Proceeding thus, we shall obtain after n steps

$$|\bar{h}_n(x)-h_n(x)| = \prod_{\nu=1}^{n}\Big|H_z\big(f^{\nu-1}(x),\, h_{n-\nu}[f^\nu(x)]+\vartheta_\nu(\bar{h}_{n-\nu}[f^\nu(x)]-h_{n-\nu}[f^\nu(x)])\big)\Big|\times$$
$$\times|\bar{h}_0[f^n(x)]-h_0[f^n(x)]|$$
$$=\prod_{\nu=1}^{n}\Big|H_z\big(f^{\nu-1}(x),\, h_{n-\nu}[f^\nu(x)]+\vartheta_\nu(\bar{h}_{n-\nu}[f^\nu(x)]-h_{n-\nu}[f^\nu(x)])\big)\Big|\times$$
$$\times|\varphi[f^n(x)]-d|.$$

By the assumption $|H_z| = |F_z/F_y| \leqslant 1$ we have $\prod|H_z| \leqslant 1$ and hence

$$|\bar{h}_n(x)-h_n(x)| < |\varphi[f^n(x)]-d|.$$

Let us take an arbitrary $\varepsilon>0$. Since $\varphi(x)\xrightarrow[x\to b]{} d$, there exists $\delta>0$ such that

$$|\varphi(x)-d|<\varepsilon \quad \text{for} \quad x\,\epsilon\,(b-\delta, b\rangle.$$

$f^n(a+\eta)\xrightarrow[n\to\infty]{} b$, and therefore there exists an index N such that for $n>N$

$$f^n(a+\eta)\,\epsilon\,(b-\delta, b\rangle.$$

Now let us take an arbitrary $x\,\epsilon\,\langle a+\eta, b\rangle$. $f^n(x)\geqslant f^n(a+\eta)$, for $f^n(x)$ is increasing with $f(x)$. Consequently, for $n>N$, $f^n(x)\,\epsilon\,(b-\delta, b\rangle$ and $|\varphi[f^n(x)]-d|<\varepsilon$, whence, for $n>N$ and $x\,\epsilon\,\langle a+\eta, b\rangle$

$$|\bar{h}_n(x)-h_n(x)|<\varepsilon,$$

i. e.

$$|\varphi(x)-h_n(x)|<\varepsilon,$$

which proves that $h_n(x) \underset{\langle a+\eta, b\rangle}{\Rightarrow} \varphi(x)$.

The second part of this theorem may be proved in a quite similar manner.

References

[1] T. Kitamura, *On the solution of some functional equations*, The Tôhoku Mathematical Journal, 49, Part 2, February 1943, p. 305-307.

[2] M. Kuczma, *On the functional equation* $\varphi(x)+\varphi[f(x)] = F(x)$, Ann. Polon. Math. 6 (1959), p. 279-285.

[3] A. H. Read, *The solution of a functional equation*, Proc. Roy. Soc. Edinburgh, Section A, 63 (1952), p. 336-345.

Reçu par la Rédaction le 10. 12. 1957

ANNALES
POLONICI MATHEMATICI
VII (1959)

On a problem of S. L. Cheng concerning sequences of functions with convergent k-th differences

by K. Urbanik (Wrocław)

In the present note we use the notation

$$\Delta_h^{(k)}f(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x+jh) \qquad (k=1,2,\ldots).$$

The aim of this note is to prove the following theorem, which is the solution of a problem raised by S. L. Cheng.

THEOREM. *Let $f_n(x)$ $(n=1,2,\ldots)$ be a sequence of Lebesgue measurable real-valued functions on the line. The convergence*

$$(*) \qquad \lim_{n\to\infty}\Delta_h^{(k)}f_n(x) = 0$$

for each h uniform with respect to x in every finite interval is equivalent to the equalities

$$(**) \qquad f_n(x) = \sum_{j=0}^{k-1}a_{jn}x^j+g_n(x) \qquad (n=1,2,\ldots),$$

where a_{jn} $(j=0,1,\ldots,k-1;\ n=1,2,\ldots)$ are constants and the sequence $g_n(x)$ $(n=1,2,\ldots)$ converges to 0 uniformly in every finite interval.

