traire, il est avantageux de choisir les valeurs des ces constantes de telle manière que les intervalles de variation permise des constantes caractéristiques M_F, k_F, $\sup|G|$, k_G, k'_G, T soient aussi grands que possible.

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The operational solution of linear differential equations with constant coefficients

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1. The operational calculus was developed by Heaviside [4] for the solution of initial value problems arising in the theory of electrical communication. He used the calculus in a formal way and occasionally was led to false results. „Shall I refuse my dinner", he said, „because I do not fully understand the process of digestion?". Several methods have since been proposed to put the operational calculus on a firm logical basis, the best known using contour integration [1] or Laplace transforms [3](1).

The drawbacks of these methods have often been remarked. They have no immediate connection with the problem in hand and they impose restrictions which are by no means necessary, although they may be satisfied in the majority of applications. Moreover one still has to verify that the proposed solution satisfies all the requirements of the problem. In many cases also the results they depend on lie deeper than those one is trying to establish. Thus the complex variable method is based on Cauchy's theorem and the Laplace transform method on Lerch's uniqueness theorem. This is particularly true of initial value problems for ordinary linear differential equations with constant coefficients, for which such paraphernalia seem quite out of place. The object of the present work is to give, for this case, a justification of Heaviside's calculus which is as simple and direct as possible.

The method is founded on the following theorem:

If $f(t)$ is any function which is continuous in the interval $0 \leqslant t < T$ and if

$$P(D) = p_0 D^n + p_1 D^{n-1} + \ldots + p_n \qquad (p_0 \neq 0)$$

is any polynomial in $D = d/dt$ with constant, real or complex, coefficients then the differential equation

$$P(D)x = f(t) \tag{1}$$

(1) Other methods are given in [2] and [6].

has one and only one solution which is defined throughout the interval $0 \leqslant t < T$ *and which vanishes together with its first* $n-1$ *derivatives at* $t = 0$.

The particular solution whose existence is asserted will be called the *rest* solution of the equation, since in mechanical problems it represents a system which is initially at rest. The theorem, as is well-known, is easily established by the method of successive approximations and holds even if the coefficients p_i are not constant. Alternatively, it can be proved directly for the special case $n = 1$ and the general case reduced to this by decomposing the polynomial $p(D)$ into its linear factors.

The use of polynomial operators, such as $p(D)$, depends only on the properties

$$[P_1(D)+P_2(D)]x = P_1(D)x+P_2(D)x,$$

$$[P_1(D)P_2(D)]x = P_1(D)[P_2(D)x],$$

which follow at once from the fact that differentiation is a linear process:

$$D(c_1x_1+c_2x_2) = c_1Dx_1+c_2Dx_2,$$

for arbitrary constants c_1, c_2. Our first object will be to extend these relations to operators which are rational functions of D.

2. If $q(D)/p(D)$ is any rational function in which the degree of the numerator does not exceed that of the denominator *we define* $[q(D)/p(D)]f$ *to be the function* $q(D)x$, *where* x *is the rest solution of the equation* $p(D)x = f(t)$.

This definition depends only on the rational function $r(D) = q(D)/p(D)$ and not on the particular representation of $r(D)$ as a quotient of polynomials. In fact any rational function $r(D)$ has a unique representation $r(D) = q(D)/p(D)$, where $q(D)$ and $p(D)$ are relatively prime polynomials and $p(D)$ has leading coefficient 1; every other representation has the form $r(D) = \pi(D)q(D)/\pi(D)p(D)$, where $\pi(D)$ is an arbitrary polynomial. If in one representation the degree of the numerator is not greater than that of the denominator then the same holds in every representation. Thus we have only to show that

$$\frac{\pi(D)q(D)}{\pi(D)p(D)}f = \frac{q(D)}{p(D)}f.$$

The left side is equal to $\pi(D)q(D)x$, where x is the rest solution of the equation $\pi(D)p(D)x = f(t)$. If we put $y = \pi(D)x$ then y is the rest solution of the equation $p(D)y = f(t)$. Therefore

