ANNALES
POLONICI MATHEMATICI
LXIX.3 (1998)

Convolution equations in the space of Laplace distributions

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Abstract. A formal solution of a nonlinear equation $P(D)u = g(u)$ in 2 variables is constructed using the Laplace transformation and a convolution equation. We assume some conditions on the characteristic set $\operatorname{Char} P$.

1. Introduction. In this paper we consider a nonlinear PDE of the form

$$P(D)u = g(u) = \sum_{j=0}^{\infty} c_j u^j, \tag{1}$$

where $P(z)$ is a (complex) polynomial of 2 variables, $D = (\partial/\partial x_1, \partial/\partial x_2)$, and g is an entire function of u with all c_j constant (complex or real).

We are interested in finding solutions of (1) represented at infinity as formal sums of Laplace transforms of Laplace distributions (see [P-Z] and [P]). In [P-Z] we have solved (1) under the assumption that the coefficients $c_j = c_j(x)$, $j = 1, 2, \ldots$, are Laplace integrals of some Laplace holomorphic functions T_j, and $c_0 \equiv 0$. In [P] we have assumed $g(0) = 0$, a condition not required here. However, we will need to assume more on the set of zeros of P.

Similarly to [P-Z] we solve the convolution equation generated by (1) with an unknown Laplace distribution T. It is clear that we will need some results on the convolution algebra structure in the space of Laplace distributions.

This paper was written by the first author after many fruitful discussions with Professor Bogdan Ziemian, who did not live to see this work completed.

1991 *Mathematics Subject Classification*: 44A10, 46F20.
Key words and phrases: Laplace distributions, Laplace transforms, convolution equations.
Supported by KBN grant 2-P03A-006-08.

2. Notation, definitions and general assumptions. Following [S-Z], for $a \in \mathbb{R}^n$ we define

$$L_a(\overline{\mathbb{R}}^n_+) = \{\phi \in C^\infty(\overline{\mathbb{R}}^n_+) : \sup_{x\in\overline{\mathbb{R}}^n_+} |e^{-ax}(\partial/\partial x)^\nu \phi(x)| < \infty,\ \nu \in \mathbb{N}^n_0\}$$

with convergence defined by the seminorms

$$\|\phi\|_{a,\nu} = \sup_{x\in\overline{\mathbb{R}}^n_+} |e^{-ax}(\partial/\partial x)^\nu \phi(x)|,$$

and for $\omega \in (\mathbb{R}\cup\{\infty\})^n$ we define

$$L_{(\omega)}(\overline{\mathbb{R}}^n_+) = \varinjlim_{a<\omega} L_a(\overline{\mathbb{R}}^n_+),$$

equipped with the inductive limit topology. The dual space $L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ is a subspace of $D'(\overline{\mathbb{R}}^n_+)$, and we call it the space of *Laplace distributions* on $\overline{\mathbb{R}}^n_+$. We write simply $L_a, L_{(\omega)}, L'_{(\omega)}$ when $n = 2$ and no confusion can arise.

Let $\operatorname{Char} P = \{z \in \mathbb{C}^2 : P(z) = 0\}$. Our basic assumptions on P are the following:

(i) $0 \notin \operatorname{Char} P$;

(ii) there exists an unbounded curve $\widetilde{Z} \subset \operatorname{Char} P$ such that after some linear transformation $A : \mathbb{C}^2 \to \mathbb{C}^2$ with $\det A \neq 0$, $Z = A(\widetilde{Z}) \subset \mathbb{R}^2_+$;

(iii) $Z = \{(x, f(x)) : x \in \mathbb{R}_+\}$ for some $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f \in \mathcal{C}^\infty$, $f'' > 0$, $f(x + y) < f(x) + f(y)$ for every $x, y \in \mathbb{R}_+$, and Z is a curve with ends at infinity;

(iv) with the notation $\widetilde{P} = P \circ A^{-1}$, we assume $\widetilde{P} \neq 0$ on $\mathbb{R}^2_+ \setminus Z$ and either $\widetilde{P}_{z_1} = \partial\widetilde{P}/\partial z_1 > 0$ and $\widetilde{P}_{z_2} = \partial\widetilde{P}/\partial z_2 \geq 0$, or $\widetilde{P}_{z_1} \geq 0$ and $\widetilde{P}_{z_2} > 0$ on $\mathbb{R}^2_+$.