Remarks. (a) H. Whitney ([2], p. 67-68) has proved the following fundamental theorem:

For each integer $k\geqslant 1$ there is a number C_k with the following property. Let I be any closed finite interval. Then for any continuous function $f(x)$ in I there is a polynomial $P(x)$ of degree at most $k-1$ such that

$$\max_{x\in I}|f(x)-P(x)| \leqslant C_k \max_{x+jh\in I;\ j=0,1,\ldots,k}|\Delta_h^{(k)}f(x)|.$$

If $f_n(x)$ $(n=1,2,\ldots)$ are continuous functions and if the convergence $(*)$ is uniform with respect to h and x in every finite square, then $(**)$ is a direct consequence of the theorem of Whitney.

(b) The theorem would fail if we omitted the hypothesis of measurability. In fact, for $k \geqslant 2$ the sequence $f(x), f(x), \ldots$, where $f(x)$ is a non-measurable function of Hamel ([1]), satisfies (∗) and does not satisfy (∗∗).

(c) As a particular case of the theorem we obtain the following well-known result:

If $f(x)$ is a Lebesgue measurable function and for each h and x $\Delta_h^{(k)} f(x) = 0$, then $f(x)$ is a polynomial of degree at most $k-1$.

Before proving the theorem we shall prove two lemmas.

Lemma 1. *If*

$$\lim_{n\to\infty} \Delta_h^{(k)} f_n(x) = 0 \tag{1}$$

for each h and x, then

$$f_n(x) = \sum_{j=0}^{k-1} a_{jn} x^j + g_n(x) \quad (n = 1, 2, \ldots),$$

where

$$\lim_{n\to\infty} g_n(w) = 0$$

for each rational w.

Proof. For every h $(0 < h \leqslant 1)$ we denote by $Q_{nh}(x)$ the polynomial of degree at most $k-1$ satisfying the equalities

$$Q_{nh}(jh) = f_n(jh) \quad (j = 0, 1, \ldots, k-2, [k/h])\ (^1).$$

By the lemma of Whitney ([2], p. 72) there are numbers $a_0^{(s)}, a_1^{(s)}, \ldots, a_l^{(s)}$ $(l = [k/h]-k)$ such that for any $0 < h \leqslant 1$

$$f_n(sh) = \sum_{j=0}^{l} a_j^{(s)} \Delta_h^{(k)} f_n(jh) + Q_{nh}(sh) \quad (s = 0, 1, \ldots, [k/h]).$$

Hence, taking into account assumption (1), we obtain the convergence

$$\lim_{n\to\infty} \big(f_n(sh) - Q_{nh}(sh)\big) = 0 \quad (0 < h \leqslant 1,\ s = 0, 1, \ldots, [k/h]). \tag{2}$$

Let r be a positive integer. From (2) it follows that

$$\lim_{n\to\infty} \left(f_n\left(sr\frac{h}{r}\right) - Q_{n,h/r}\left(sr\frac{h}{r}\right)\right) = 0 \quad (0 < h \leqslant 1,\ s = 0, 1, \ldots, k-1).$$

Consequently, in view of (2),

$$\lim_{n\to\infty} \big(Q_{nh}(sh) - Q_{n,h/r}(sh)\big) = 0 \quad (0 < h \leqslant 1,\ s = 0, 1, \ldots, k-1;\ r = 1, 2, \ldots).$$

(¹) $[x]$ denotes the greatest integer $\leqslant x$.