$$\frac{\pi(D)q(D)}{\pi(D)p(D)}f = \pi(D)q(D)x = q(D)y = \frac{q(D)}{p(D)}f.$$

If $r_1(D)$ and $r_2(D)$ are rational functions in which the degree of the numerator is not greater than that of the denominator then $r_1(D)+r_2(D)$ and $r_1(D)r_2(D)$ are rational functions of the same sort. For any continuous function f we have the following two fundamental properties:

$$(2)\qquad [r_1(D)+r_2(D)]f = r_1(D)f+r_2(D)f,$$

$$(3)\qquad [r_1(D)r_2(D)]f = r_1(D)[r_2(D)f].$$

For let $r_1(D) = q_1(D)/p_1(D)$ and $r_2(D) = q_2(D)/p_2(D)$ be representations of $r_1(D)$ and $r_2(D)$ as quotients of polynomials. Then the left side of (2) is equal to

$$[q_1(D)p_2(D)+q_2(D)p_1(D)]\tilde{x},$$

where $\tilde{x}$ is the rest solution of the equation $p_1(D)p_2(D)x = f(t)$. Put $x_1 = p_2(D)\tilde{x}$ and $x_2 = p_1(D)\tilde{x}$. Then x_1 is the rest solution of the equation $p_1(D)x = f(t)$ and x_2 is the rest solution of the equation $p_2(D)x = f(t)$. Therefore

$$[r_1(D)+r_2(D)]f = q_1(D)x_1+q_2(D)x_2 = \frac{q_1(D)}{p_1(D)}f+\frac{q_2(D)}{p_2(D)}f.$$

This proves (2). Similarly the left side of (3) is equal to $q_1(D)q_2(D)\tilde{x}$. If we put $y = p_1(D)\tilde{x}$ and $z = q_2(D)\tilde{x}$ then y is the rest solution of the equation $p_2(D)x = f(t)$ and z is the rest solution of the equation $p_1(D)z = q_2(D)y$. Therefore

$$[r_1(D)r_2(D)]f = q_1(D)z = \frac{q_1(D)}{p_1(D)}[q_2(D)y] = \frac{q_1(D)}{p_1(D)}\left[\frac{q_2(D)}{p_2(D)}f\right].$$

This proves (3). The relations (2) and (3) extend by induction to sums and products of any number of terms.

3. If $\varrho(D)$ is an arbitrary rational function it can be uniquely expressed in the form

$$\varrho(D) = \pi(D)+r(D),$$

where $\pi(D)$ is a polynomial without constant term and $r(D)$ is a rational function whose numerator is not of greater degree than its denominator. We will refer to $\pi(D)$ and $r(D)$ as the *polynomial part* and the *proper rational part* of the rational function $\varrho(D)$. If $\pi(D) = 0$ then $\varrho(D)$ will be called a *proper rational function.* If $f(t)$ is continuously differentiable a number of times equal to the degree of $\pi(D)$ we can define

$$\varrho(D)f = \pi(D)f+r(D)f.$$

This is consistent with the definition already given for proper rational operators and with the definition of polynomial operators. It is easily seen that the law (2) continues to hold: *if $\varrho_1(D)f$ and $\varrho_2(D)f$ are defined then so is $[\varrho_1(D)+\varrho_2(D)]f$ and*

$$[\varrho_1(D)+\varrho_2(D)]f = \varrho_1(D)f+\varrho_2(D)f. \tag{4}$$

However (3) is no longer valid in general. Instead we have: *if $\varrho_1(D)[\varrho_2(D)f]$ is defined then so is $[\varrho_1(D)\varrho_2(D)]f$ and*

$$[\varrho_1(D)\varrho_2(D)]f = \varrho_1(D)[\varrho_2(D)f]+[\varrho_1(D)\varrho_2(D|f)]1, \tag{5}$$

where $\varrho_2(D|f)$ denotes the polynomial part, without constant term, in the expansion of

$$\varrho_2(D)\left[f(0)+\frac{f'(0)}{D}+\frac{f''(0)}{D^2}+\ldots\right]$$

according to powers of D. The series between the brackets may be regarded as the Taylor series of the function $f(t)$. In fact $D^{-\nu}1 = t^\nu/\nu!$, since $x = t^\nu/\nu!$ is the rest solution of the equation $D^\nu x = 1$.