To simplify notation we use the same letter P for $\widetilde{P}$.

By (iii) it is obvious that $2Z \cap Z = \emptyset$. Here and subsequently jZ stands for the algebraic sum $Z + \ldots + Z$ with j summands.

3. Properties of Z

Lemma 1. *Let assumptions* (i)–(iv) *hold. Then for* $j \geq 2$,

$$jZ = \{(x, y) \in \mathbb{R}^2_+ : P(x/j, y/j) \geq 0\}.$$

Proof. If $(x, y) \in jZ$, then $x = x_1 + \ldots + x_j$ and $y = f(x_1) + \ldots + f(x_j)$. The convexity of f and monotonicity of P in y give

$$P\left(\frac{x}{j}, \frac{y}{j}\right) = P\left(\frac{x_1}{j} + \ldots + \frac{x_j}{j}, \frac{f(x_1)}{j} + \ldots + \frac{f(x_j)}{j}\right) \geq P\left(\frac{x}{j}, f\left(\frac{x}{j}\right)\right) = 0.$$

Conversely, if $P(x/j, y/j) \geq 0$, then $y \geq jf(x/j)$. Writing $x_k = x/j$ for $k = 3, \ldots, j$ and $x_1 = \alpha x/j$, $x_2 = (2 - \alpha)x/j$ for $\alpha \in (0, 2)$ we obtain

$x = x_1 + \ldots + x_j$. We proceed to show that there exists α such that $y = f(x_1) + \ldots + f(x_j)$. Let $F(\alpha) = f(\alpha x/j) + f((2-\alpha)x/j) - 2f(x/j)$ and let $\beta = y - jf(x/j)$. It is easy to check that $F(1) = 0$ and $\lim F(\alpha) = \infty$ at 0 and 2. By continuity of F, there exists α such that $F(\alpha) = \beta$. This proves the lemma.

Clearly, if $k < j$ then $jZ \subset kZ$.

LEMMA 2. *For every* $k = 2, 3, \ldots,$

$$kZ \setminus (k+1)Z \neq \emptyset$$

and $\mathbb{R}^2_+ = \bigcup_{k=2}^{\infty} (\mathbb{R}^2_+ \setminus kZ)$.

Proof. Taking $x = kt$ for some $t \in \mathbb{R}_+$, and $y = kf(t)$ we get $(x, y) \in kZ$ and $P(x/k, y/k) = P(t, f(t)) = 0$. But

$$P\left(\frac{x}{k+1}, \frac{y}{k+1}\right) = P\left(\frac{k}{k+1}t, \frac{k}{k+1}f(t)\right) < P(t, f(t)) = 0.$$

This gives $(x, y) \notin (k+1)Z$.

Let $(x, y) \in \mathbb{R}^2_+$. Since $P(x/k, y/k)$ tends to $P(0,0) < 0$ as $k \to \infty$, we have $P(x/k, y/k) < 0$ for k sufficiently large, and by Lemma 1, $(x, y) \notin kZ$.