Since $Q_{nh}(x)$ are polynomials of degree at most $k-1$, the last formula implies

$$\lim_{n\to\infty} \big(Q_{nh}(x) - Q_{n,h/r}(x)\big) = 0 \quad (0 < h \leqslant 1,\ r = 1, 2, \ldots)$$

uniformly in every finite interval. Hence for every pair of non-negative integers $p < q$

$$\lim_{n\to\infty} \big(Q_{n1}(x) - Q_{n,1/q}(x)\big) = 0, \quad \lim_{n\to\infty} \big(Q_{n,p/q}(x) - Q_{n,1/q}(x)\big) = 0,$$

which implies

$$\lim_{n\to\infty} \big(Q_{n1}(x) - Q_{n,p/q}(x)\big) = 0$$

uniformly in every finite interval. Thus, in virtue of (2), for every rational w $(0 \leqslant w \leqslant 1)$

$$\lim_{n\to\infty} \big(f_n(w) - Q_{n1}(w)\big) = 0. \tag{3}$$

In an analogous way we find that there are polynomials $G_n(x)$ and $H_n(x)$ $(n = 1, 2, \ldots)$ for degree at most $k-1$ such that for rational w $(\frac{1}{2} \leqslant w \leqslant \frac{3}{2})$

$$\lim_{n\to\infty} \big(f_n(w) - G_n(w)\big) = 0 \tag{4}$$

and for rational w $(-\frac{1}{2} \leqslant w \leqslant \frac{1}{2})$

$$\lim_{n\to\infty} \big(f_n(w) - H_n(w)\big) = 0. \tag{5}$$

Hence and from (3) it follows that for every x

$$\lim_{n\to\infty} \big(Q_{n1}(x) - G_n(x)\big) = 0, \quad \lim_{n\to\infty} \big(Q_{n1}(x) - H_n(x)\big) = 0.$$

Consequently, in virtue of (4) and (5), relation (3) holds for every rational w $(-\frac{1}{2} \leqslant w \leqslant \frac{3}{2})$. By iterating this procedure we finally obtain (3) for every rational w. Putting $g_n(x) = f_n(x) - Q_{n1}(x)$ $(n = 1, 2, \ldots)$ we obtain the assertion of the lemma.

Lemma 2. *If for every finite interval I*

$$\sup_{x\in I} \limsup_{n\to\infty} |z_n(x)| < \infty, \quad \lim_{n\to\infty} \Delta_h^{(k)} z_n(x) = 0 \tag{6}$$

for each h and x, and

$$\lim_{n\to\infty} z_n(w) = 0 \tag{7}$$

for each rational w, then the sequence $z_n(x)$ $(n = 1, 2, \ldots)$ converges to 0 for each x.

Proof. Given an arbitrary number x, there are rational numbers w_r $(r = 1, 2, \ldots)$ such that

(8) $$x < w_r \quad (r = 1, 2, \ldots),$$

(9) $$h_r = \frac{w_r - x}{k!} < \frac{1}{r} \quad (r = 1, 2, \ldots).$$

The definition of $\Delta_h^{(k)} z_n(x)$ gives the following equality:

(10) $$z_n(w_r) - z_n(x) = \frac{(-1)^k}{r} \sum_{s=1}^{r} \{\Delta_{sh_r}^{(k)} z_n(w_r) - \Delta_{sh_r}^{(k)} z_n(x)\} -$$
$$- \frac{1}{r} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \sum_{s=1}^{r} \big(z_n(w_r + jsh_r) - z_n(x + jsh_r)\big)$$
$$(n = 1, 2, \ldots; r = 1, 2, \ldots).$$

Moreover, in view of (9), the equality

$$\sum_{s=1}^{r} \big(z_n(w_r + jsh_r) - z_n(x + jsh_r)\big)$$
$$= \sum_{s=r-k!/r+1}^{r} z_n(w_r + jsh_r) - \sum_{s=1}^{k!/j} z_n(x + jsh_r) \quad (n = 1, 2, \ldots; r \geqslant k!)$$

holds. Hence, according to (6), (7) and (10), it follows that

(11) $$\limsup_{n\to\infty} |z_n(x)| \leqslant \limsup_{n\to\infty} |z_n(x) - z_n(w_r)| + \limsup_{n\to\infty} |z_n(w_r)|$$
$$\leqslant \frac{1}{r} \sum_{j=1}^{k} \binom{k}{j} \Big\{ \sum_{s=r-k!/j+1}^{r} \limsup_{n\to\infty} |z_n(w_r + jsh_r)| +$$
$$+ \sum_{s=1}^{k!/j} \limsup_{n\to\infty} |z_n(x + jsh_r)| \Big\} \quad (r \geqslant k!).$$