To prove (5) suppose first that $\varrho_2(D) = r_2(D) = q_2(D)/p_2(D)$ is a proper rational function; therefore $\varrho_2(D|f) = 0$. If $\varrho_1(D) = \pi_1(D)+ +r_1(D)$ then by (4)

$$[\varrho_1(D)\varrho_2(D)]f = [\pi_1(D)r_2(D)]f+[r_1(D)r_2(D)]f.$$

Let

$$\pi_1(D)q_2(D) = s(D)p_2(D)+t(D),$$

where $s(D)$ is a polynomial without constant term and $t(D)$ is a polynomial of degree not greater than that of $p_2(D)$. Then

$$[\pi_1(D)r_2(D)]f = s(D)f+\frac{t(D)}{p_2(D)}f = s(D)p_2(D)\cdot\frac{1}{p_2(D)}f+t(D)\cdot\frac{1}{p_2(D)}f$$
$$= \pi_1(D)q_2(D)\cdot\frac{1}{p_2(D)}f = \pi_1(D)\cdot r_2(D)f.$$

Hence

$$[\varrho_1(D)\varrho_2(D)]f = \pi_1(D)\cdot r_2(D)f+r_1(D)\cdot r_2(D)f = \varrho_1(D)\cdot\varrho_2(D)f,$$

in accordance with (5). Suppose next that $\varrho_1(D) = \pi_1(D)$ has only a polynomial part. Then $\varrho_1(D)\varrho_2(D|f)$ is a polynomial without constant term and hence $[\varrho_1(D)\varrho_2(D|f)]1 = 0$. On the other hand if $\varrho_2(D) = = \pi_2(D)+r_2(D)$ then

$$[\varrho_1(D)\varrho_2(D)]f = [\pi_1(D)\pi_2(D)]f+[\pi_1(D)r_2(D)]f$$
$$= \pi_1(D)\cdot\pi_2(D)f+\pi_1(D)\cdot r_2(D)f = \varrho_1(D)\cdot\varrho_2(D)f.$$

Thus (5) holds also in this case. Therefore, since both sides of (5) involve ϱ_1 and ϱ_2 linearly, we can suppose that $\varrho_1(D)$ is a proper rational function and $\varrho_2(D)$ a polynomial without constant term. In fact, we can suppose $\varrho_2(D) = D^\nu$ $(\nu \geqslant 1)$. Again, if $\varrho_1(D) = q_1(D)/p_1(D)$ then by what has already been established:

$$[\varrho_1(D)\varrho_2(D)]f = q_1(D)\cdot\left[\frac{1}{p_1(D)}\varrho_2(D)\right]f,$$

$$\varrho_1(D)[\varrho_2(D)f] = q_1(D)\cdot\frac{1}{p_1(D)}[\varrho_2(D)f],$$

$$[\varrho_1(D)\varrho_2(D|f)]1 = q_1(D)\cdot\left[\frac{1}{p_1(D)}\varrho_2(D|f)\right]1.$$

Therefore if (5) holds with $\varrho_1(D) = 1/p_1(D)$ it holds also with $\varrho_1(D) = q_1(D)/p_1(D)$. Thus we have only to show that

$$\left[\frac{D^\nu}{p(D)}\right]f = \frac{1}{p(D)}[D^\nu f]+\left[\frac{f(0)D^\nu+f'(0)D^{\nu-1}+\ldots+f^{(\nu-1)}(0)D}{p(D)}\right]1. \tag{5'}$$

We will prove this by induction on ν. If $\nu = 1$ the relation to be proved reads:

$$\left[\frac{D}{p(D)}\right]f = \frac{1}{p(D)}[Df]+\left[\frac{f(0)D}{p(D)}\right]1. \tag{5''}$$

