segment connecting the points 0 and v. Now let us consider the function

$$g(a, t) = \sum_{n=0}^{\infty}\frac{t^{n+2^{-q}}a^{n\cdot 2^q+1}}{\Gamma(n+2^{-q}+1)}. \tag{7}$$

According to the above sense of raising to a power if $a = \varrho e^{i\beta}$ $(0 \leqslant \beta < 2\pi)$, we have

$$(a^{2^q})^{2^{-q}} = [\varrho^{2^q}e^{i2^q\beta}]^{2^{-q}} = \varrho\,[e^{i(2^q\beta - l2\pi)}]^{2^{-q}} = ae^{-il2^{-q+1}\cdot\pi} \tag{8}$$

where

$$l = \operatorname{Ent}\left(\frac{2^{q-1}\beta}{\pi}\right) \quad (\beta = \arg a). \tag{9}$$

Comparing (4), (7) and (8) we receive

$$g(a, t) = \varepsilon f(t\cdot a^{2^q}) \tag{10}$$

where

$$\varepsilon = e^{-il2^{-q+1}\pi} \tag{11}$$

and l is defined by (9). It follows from formula (11) that

$$\varepsilon^{2^q} = 1. \tag{12}$$

Taking into account (6), (7) and (10) we have

$$\sum_{n=0}^{\infty}\frac{t^{n+2^{-q}}a^{n\cdot 2^q+1}}{\Gamma(n+2^q+1)} = \frac{\varepsilon e^{a^{2^q}\cdot t}}{\Gamma(2^{-q})}\int_0^{a^{2^q}\cdot t} e^{-u}u^{2^{-q}-1}du. \tag{13}$$

Likewise for $1 \leqslant r < 2^q$ we have

$$\sum_{n=0}^{\infty}\frac{t^{n+r\cdot 2^{-q}}a^{n\cdot 2^q+r}}{\Gamma(n+r\cdot 2^{-q}+1)} = \frac{\varepsilon^r e^{a^{2^q}\cdot t}}{\Gamma(r2^{-q})}\int_0^{a^{2^q}\cdot t} e^{-u}u^{r\cdot 2^{-q}-1}du. \tag{14}$$

And we also have the obvious equality

$$\sum_{n=0}^{\infty}\frac{t^n a^{n\cdot 2^q}}{\Gamma(n+1)} = e^{a^{2^q}\cdot t}. \tag{15}$$

Adding by parts equality (15) and equalities (14) for $r = 1, 2, \ldots, 2^q-1$ we receive equality (3), where ε is defined by (11) and satisfies condition (12). Let us observe that $\varepsilon = 1$ if $l = 0$. In virtue of formula (9) the equality $l = 0$ is equivalent to inequality $(*)$. In this way the lemma has been proved completely.

THEOREM I. *The method of Toeplitz defined by the matrix*

$$a_{mn} = 2^{-m}e^{-m}\frac{m^{n\cdot 2^{-m}}}{\Gamma(n\cdot 2^{-m}+1)} \tag{17}$$

is permanent, i. e. it limits convergent sequences to their ordinary limits.

Proof. As we know (see for instance [3], p. 117) the following conditions are necessary and sufficient for a Toeplitz method to be permanent (regular):

1° $\lim_{m\to\infty} a_{mn} = 0$ for $n = 0, 1, 2, \ldots$,

2° $\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1$,

3° $\sum_{n=0}^{\infty} |a_{mn}| \leqslant K < \infty$ where K does not depend on m.

Condition 1° is plainly satisfied. We receive the sum $\sum_{n=0}^{\infty} a_{mn}$ appearing in condition 2° by putting in formula (3) $a = 1$, $t = q = m$ and multiplying both sides of the equality obtained by $2^{-m}e^{-m}$; for $a = 1$ we plainly have $\varepsilon = 1$. Hence we have

(18) $$\sum_{n=0}^{\infty} a_{mn} = 2^{-m}\left[1 + \sum_{r=1}^{2^m-1} \frac{1}{\Gamma(r\cdot 2^{-m})} \int_0^m e^{-u} u^{r\cdot 2^{-m}-1} du\right].$$

Using a well-known formula

(19) $$\Gamma(a) = \int_0^{\infty} e^{-u} u^{a-1} du$$

we have the inequality

(20) $$1 - d_m \leqslant \sum_{n=0}^{\infty} a_{mn} \leqslant 1$$

where

(21) $$d_m = 2^{-m} \sum_{r=1}^{2^m-1} \frac{1}{\Gamma(r\cdot 2^{-m})} \int_m^{\infty} e^{-u} u^{r\cdot 2^{-m}-1} du.$$

Further, we notice that for $0 < r < 2^m$ we have

(22) $$\int_m^{\infty} e^{-u} u^{r\cdot 2^{-m}-1} du < \int_m^{\infty} e^{-u} du = e^{-m}.$$

On the other hand, we know (see for instance [2], p. 29) that

(23) $$\Gamma(a) > 1 \quad \text{for} \quad 0 < a < 1.$$

It follows from equalities (21), (22) and (23) that $0 < d_m < e^{-m}$ and taking into account equality (20) we have

(24) $$1 - e^{-m} < \sum_{n=0}^{\infty} a_{mn} < 1.$$

It follows from inequality (24) that condition 2° of the permanence of the method is satisfied.