Since, according to (8) and (9),

$$x < x + jsh_r < w_r + jsh_r \leqslant x + 2k!$$
$$(j = 1, 2, \ldots, k;\ s = 1, 2, \ldots, r;\ r = 1, 2, \ldots)$$

we have, in virtue of (11), the following inequality:

$$\limsup_{n\to\infty} |z_n(x)| \leqslant \frac{1}{r} \sum_{j=1}^{k} \binom{k}{j} 2 \frac{k!}{j} M \leqslant \frac{1}{r} 2^{k+1} k! M \quad (r \geqslant k!),$$

where

$$M = \sup_{x \leqslant y \leqslant x+2k!} \limsup_{n\to\infty} |z_n(y)| < \infty.$$

Hence, letting $r \to \infty$, we obtain the assertion of the lemma.

Proof of the theorem. The sufficiency of (**) is obvious. To prove the necessity of (**) it is sufficient to prove that the sequence $g_n(x)$ $(n = 1, 2, \ldots)$ defined by lemma 1 converges to 0 uniformly in every finite interval. Suppose the contrary, i. e. that there exists an interval I_0 such that for

(12) $$M_n = \sup_{x \in I_0} |g_n(x)| \quad (n = 1, 2, \ldots)$$

we have

$$\limsup_{n\to\infty} M_n > 0.$$

Since we can choose a convergent subsequence $M_{m_n} > 0$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} M_{m_n} > 0$, for the sake of simplicity we shall assume — without restricting the generality of our considerations — that

(13) $$M_n > 0 \quad (n = 1, 2, \ldots),$$

(14) $$0 < \lim_{n\to\infty} M_n \leqslant \infty.$$

Now we shall prove that M_{n_k} are finite for a subsequence $n_1 < n_2 < \ldots$ Suppose the contrary, i. e. that there exists a sequence of points y_{mn} $(m = 1, 2, \ldots; n \geqslant n_0)$ belonging to I_0 such that

(15) $$|g_n(y_{mn})| \geqslant 2^k m + 1 \quad (m = 1, 2, \ldots; n \geqslant n_0).$$

Using the notation

(16) $$d_n(h) = \sup_{m \geqslant 1} |\Delta_h^{(k)} g_n(y_{mn})| \quad (n \geqslant n_0)$$

from the formula

(17) $$g_n(y) = \Delta_h^{(k)} g_n(y) - \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} g_n(y + jh)$$

we obtain the inequality

(18) $$|g_n(y_{mn})| \leqslant d_n(h) + 2^k \max_{1 \leqslant j \leqslant k} |g_n(y_{mn} + jh)| \quad (m = 1, 2, \ldots; n \geqslant n_0).$$

By assumption the functions $f_n(x)$ are measurable. Consequently, the functions $d_n(h)$ are also measurable. (The functions $d_n(h)$ may take on the values ∞). Putting

$$A_n = \{h: 0 < h \leqslant 1, d_n(h) \leqslant 1\} \quad (n = 1, 2, \ldots)$$

we have, according to (*) and (16),

$$\lim_{n\to\infty} \operatorname{mes} A_n = 1, \tag{19}$$

where $\operatorname{mes} A$ denotes the Lebesgue measure of A. Further, in view of (18), we obtain the inequality

$$\max_{1\leqslant j\leqslant k} |g_n(y_{mn}+jh)| \geqslant m \quad \text{for} \quad h\in A_n, \quad m = 1, 2, \ldots, \quad n \geqslant n_0. \tag{20}$$

Put

$$B_{mj}^{(n)} = \{y_{mn}+jh\colon 0 < h \leqslant 1,\ |g_n(y_{mn}+jh)| \geqslant m\} \tag{21}$$
$$(j = 1, 2, \ldots, k;\ m = 1, 2, \ldots;\ n \geqslant n_0).$$