The function $y = [D/p(D)]f$ is a solution of the equation $p(D)y = Df$ and it satisfies the initial conditions $D^iy = 0$ $(i = 0, \ldots, n-2)$, $p_0D^{n-1}y = f(0)$, where p_0 is the leading coefficient and n the degree of $p(D)$. The function $z = [f(0)D/p(D)]1$ is a solution of the homogeneous equation $p(D)z = 0$ and it satisfies the same initial conditions. Therefore $x = y-z$ is the rest solution of the equation $p(D)x = Df$; that is,

$$x = \frac{1}{p(D)}[Df].$$

This proves (5''). Suppose the relation (5') holds as written. Then on differentiating we get

$$\left[\frac{D^{\nu+1}}{p(D)}\right]f = \frac{D}{p(D)}[D^\nu f]+\left[\frac{f(0)D^{\nu+1}+f'(0)D^\nu+\ldots+f^{(\nu-1)}(0)D^2}{p(D)}\right]1$$
$$= \frac{1}{p(D)}[D^{\nu+1}f]+\left[\frac{f(0)D^{\nu+1}+f'(0)D^\nu+\ldots+f^{(\nu)}(0)D}{p(D)}\right]1$$

by (5'') with f replaced by $D^\nu f$. Hence (5') holds for any $\nu \geqslant 1$. This completes the proof.

4. Consider now the homogeneous differential equation $p(D)x = 0$ with arbitrary initial conditions $D^i x = x_0^{(i)}$ $(i = 0, \ldots, n-1)$ at $t = 0$. If x is a solution of this initial value problem it is differentiable any finite number of times and by (5)

$$x = \left[\frac{1}{p(D)}p(D)\right]x = \frac{1}{p(D)}[p(D)x] + \left[\frac{p(D|x)}{p(D)}\right]1 = \left[\frac{p(D|x)}{p(D)}\right]1$$

where $p(D|x)$ is the polynomial part of

$$p(D)\left(x_0 + \frac{x_0'}{D} + \ldots + \frac{x_0^{(n-1)}}{D^{n-1}}\right).$$

Hence the solution, if it exists, is unique. Conversely we can show that the function $x = [p(D|x)/p(D)]1$ is in fact a solution. In the first place

$$p(D)x = p(D|x)1 = 0.$$

Secondly, if y is any function such that $D^i y = x_0^{(i)}$ $(i = 0, \ldots, n-1)$ at $t = 0$, for example

$$y = x_0 + x_0'\frac{t}{1!} + \ldots + x_0^{(n-1)}\frac{t^{n-1}}{(n-1)!},$$

then by (5) again

$$y = \left[\frac{1}{p(D)}p(D)\right]y = \frac{1}{p(D)}[p(D)y] + x.$$

Therefore when $t = 0$

$$x_0^{(i)} = D^i y = D^i x \qquad (i = 0, \ldots, n-1).$$

If we add to x the rest solution of the inhomogeneous equation (1) we obtain the solution of the inhomogeneous equation subject to the same initial conditions. Thus we can sum up as follows:

The differential equation $p(D)x = f(t)$ has one and only one solution which satisfies the initial conditions $D^i x = x_0^{(i)}$ $(i = 0, \ldots, n-1)$ at $t = 0$, for arbitrary values of the constants $x_0^{(i)}$. This solution can be expressed in the form

$$x = \frac{1}{p(D)}f + \frac{p(D|x)}{p(D)}1, \tag{6}$$

where $p(D|x)$ denotes the polynomial part of the rational function

$$p(D)\left(x_0 + \frac{x_0'}{D} + \ldots + \frac{x_0^{(n-1)}}{D^{(n-1)}}\right).$$

5. We have considered the effect of different rational functions of D operating on the same function f. We are going to see now what happens when the same rational operator acts on different functions f. Let $r(D)$ be any proper rational function. Then we will show that

$$r(D)(c_1 f_1 + c_2 f_2) = c_1 \cdot r(D)f_1 + c_2 \cdot r(D)f_2, \tag{7}$$

$$r(D)(f_1 * f_2) = [r(D)f_1] * f_2 \tag{8}$$

where c_1 and c_2 are arbitrary constants, f_1 and f_2 are continuous functions and $f_1 * f_2$ denotes the *convolution product* of f_1 and f_2, i. e. the function

$$\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau.$$

The first of these relations follows immediately from the superposition principle. To prove the second put $r(D) = q(D)/p(D)$, $g = 1/p(D)f_1$, and

$$h = g * f_2 = \int_0^t g(t-\tau)f_2(\tau)\,d\tau.$$

The function g vanishes together with its first $n-1$ derivatives at $t = 0$, if n is the degree of $p(D)$. Differentiating under the sign of integration we obtain