We notice that all the elements of the matrix a_{mn} are positive, whence it also follows from inequality (24) that the permanent condition 3° is satisfied and $K = 1$. In this way the proof of the theorem I has been given.

LEMMA 2. *The functions defined by the formula*

(25) $$\eta(b, a, t) = e^{bt} \frac{1}{\Gamma(a)} \int_0^{bt} e^{-u} u^{a-1} du$$

(where the integration-path on the right is a straight line) are uniformly bounded by number 6 for all real $t \geqslant 0$, a real, satisfying the inequality $0 < a < 1$ and b complex with their real part $\operatorname{re} b \leqslant 0$.

Proof. We notice that $\eta(0, a, t) = 0$, whence we may assume that $b \neq 0$. Likewise we may assume without reducing generality that

(26) $$|b| = 1.$$

For it can be seen from the formula

(27) $$\eta(b, a, t) = \eta(b_1, a, \varrho t) \quad \text{where} \quad b_1 = b/\varrho, \quad \varrho = |b|$$

that if functions (25) are bounded by number M for b_1 lying on the circle $|b_1| = 1$, then they are also bounded by the same number M for all b.

In the integral on the right-hand side of equality (25) we substitute $u = bv$ and receive

(28) $$\eta(b, a, t) = e^{bt} \frac{b^a}{\Gamma(a)} \int_0^t e^{-bv} v^{a-1} dv.$$

In order to assess the integral on the right-hand side of formula (28) we divide it into a sum of two integrals — from zero to one and from one to t (provided $t > 1$). We assess the first of the integrals

$$\left|\int_0^1 e^{-bv} v^{a-1} dv\right| \leqslant e \int_0^1 v^{a-1} dv = \frac{1}{a} e;$$

hence, taking into account (26) and the supposition $\operatorname{re} b \leqslant 0$ we have

(29) $$\left| e^{bt} \frac{b^a}{\Gamma(a)} \int_0^1 e^{-bv} v^{a-1} dv \right| \leqslant \frac{e^{1+t\operatorname{re} b}}{a\Gamma(a)} \leqslant \frac{e}{\Gamma(a+1)}.$$

Now we want to assess an analogous term, in which appears an integral from one to t. By applying to the integral in question the formula for integration by parts, we have

$$\int_1^t e^{-bv}v^{a-1}dv = \frac{1}{b}(e^{-b}-t^{a-1}e^{-bt}) - \frac{1-a}{b}\int_1^t e^{-bv}v^{a-2}dv;$$

taking into account (26) and writing $\mathrm{re}\, b = -\gamma \leqslant 0$ we have

$$\left|\int_1^t e^{-bv}v^{a-1}dv\right| \leqslant e^{\gamma}+t^{a-1}e^{\gamma t}+(1-a)e^{\gamma t}\int_1^t v^{a-2}dv,$$

whence after calculating the integral on the right-hand side and reducing

(30) $$\left|\int_1^t e^{-bv}v^{a-1}dv\right| \leqslant e^{\gamma}+e^{\gamma t} \leqslant 2e^{\gamma t} \quad (t>1).$$

Now if $t \leqslant 1$, then reasoning in the same way as we did when deriving formula (29) we receive

$$|\eta(b, a, t)| \leqslant \frac{e}{\Gamma(a+1)} \leqslant 4$$

for $1/\Gamma(a+1) < 1{,}2$ when $a > 0$ (see for instance [2], p. 27). Now if $t > 1$, then taking into account (23), (25), (28), (29) and (30) we have

$$|\eta(b, a, t)| \leqslant \frac{e}{\Gamma(a+1)} + \frac{2}{\Gamma(a)} \leqslant 6$$

and ultimately

(31) $$|\eta(b, a, t)| \leqslant 6$$

for all values of the variables b, a and t given in the lemma.

THEOREM II. *Toeplitz's method defined by matrix* (17) *limits the geometrical sequence* (a^n) *all over the complex plane, namely to* 1 *for* $a = 1$, *to* ∞ *for* a *real and greater than* 1 *and to zero for any* a *other.*

Proof. We notice that the transform (1) of the geometrical sequence is

(32) $$\eta_m = 2^{-m}e^{-m}\sum_{n=0}^{\infty}\frac{m^{n\cdot 2^{-m}}a^n}{\Gamma(n\cdot 2^{-m}+1)}.$$

It follows from Stirling's formula,

(33) $$\Gamma(s) \sim \sqrt{2\pi}e^{-s}s^{s-1/2},$$

that the series on the right-hand side of equality (32) is convergent for any a and for any m.

