ANNALES
POLONICI MATHEMATICI
LXIX.3 (1998)

On the graded Betti numbers for large finite subsets of curves

by E. Ballico (Trento)

Abstract. We prove a recent conjecture of S. Lvovski concerning the periodicity behaviour of top Betti numbers of general finite subsets with large cardinality of an irreducible curve $C \subset \mathbb{P}^n$.

Introduction. In [5], S. Lvovski made a nice Periodicity Conjecture (see [5], Conjecture 1.2, or the case $B = \emptyset$ of our theorem below for its statement) about the graded Betti numbers of general subsets with large cardinality of a fixed irreducible curve $C \subset \mathbb{P}^n$. He proved it in some cases (e.g. for the lower Betti diagram ([5], Prop. 5.1) or if C is a rational normal curve). He raised also a generalization of this conjecture (see [5], Conjecture 1.3) which inspired the statement of our theorem. We will not use the methods introduced in [5] and hence we will be able to work over an uncountable algebraically closed field $\mathbf{K}$ with arbitrary characteristic.

To state our results we need to introduce the following notations. Set $\mathbb{P}^n := \mathrm{Proj}(R)$ with $R := \mathbf{K}[T_0, \ldots, T_n]$. For any closed subscheme Z of $\mathbb{P}^n$, let $\mathbf{I}_Z \subset R$ be its homogeneous ideal and $L_*(Z)$ its minimal graded free resolution. Hence $L_0(Z) = R$ and $L_m(Z) = 0$ for $m > n$. If $L_i = \bigoplus_j R(-i-j)^{b_{ij}(Z)}$, then the non-negative integers $b_{ij}(Z)$ will be called the *graded Betti numbers* of Z. Our notations for graded Betti numbers agree with those adopted by the program Macaulay and in [5]. Set $\delta(Z) := \max\{j :$ there is i with $b_{ij}(Z) \neq 0\}$. Call $\delta(Z)$ the *index of regularity* of Z.

For a very good introduction to the Koszul cohomology of finite sets and linearly normal curves, see [2], §1, or [3], Introduction and §1, or [4], §1. We only need the following fact. Let Ω^j, $0 \le j \le n$, be the sheaf of exterior j-forms on $\mathbb{P}^n$. Fix integers n, i, k with $n \ge 2$, $1 \le i \le n$ and $k > 0$. Let $A \subseteq \mathbb{P}^n$ be a closed subscheme such that $h^i(A, \mathcal{O}_A(z)) = 0$ for every $i > 0$

1991 *Mathematics Subject Classification*: 14N05, 14H99, 13D40, 13P99.
Key words and phrases: Koszul cohomology, irreducible curve, Betti diagram.

and every $z \geq k - i$; this condition is satisfied if $\dim(A) = 0$. We have $b_{ij}(A) = 0$ for every $j \geq k$ if and only if $h^1(\mathbb{P}^n, \Omega^i(i+k) \otimes \mathbf{I}_A) = 0$.

Theorem. *Let $C \subset \mathbb{P}^n$ be an irreducible curve and $B \subset C$ an effective (or empty) Cartier divisor of C. Set $d := \deg(C)$. Let $\{P_i\}_{i\in\mathbb{N}}$ be a generic sequence of points of C and for every $m > 0$ set $X_m := B \cup \{P_1, \ldots, P_m\}$. Set $t_m := \min\{j > \delta(C)$: there is i with $b_{ij}(X_m) \neq 0\}$. Then there in an integer m' such that for all integers $m \geq m'$ we have $t_{m+d} = t_m + 1$ and if $j \geq t_m$, then $b_{i,j+1}(X_{m+d}) = b_{ij}(X_m)$ for all i. Moreover, the periodic pattern appearing for large m in the Betti diagram of X_m depends only on the integers $d, g := p_a(C)$, $\delta(C)$ and $b := \mathrm{length}(B)$.*

Indeed our proof of this theorem will give some information on the graded Betti numbers of X_m for large m. Furthermore, the proof will show that the cases $m+b+1-g \equiv 0 \bmod d$ and $m+b+1-g \equiv 1 \bmod d$ are "easier" than the cases with $m+b+1-g \equiv i \bmod d$ and $2 \leq i < d$.