$$Dh = (Dg)*f_2 + g(0)f_2(t) = (Dg)*f_2.$$

By repeated differentiation it follows in the same way that $D^i h = (D^i g)*f_2$ for $i \leqslant n$. Hence

$$q(D)h = q(D)g*f_2, \qquad p(D)h = p(D)g*f_2 = f_1*f_2.$$

Moreover h vanishes together with its first $n-1$ derivatives at $t = 0$. Therefore it is the rest solution of the equation $p(D)x = f_1*f_2$ and

$$r(D)(f_1*f_2) = q(D)h = [q(D)g]*f_2 = [r(D)f_1]*f_2.$$

If we take $f_1(t) = 1$ in (8) we get $[r(D)/D]f = [r(D)1]*f$, since

$$1*f = \int_0^t f(\tau)\,d\tau = \frac{1}{D}f.$$

In particular for $r(D) = D/p(D)$ it follows that

The rest solution of the equation $p(D)x = f(t)$ can be expressed in the form

$$\frac{1}{p(D)}f = \left[\frac{D}{p(D)}\right]1*f. \tag{9}$$

A further property of rational operators deals with multiplication by an exponential function:

$$r(D)[e^{-\lambda t}f(t)] = e^{-\lambda t}\cdot r(D-\lambda)f. \tag{10}$$

To prove this we use the relation

$$D^\nu(e^{-\lambda t}x) = e^{-\lambda t}\cdot(D-\lambda)^\nu x \tag{11}$$

which holds for any integer $\nu \geqslant 0$ and is simply a special case of Leibniz's rule for finding the derivatives of a product of two functions. Let $r(D) = q(D)/p(D)$. If x is the rest solution of the equation $p(D-\lambda)x = f(t)$ then by (11) $y = e^{-\lambda t}x$ is the rest solution of the equation $p(D)y = e^{-\lambda t}f(t)$. Moreover $e^{-\lambda t}q(D-\lambda)x = q(D)y$. Therefore

$$e^{-\lambda t}\cdot\frac{q(D-\lambda)}{p(D-\lambda)}f = e^{-\lambda t}\cdot q(D-\lambda)x = q(D)y = \frac{q(D)}{p(D)}[e^{-\lambda t}f(t)].$$

These relations can easily be extended to arbitrary rational operators.

6. The solution of the differential equation (1), subject to arbitrary initial conditions, is reduced by (6) and (9) to the evaluation of $r(D)1$, where $r(D) = q(D)/p(D)$ is a proper rational function. This evaluation will be carried out in the present section.

We have already seen in 3 that $D^{-\nu}1 = t^\nu/\nu!$. Also,

$$e^{\lambda t} = \frac{D}{D-\lambda}1,$$

since $x = (e^{\lambda t}-1)/\lambda$ is the rest solution of the equation $(D-\lambda)x = 1$. Therefore, using (10),

$$\frac{D}{(D-\lambda)^\nu}1 = \frac{1}{(D-\lambda)^{\nu-1}}\left[\frac{D}{D-\lambda}1\right] = \frac{1}{(D-\lambda)^{\nu-1}}e^{\lambda t}$$
$$= e^{\lambda t}\cdot\frac{1}{D^{\nu-1}}1 = \frac{e^{\lambda t}t^{\nu-1}}{(\nu-1)!}\,(^2).$$

But by the theory of partial fractions the rational function $r(D)/D$ can be expressed in the form

$$\frac{r(D)}{D} = \sum_{i=1}^{m}\left\{\frac{c_{i1}}{D-\lambda_i}+\frac{c_{i2}}{(D-\lambda_i)^2}+\ldots+\frac{c_{in_i}}{(D-\lambda_i)^{n_i}}\right\}$$

(2) The value of $D/(D-\lambda)^\nu 1$ can also be obtained from that of $D/(D-\lambda)1$ by differentiation with respect to λ.

where the c's and λ's are complex constants. Hence

$$r(D)1 = \sum_{i=1}^{m}\left\{c_{i1}+c_{i2}\frac{t}{1!}+\ldots+c_{in_i}\frac{t^{n_i-1}}{(n_i-1)!}\right\}e^{\lambda_i t}.$$

Thus we have the following rule:

To obtain $r(D)1$ expand the rational function $r(D)/D$ in partial fractions and replace $(D-\lambda)^{-\nu}$ by $e^{\lambda t}t^{\nu-1}/(\nu-1)!$.