Taking into account (20), we obtain the inequality

$$\max_{1\leqslant j\leqslant k} \operatorname{mes} B_{mj}^{(n)} \geqslant \frac{1}{k} \operatorname{mes} A_n \quad (m = 1, 2, \ldots;\ n \geqslant n_0). \tag{22}$$

Let U_0 be a finite interval containing all points of the form $x+jh$ ($x\in I_0$, $j = 1, 2, \ldots, k$; $0 < h \leqslant 1$). Define the sets

$$C_m^{(n)} = \{x\colon x\in U_0,\ |g_n(x)| \geqslant m\} \quad (m = 1, 2, \ldots;\ n \geqslant n_0). \tag{23}$$

Obviously, in view of (21), $C_m^{(n)} \supset B_{mj}^{(n)}$ ($j = 1, 2, \ldots, k$; $m = 1, 2, \ldots$; $n \geqslant n_0$), which implies, according to (22),

$$\operatorname{mes} C_m^{(n)} \geqslant \max_{1\leqslant j\leqslant k} \operatorname{mes} B_{mj}^{(n)} \geqslant \frac{1}{k} \operatorname{mes} A_n \quad (m = 1, 2, \ldots;\ n \geqslant n_0).$$

Consequently, taking into account (19), we have

$$\operatorname{mes} \bigcap_{m=1}^{\infty} C_m^{(n)} > 0$$

for sufficiently large n. There are then an integer $n \geqslant n_0$ and a point $u\in U_0$ such that $|g_n(u)| = \infty$, which contradicts the assumption that $g_n(x)$ is real-valued function. Thus M_n is finite. Therefore in the sequel we shall assume — without restricting the generality of our considerations — that

$$M_n < \infty \quad (n = 1, 2, \ldots). \tag{24}$$

By x_n we denote a point belonging to I_0 such that

$$|g_n(x_n)| \geqslant \tfrac{1}{2} M_n \quad (n = 1, 2, \ldots). \tag{25}$$

Hence and from (17), setting

$$e_n(h) = |\Delta_h^{(k)} g_n(x_n)| \quad (n = 1, 2, \ldots), \tag{26}$$

we obtain the inequality

$$\max_{1\leqslant j\leqslant k} |g_n(x_n+jh)| \geqslant \frac{1}{2^{k+1}} M_n - \frac{e_n(h)}{2^k} \quad (n = 1, 2, \ldots). \tag{27}$$

Putting

$$E_n = \{h\colon 0 < h \leqslant 1,\ e_n(h) \leqslant \tfrac{1}{4} M_n\} \quad (n = 1, 2, \ldots)$$

we have, according to (*), (14) and (26),

$$\lim_{n\to\infty} \operatorname{mes} E_n = 1. \tag{28}$$

Moreover, in view of (27), we obtain the inequality

$$\max_{1\leqslant j\leqslant k} |g_n(x_n+jh)| \geqslant \frac{1}{2^{k+2}} M_n \quad \text{for} \quad h\in E_n, \quad n = 1, 2, \ldots \tag{29}$$

Define the sets

$$D_{nj} = \left\{x_n+jh\colon 0 < h \leqslant 1,\ |g_n(x_n+jh)| \geqslant \frac{1}{2^{k+2}} M_n\right\}$$
$$(j = 1, 2, \ldots, k;\ n = 1, 2, \ldots),$$

$$D_n = \left\{x\colon x\in U_0,\ |g_n(x)| \geqslant \frac{1}{2^{k+2}} M_n\right\} \quad (n = 1, 2, \ldots). \tag{30}$$

Obviously, $D_n \supset D_{nj}$ ($j = 1, 2, \ldots, k$; $n = 1, 2, \ldots$) and, in view of (29),

$$\max_{1\leqslant j\leqslant k} \operatorname{mes} D_{nj} \geqslant \frac{1}{k} \operatorname{mes} E_n \quad (n = 1, 2, \ldots)$$

which implies, according to (28),

$$\operatorname{mes} \limsup_{n\to\infty} D_n \geqslant \liminf_{n\to\infty} \operatorname{mes} D_n \geqslant 1/k.$$