The author was partially supported by MURST and GNSAGA of CNR (Italy) and by Max-Planck-Institut für Mathematik in Bonn. He wants to thank the Max-Planck-Institut for excellent working atmosphere.

The proof. If $P_i \in C_{\mathrm{reg}}$, $i \geq 1$, and $m \geq 1$, set $Y\{m\} := \sum_{1\leq i\leq m} P_i$ and $X\{m\} := B + Y\{m\}$. Hence $Y_m := \bigcup_{1\leq i\leq m} P_i$ and $X_m := B \cup Y_m$ are 0-dimensional closed subschemes of C, $Y\{m\}$ is the effective degree m Cartier divisor of C associated with Y_m and $X\{m\}$ is the effective degree $m+b$ Cartier divisor of C associated with X_m. Note that if $m \geq g$ for general P_i the line bundle $\mathcal{O}_C(Y\{m\})$ (resp. $\mathcal{O}_C(X\{m\})$) is a general line bundle of degree m (resp. degree $b+m$) on C. A general $L \in \mathrm{Pic}^w(C)$ has $h^1(C,L) = 0$ (resp. $h^0(C,L) \neq 0$, resp. it is spanned) if and only if $w \geq g-1$ (resp. $w \geq g$, resp. $w \geq g+1$). Hence for every integer z we have $h^1(C, \mathcal{O}_C(z)(-X\{m\})) = 0$ if and only if $zd \geq b+m+g-1$ and $h^0(C, \mathcal{O}_C(z)(-X\{m\})) \neq 0$ if and only if $zd \geq b+m+g$. Furthermore, $\mathcal{O}_C(z)(-X\{m\})$ is spanned by its global sections if and only if $zd \geq b+m+g+1$, i.e. if and only if $h^0(C, \mathcal{O}_C(z)(-X\{m\})) = zd - b - m + 1 - g \geq 2$.

Fix an integer i with $0 \leq i < d$. We are interested in the integers m with $m \equiv i \bmod d$. Let $\alpha(i,m)$ be the first integer $> \delta(C)$ with $m \geq d$ and $d\alpha(i,m) - m - b \geq g$. The proof of our theorem will show that the inequality $m \geq d$ can be easily weakened. For general X_m we have $h^0(\mathbb{P}^n, \mathbf{I}_{X_m}(t)) = h^0(\mathbb{P}^n, \mathbf{I}_C(t))$ if and only if $t < \alpha(i,m)$. Hence $t_m = \alpha(i,m)$. Note that $\alpha(i, m+d) = \alpha(i,m) + 1$. Thus we have proved that for general $\{P_i\}_{i\in\mathbb{N}}$ we have $t_{m+d} = t_m + 1$, i.e. the first assertion of the theorem.

Furthermore, the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(X_m, \mathbf{I}_{X_m}(t))$ is surjective if and only if $t \geq \alpha(i,m)$. By Castelnuovo–Mumford's regularity theorem, we have $\alpha(i,m) \leq \delta(X_m) \leq \alpha(i,m) + 1$.

If $0 \le d\alpha(i,m) - b - m - g + 1 \le 1$, then $\mathbf{I}_{X_m}$ cannot be generated by forms of degree $\le \alpha(i,m)$ and hence we have $\delta(X_m) = \alpha(i,m)+1$. The last part of our proof concerning the values of $h^0(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)) \otimes \mathbf{I}_{X_m})$ and $h^0(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)+1) \otimes \mathbf{I}_{X_{m+d}})$ will show even in this case the equality $b_{i,j+1}(X_{m+d}) = b_{ij}(X_m)$, completing the proof of our theorem in this case.