This completes our solution of a single differential equation. However, a few remarks may be made concerning the practical application of the method. First, it is usually not necessary to perform the complete decomposition of $r(D)/D$ into partial fractions. It is sufficient to express $r(D)$ as a sum of rational functions $r_i(D)$ for each of which $r_i(D)1$ is already known. The value of $r(D)1$ for a variety of different rational functions $r(D)$ may be read off directly from tables of Laplace transforms. For on comparing the solutions given by the two methods it will be seen that if

$$r(s) = s\int_0^\infty e^{-st}\varphi(t)\,dt$$

then $\varphi(t) = r(D)1$.

Secondly, if $f(t)$ is itself a function of the form $r(D)1$ the two terms in (6) can be combined and need not be evaluated separately. In general, however, the solution will involve quadratures, as indicated by the formula (9). The function $f(t)$ has been supposed continuous but (9) still holds if it is only required to be integrable, provided we then define a solution of the equation (1) as the derivative of any solution of the equation

$$p(D)x = \int_0^t f(\tau)\,d\tau+\text{const}.$$

If the function actually required is $q(D)x$, where $q(D)$ is a polynomial of lower degree than $p(D)$, it can be obtained directly without first finding x. In fact

$$q(D)x = \frac{q(D)p(D|x)}{p(D)}1+\frac{q(D)}{p(D)}f = \frac{q(D)p(D|x)}{p(D)}1+\left[\frac{Dq(D)}{p(D)}1\right]*f.$$

Similarly if the equation to be solved has the form $p(D)x = q(D)g$ the solution is given by

$$x = \left[\frac{1}{p(D)}p(D)\right]x = \frac{1}{p(D)}[q(D)g]+\frac{p(D|x)}{p(D)}1 = \frac{q(D)}{p(D)}g+\frac{p(D|x)-q(D|g)}{p(D)}1.$$

7. These results can be extended without difficulty to systems of differential equations. Indeed the advantages of operational methods in dealing with initial value problems are even greater for systems. Let f be a column vector whose coordinates $f_1, \ldots, f_m$ are continuous functions of t for $0 \leqslant t < T$, and let $R(D) = [r_{ik}(D)]$ be an $m \times m$ matrix of proper rational functions. Then we define $R(D)f$ to be the vector whose coordinates are the functions

$$\sum_{k=1}^{m} r_{ik}(D) f_k \qquad (i = 1, \ldots, m).$$

The properties

$$[R_1(D)+R_2(D)]f = R_1(D)f+R_2(D)f,$$

$$[R_1(D)R_2(D)]f = R_1(D)[R_2(D)f]$$

carry over at once from scalar to vector functions. If $\varrho(D)$ is a matrix of arbitrary rational functions it can be uniquely expressed in the form

$$\varrho(D) = \pi(D)+R(D),$$

where $\pi(D)$ is a matrix of polynomials without constant term and $R(D)$ is a matrix of proper rational functions. If the elements of $\pi(D)$ are of degree $\leqslant n$ and if the coordinates of f are continuously differentiable at least n times we define

$$\varrho(D)f = \pi(D)f+R(D)f.$$

It is easily shown that if $\varrho_1(D)f$ and $\varrho_2(D)f$ are both defined then so is $[\varrho_1(D)+\varrho_2(D)]f$ and

$$[\varrho_1(D)+\varrho_2(D)]f = \varrho_1(D)f+\varrho_2(D)f.$$

Again, suppose the polynomial part of $\varrho_1(D)$ is of degree n and let $\varrho_2(D)$ equal D^r times a matrix of proper rational functions. Then $\varrho_1(D)[\varrho_2(D)f]$ and $[\varrho_1(D)\varrho_2(D)]f$ are both defined if $n+r > 0$ and f is continuously differentiable $n+r$ times, or if $n+r \leqslant 0$ and f is continuous. Moreover

$$[\varrho_1(D)\varrho_2(D)]f = \varrho_1(D)[\varrho_2(D)f]+[\varrho_1(D)\varrho_2(D|f)]1,$$

where the vector $\varrho_2(D|f)$ is the polynomial part, without constant term, in the expansion of

$$\varrho_2(D)\left[f(0)+\frac{f'(0)}{D}+\frac{f''(0)}{D^2}+\ldots\right]$$

according to powers of D and $[\varrho_1(D)\varrho_2(D|f)]1$ denotes the vector function which is produced when the vector differential operator $\varrho_1(D)\varrho_2(D|f)$ acts on the scalar function 1. To prove these relations we simply compare coordinates on both sides and use the corresponding scalar results.