There is then, in virtue of definition (30), a point x_0 such that

$$\limsup_{n\to\infty} |z_n(x_0)| \geqslant 1/2^{k+2}, \tag{31}$$

where $z_n(x) = g_n(x)/M_n$ $(n = 1, 2, \ldots)$. Moreover, in virtue of (*), (12), (13), (14) and (24), we have

(32) $$\sup_{x\in I_0} |z_n(x)| = 1 \quad (n = 1, 2, \ldots),$$

(33) $$\lim_{n\to\infty} \Delta_h^{(k)} z_n(x) = 0$$

and for each rational w

(34) $$\lim_{n\to\infty} z_n(w) = 0.$$

Further, if $x+jh \in I_0$ $(j = 0, 1, \ldots, k-1)$, then, according to (32),

$$|z_n(x+kh)| \leqslant |\Delta_h^{(k)} z_n(x)| + \sum_{j=0}^{k-1} \binom{k}{j} |z_n(x+jh)|$$
$$\leqslant |\Delta_h^{(k)} z_n(x)| + 2^k \quad (n = 1, 2, \ldots).$$

Hence and from (33) it follows immediately that

$$\limsup_{n\to\infty} |z_n(x+kh)| \leqslant 2^k.$$

By iterating of this procedure we finally obtain for every finite interval I the inequality

$$\sup_{x\in I} \limsup_{n\to\infty} |z_n(x)| < \infty.$$

Hence and from (33) and (34), applying lemma 2, we obtain the convergence $\lim_{n\to\infty} z_n(x) = 0$ for each x, which contradicts (31). The theorem is thus proved.

References

[1] G. Hamel, *Eine Basis aller Zahlen und die unstetige Lösungen der Funktionalgleichung* $f(x+y) = f(x)+f(y)$, Math. Ann. 60 (1905), p. 459-462.

[2] H. Whitney, *On functions with bounded* n^{th} *differences*, Journal de Math. pures et appliquées 36 (1957), p. 67-95.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 22. 5. 1958

ANNALES
POLONICI MATHEMATICI
VII (1959)

On a certain method of Toeplitz

by L. Włodarski (Łódź)

When considering a method of summability we come across a question of basic importance, namely that of the domain in which that method sums the analitical expansion $\sum a_n z^n$ of the function $f(z)$ to the function $f(z)$. The limitability of the geometrical sequence (a^n) plays a decisive part in considerations of this kind. The range of classical methods, as far as the limitability of a geometrical sequence is concerned, is rather restricted. The mean methods (the methods of Hölder and Cesàro), and the continuous methods of Abel-Poisson limit a geometrical sequence within the closed circle $|a| \leqslant 1$. The method of Euler (E, k) limits a geometrical sequence within an open circle $|a+k| < k+1$, adding the point $a = 1$ (see for instance [1], p. 178 below), whereas the classical method of Borel limits a geometrical sequence within the open half-plane $\operatorname{re} a < 1$, adding the point $a = 1$ (see [1], p. 183, th. 128).

In this paper we define a permanent method of Toeplitz which limits a geometrical sequence all over the complex plane, namely for $a = 1$ to one, for a real greater than one to ∞, and for any other complex a to zero. In this way the method in question sums the geometrical series $\sum z^n$ to the function $1/(1-z)$ all over the complex plane, with the exception of real numbers $z \geqslant 1$.

A sequence transformed by this method we define as follows:

(1) $$\eta_m = 2^{-m} e^{-m} \sum_{n=0}^{\infty} \frac{m^{n\cdot 2^{-m}}}{\Gamma(n\cdot 2^{-m}+1)} \xi_n.$$

The construction of this method is connected with Borel's continuous method B_k ([4], p. 143) defined by the formula

$$B_k(t, x) = 2^k e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\cdot 2^k}}{\Gamma(n\cdot 2^k+1)} \xi_n.$$