The validity of the theorem for $a = 1$ follows from the permanence of the method (theorem I) and particularly from condition 2°.

Now let us suppose that a is real and greater than one. Then transform (32), in virtue of formula (3), can be written in the following way:

(34) $$\eta_m = 2^{-m}e^{(a^{2^m}-1)m}\left[1+\sum_{r=1}^{2^m-1}\frac{1}{\Gamma(r\cdot 2^{-m})}\int_0^{a^{2^m}\cdot m} e^{-u}u^{r\cdot 2^{-m}-1}du\right].$$

It can be distinctly seen from the shape of the transform that $\lim_{m\to\infty}\eta_m = \infty$, which proves the theorem in this case.

Let us now consider what happens if a is not a real number $\geqslant 1$. We perceive once more that if $|a| < 1$, then the theorem follows from the permanence of the method, for in this case $\lim_{n\to\infty} a^n = 0$. Thus the only case to be considered is

(35) $$0 < \arg a < 2\pi, \quad |a| \geqslant 1.$$

Let a be a complex number satisfying conditions (35). Thus there exists such an m_0 that

(36) $$\arg a > \pi 2^{-m+1} \quad \text{for} \quad m > m_0.$$

In virtue of formula (3) we can write transform (32) as follows:

(37) $$\eta_m = 2^{-m}e^{(a^{2^m}-1)m}\left[1+\sum_{r=1}^{2^m-1}\frac{\varepsilon^r}{\Gamma(r\cdot 2^{-m})}\int_0^{a^{2^m}\cdot m} e^{-u}u^{r\cdot 2^{-m}-1}du\right]$$

where in virtue of lemma 1

(38) $$\varepsilon \neq 1 \quad \text{for} \quad m > m_0.$$

In the further considerations a is fixed, whereas m is variable, yet we continuously assume that

(39) $$m > m_0.$$

According to various values of m we distinguish two possible cases:

I° $\operatorname{re} a^{2m} \leqslant 0$. In this case, since $|\varepsilon| = 1$, applying lemma 2 to the terms on the right-hand side of formula (37), we receive the following assessment for η_m:

$$|\eta_m| \leqslant 6e^{-m}. \tag{40}$$

Let us now consider the second possible case:

II° $\operatorname{re} a^{2m} > 0$. In virtue of supposition (39) formula (38) is satisfied, whence, according to lemma 1, ε is one of the roots $\sqrt[2m]{1}$ but different from 1. In this case

$$1 + \sum_{r=1}^{2m-1} \varepsilon^r = 0. \tag{41}$$

Taking advantage of the above relationship we can write formula (37) as follows:

$$\eta_m = 2^{-m} e^{(a^{2m}-1)m} \sum_{r=1}^{2m-1} \varepsilon^r \Big[-1 + \frac{1}{\Gamma(r \cdot 2^{-m})} \int_0^{a^{2m}\cdot m} e^{-u} u^{r\cdot 2^{-m}-1} du \Big]. \tag{42}$$

With regard to a well-known formula

$$\Gamma(a) = \int_0^{b\cdot\infty} e^{-u} u^{a-1} du, \tag{43}$$

true for complex b the real part of which $\operatorname{re} b$ is positive (see for instance [4], lemma 4, p. 157), we may put relationship (42) in case II° in this way:

$$\eta_m = 2^{-m} e^{(a^{2m}-1)m} \sum_{r=1}^{2m-1} \frac{-\varepsilon^r}{\Gamma(r\cdot 2^{-m})} \int_{a^{2m}\cdot m}^{a^{2m}\cdot\infty} e^{-u} u^{r\cdot 2^{-m}-1} du. \tag{44}$$

It follows from the supposition made (35) that the terms $u^{r\cdot 2^{-m}-1}$ appearing in the integrals in formula (44) are bounded with regard to the absolute value by the number 1, whence each of the integrals on the right-hand side of (44) is bounded with regard to the absolute value by the number $e^{-m \operatorname{re} a^{2m}}$.

According to (23) and the above considerations in this case we receive the following assessment from formula (44):

$$|\eta_m| \leqslant e^{-m}. \tag{45}$$

So we see that ultimately in virtue of formulas (40) and (45) for any complex a, satisfying conditions (35), there exists such an m_0 that the inequality

$$|\eta_m| \leqslant 6e^{-m} \quad \text{for} \quad m > m_0 \tag{46}$$

holds which proves the theorem in this case.

In this way the theorem has been proved for all complex a, which means that the proof of theorem II has been given.

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Reçu par la Rédaction le 21. 2. 1958