Consider now the system of linear differential equations

$$P(D)x = f(t), \tag{12}$$

where

$$P(D) = P_0D^n+P_1D^{n-1}+\ldots+P_n \tag{13}$$

is a polynomial in D whose coefficients P_i are constant $m \times m$ matrices. We will suppose also that P_0 is non-singular. This is the case of practical importance and the only one in which the initial conditions can be prescribed arbitrarily. From algebra it is known that under this assumption there is a unique matrix of rational functions $P^{-1}(D)$ such that

$$P^{-1}(D)P(D) = P(D)P^{-1}(D) = \text{the } m\text{-rowed identity matrix.}$$

Moreover the expansion of $P^{-1}(D)$ in descending powers of D begins with the term $P_0^{-1}D^{-n}$:

$$P^{-1}(D) = P_0^{-1}D^{-n}+D^{-n-1} \text{ (a matrix of proper rational functions).}$$

We will show first that the system (12) has one and only one solution which is defined throughout the interval $0 \leqslant t < T$ and which satisfies the initial conditions $D^i x = 0$ $(i = 0, \ldots, n-1)$ at $t = 0$. In fact if x is such a solution then

$$P^{-1}(D)f = P^{-1}(D)[P(D)x] = x-[P^{-1}(D)P(D|x)]1 = x,$$

since $P(D|x) = 0$. Conversely, if $x = P^{-1}(D)f$ then x is continuously differentiable n times and $D^i x = 0$ at $t = 0$ for $i < n$. Moreover

$$P(D)x = P(D)[P^{-1}(D)f] = f,$$

since $P^{-1}(D)$ is without polynomial part. Thus x is the required solution.

Consider next the homogeneous system

$$P(D)x = 0. \tag{14}$$

If x is a solution such that $D^i x = x_0^{(i)}$ $(i = 0, \ldots, n-1)$ at $t = 0$ then it is differentiable any number of times and

$$x = [P^{-1}(D)P(D)]x = [P^{-1}(D)P(D|x)]1,$$

since $P(D)x = 0$. Conversely the function $x = [P^{-1}(D)P(D|x)]1$ is a solution of (14), since

$$P(D)x = [P(D)P^{-1}(D)P(D|x)]1 = P(D|x)1 = 0.$$

It also satisfies the initial conditions. For

$$D^i x = [D^i P^{-1}(D)P(D|x)]1.$$

But

$$P(D|x) = P(D)\left(x_0 + \frac{x_0'}{D} + \ldots + \frac{x_0^{(n-1)}}{D^{n-1}}\right) + \text{proper rational function}$$

and therefore, if $i < n$,

$$D^i P^{-1}(D)P(D|x) = (\text{polynomial without constant term}) + x_0^{(i)} + \\ + D^{-1}\,(\text{proper rational function}).$$

Operating on 1 the first term gives zero, the second term gives back $x_0^{(i)}$, and the third term gives a function which vanishes at $t = 0$. Thus $D^i x|_{t=0} = x_0^{(i)}$.

Summing up, we can say:

The system (12) *has one and only one solution which satisfies the initial conditions $D^i x = x_0^{(i)}$ $(i = 0, \ldots, n-1)$ at $t = 0$ for arbitrary initial vectors $x_0^{(i)}$. This solution is given by the formula*

$$x = P^{-1}(D)f + [P^{-1}(D)P(D|x)]1, \tag{15}$$

where $P^{-1}(D)$ is the inverse of the matrix $P(D)$ and $P(D|x)$ denotes the polynomial part of the vector

$$P(D)\left(x_0 + \frac{x_0'}{D} + \ldots + \frac{x_0^{(n-1)}}{D^{n-1}}\right).$$