If $2 \le d\alpha(i,m) - b - m - g + 1 < d$, then $\alpha(i,m)$ is the first integer t such that the part of degree t of $\mathbf{I}_{X_m}$ has X_m as scheme-theoretic 0-locus. If for all integers $m \ge m'$ with $m \equiv i \bmod d$ we have $\delta(X_m) = \alpha(i,m)+1$, then we have proved the assertion on the index of regularity in the statement of the theorem for the integers m in the congruence class of i modulo d. Hence we may assume the existence of an integer $m \ge m'$ with $m \equiv i \bmod d$ such that $\delta(X_m) = \alpha(i,m)$.

It is sufficient to prove that $\delta(X_{m+d}) = \alpha(i,m)$. As explained before the statement of the theorem, the assertion $\delta(X_m) = \alpha(i,m)$ (resp. $\delta(X_{m+d}) = \alpha(i,m)+1$) is equivalent to the fact that for all integers t with $1 \le t \le n$ we have $h^1(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)) \otimes \mathbf{I}_{X_m}) = 0$ (resp. $h^1(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)+1) \otimes \mathbf{I}_{X_{m+d}}) = 0$). By semicontinuity it is sufficient to show the vanishing of these cohomology groups for a very special finite subset, X_{m+d}, of C with $B \subseteq X_m \subseteq X_{m+d}$ and $\operatorname{card}(X_{m+d}) = \operatorname{card}(X_m) + d = b+m+d$. We take a general hyperplane H and prove that $h^1(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)+1) \otimes \mathbf{I}_{X_{m+d}}) = 0$ when X_{m+d} is the union of X_m and of the hyperplane section $H \cap C$ of C. Note that $\Omega^t(t+\alpha(i,m)+1)|H \cong \Omega^t_H(t+\alpha(i,m)+1) \oplus \Omega^{t-1}_H(t+\alpha(i,m))$.

Since we assumed $\alpha(i,m) \ge d$, the graded Betti numbers of $C \cap H$ in H are all lower than $c(i,m)$. Thus again by Koszul cohomology we have $H^1(H, \mathbf{I}_{C\cap H,H} \otimes \Omega^t(t+\alpha(i,m)+1)|H) = 0$. Thus we obtain $\delta(X_{m+d}) = \alpha(i,m)+1$ using the following short exact sequence:

$$0 \to \mathbf{I}_{X_m} \otimes \Omega^t(t+\alpha(i,m)) \to \mathbf{I}_{X_{m+d}} \otimes \Omega^t(t+\alpha(i,m)+1) \to \mathbf{I}_{C\cap H,H} \otimes (\Omega^t(t+\alpha(i,m)+1)|H) \to 0.$$

The top graded Betti numbers of X_m (resp. X_{m+d}) are uniquely determined by the ordered set of $t+1$ integers $h^0(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)) \otimes \mathbf{I}_{X_m})$ (resp. $h^0(\mathbb{P}^n, \Omega^t(t+\alpha(i,m)+1) \otimes \mathbf{I}_{X_{m+d}})$), $0 \le t \le n$, and hence we obtain in both cases the statements on $\delta(X_m) - \delta(X_{m+d})$ and on the graded Betti numbers of X_m and X_{m+d}.

References

[1] M. Green, *Koszul cohomology*, J. Differential Geom. 19 (1984), 125–171.

[2] —, *Koszul cohomology and geometry*, in: Lectures on Riemann Surfaces (Trieste, 1987), World Sci., 1989, 177–200.

[3] M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. 67 (1988), 301–314.

[4] R. Lazarsfeld, *A sampling of vector bundles techniques in the study of linear series*, in: Lectures on Riemann Surfaces (Trieste, 1987), World Sci., 1989, 500–559.
[5] S. Lvovski, *On graded Betti numbers for finite subsets of curves*, preprint 97-31, Max-Planck-Institut.

Department of Mathematics
University of Trento
38050 Povo (TN), Italy
E-mail: ballico@science.unitn.it

Reçu par la Rédaction le 17.11.1997
Révisé le 16.2.1998