8. Rational operators have long been used in a somewhat different sense from the one above. Boole represented any solution of the equation $p(D)x = f(t)$ by the symbol $[1/p(D)]f$. He also showed that from the decomposition of $1/p(D)$ into partial fractions:

$$\frac{1}{p(D)} = \sum_{i=1}^{m}\left\{\frac{c_{i1}}{D-\lambda_i} + \frac{c_{i2}}{(D-\lambda_i)^2} + \ldots + \frac{c_{in_i}}{(D-\lambda_i)^{n_i}}\right\}$$

a solution $x = x(t)$ could be at once obtained:

$$x(t) = \sum_{i=1}^{m}\left\{c_{i1} + c_{i2}\frac{t}{1!} + \ldots + c_{in_i}\frac{t^{n_i-1}}{(n_i-1)!}\right\}e^{\lambda_i t} * f(t).$$

The proofs of Boole's result given in textbooks, for example in Kamke [5], are valid only if f is differentiable a sufficient number of times, since they apply polynomial operators of degree greater than ν to the solutions of the equation $(D-\lambda)^\nu x = f(t)$. Our proof merely requires $f(t)$ to be continuous and shows that $x(t)$ is in fact the rest solution. More important, we have put the operational calculus itself on a firm foundation. This is the essential part of Heaviside's work, the expansion formula being merely a particular application.

9. The same method can be applied almost without change to the solution of linear difference equations with constant coefficients (cf [7]). The basic theorem now reads:

If $f(t)$ is any function which is defined on the set $t = 0, 1, 2, \ldots$ and if

$$p(\Delta) = p_0\Delta^n + p_1\Delta^{n-1} + \ldots + p_n \quad (p_0 \neq 0)$$

is any polynomial in Δ with constant coefficients then the difference equation $p(\Delta)x = f(t)$ has one and only one solution which vanishes together with its first $n-1$ differences at $t = 0$.

Since Δ is a linear transformation and $\Delta 1 = 0$ the argument of sections 2 and 3 establishes the formulae

$$[\varrho_1(\Delta) + \varrho_2(\Delta)]f = \varrho_1(\Delta)f + \varrho_2(\Delta)f,$$

$$[\varrho_1(\Delta)\varrho_2(\Delta)]f = \varrho_1(\Delta)[\varrho_2(\Delta)f] + [\varrho_1(\Delta)\varrho_2(\Delta|f)]1$$

where $\varrho(\Delta|f)$ denotes the polynomial part, without constant term, in the expansion of

$$\varrho(\Delta)\left[f(0) + \frac{f^{(1)}(0)}{\Delta} + \frac{f^{(2)}(0)}{\Delta} + \ldots\right], \qquad f^{(i)}(0) = \Delta^i f(0).$$

If we define the convolution product ot two functions $f_1(t)$ and $f_2(t)$ to be the function

$$f_1 * f_2(t) = \begin{cases} 0 & \text{for} \quad t = 0, \\ \sum_{\tau=0}^{t-1} f_1(t-1-\tau)f_2(\tau) & \text{for} \quad t = 1, 2, \ldots \end{cases}$$

we will have

$$\Delta(f_1 * f_2) = (\Delta f_1) * f_2 + f_1(0)f_2(t).$$

It then follows as before that for any proper rational function $r(\Delta)$

$$r(\Delta)(f_1 * f_2) = [r(\Delta)f_1] * f_2.$$

In particular, since $1*f = \Delta^{-1}f$,

$$\frac{1}{p(\Delta)}f = \left[\frac{\Delta}{p(\Delta)}1\right]*f.$$

The analogue of (10) is the relation

$$r(\Delta)(a^t x) = a^t \cdot r(a\Delta + a - 1)x.$$

Using this relation we obtain from

$$\Delta^{-\nu}1 = \binom{t}{\nu} \quad \text{(binomial coefficient)}$$

and

$$\frac{\Delta}{\Delta - \lambda}1 = (\lambda+1)^t$$

the more general result

$$\frac{\Delta}{(\Delta-\lambda)^\nu}1 = \binom{t}{\nu-1}(\lambda+1)^{t-\nu+1}.$$

The value of $r(\Delta)1$ for any proper rational function $r(\Delta)$ can now be obtained by decomposing $r(\Delta)/\Delta$ into partial fractions. No tabulation of these values appears to have been made.

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