4. Convolution equation. The function $\phi_x(z) = e^{-xz}$ belongs to the space $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ for every $\omega \in \mathbb{R}^n$ with $\omega > -x$ ($x \in \mathbb{R}^n$ fixed). Our aim is to find a solution u of equation (1) in the form

$$u(x) = T[\phi_x] = T[e^{-xz}], \tag{2}$$

with T being a Laplace distribution on $\overline{\mathbb{R}}^2_+$. Applying $P(D)$ and g to u in the form (2) we get the convolution equation

$$P(z)T = g_*(T) = \sum_{j=0}^{\infty} c_j T^{*j}, \tag{3}$$

where $T^{*0} = \delta_0$ (Dirac delta at $(0,0)$). We are looking for a solution T of (3) in the form of a formal series of Laplace distributions

$$T = \sum_{k=0}^{\infty} T_k. \tag{4}$$

For convenience we consider a slightly modified equation

$$P(z)T = \varepsilon \sum_{j=0}^{\infty} c_j T^{*j} \tag{5}$$

with $\varepsilon > 0$ and we look for T in the form

$$T = \sum_{k=0}^{\infty} \varepsilon^k T_k. \tag{6}$$

Inserting (6) in (5) we get

$$\sum_{k=0}^{\infty} \varepsilon^k P(z) T_k = \varepsilon c_0 \delta_0 + \sum_{k=1}^{\infty} \Big[\sum_{j=1}^{\infty} c_j \sum_{k_1+\ldots+k_j=k-1} T_{k_1} * \ldots * T_{k_j} \Big] \varepsilon^k.$$

Hence, comparing the summands with the same power of ε we obtain the recurrence system

$$P(z)T_0 = 0, \tag{7}$$

$$P(z)T_1 = c_0\delta_0 + \sum_{j=1}^{\infty} c_j T_0^{*j} = g_*(T_0), \tag{8}$$

$$P(z)T_k = \sum_{j=1}^{\infty} c_j \sum_{k_1+\ldots+k_j=k-1} T_{k_1} * \ldots * T_{k_j}. \tag{9}$$

LEMMA 3. *Let $\omega \in \mathbb{R}^2_+$ and $\Phi \in L_{(-\omega)}$. If T_0 is defined by*

$$T_0[\phi] = \int_Z \phi(x)\Phi(x)\, dx \tag{10}$$

for Z defined in (iii) *and for $\phi \in L_{(\omega)}$, then $T_0 \in L'_{(\omega)}$ and T_0 solves* (7).

Proof. The proof is immediate.

Observe that $\operatorname{supp} T_0 \subset Z$ and for $\phi \in L_a$, $a < \omega$,

$$|T_0[\phi]| \le \int_Z |\phi(x)e^{-ax}| \cdot |\Phi(x)e^{ax}|\, dx \le \|\phi\|_{a,0} K_a$$

where $K_a = \int_Z |\Phi(x)|e^{ax}\, dx$.

Let $L'_{(\omega)}(Z)$ denote the subspace of $L'_{(\omega)}$ defined by

$$L'_{(\omega)}(Z) = \Big\{ S \in L'_{(\omega)} : S = \sum_{k=0}^{\infty} S_k,\ S_0 = a\delta_0,\ \operatorname{supp} S_k \subset kZ,\ a \in \mathbb{C} \Big\}.$$

LEMMA 4. *$L'_{(\omega)}(Z)$ is a convolution algebra, i.e. if $S, R \in L'_{(\omega)}(Z)$ then $S * R \in L'_{(\omega)}(Z)$.*

Proof. This follows immediately from the properties of Z and of convolution. Namely if $S = \sum_{j=0}^{\infty} S_j$ and $R = \sum_{j=0}^{\infty} R_j$ then

$$S * R = \sum_{j=0}^{\infty} \sum_{p=0}^{j} S_p * R_{j-p}$$

and for every $p \le j$, $\operatorname{supp}(S_p * R_{j-p}) = \operatorname{supp} S_p + \operatorname{supp} R_{j-p} \subset pZ + (j-p)Z = jZ$.

Moreover, if

$$S_p[\phi] = \int_{Z^p} \phi(x_1 + \ldots + x_p)\Phi_p(x_1, \ldots, x_p)\, dx_1 \ldots dx_p,$$

$$R_l[\phi] = \int_{Z^l} \phi(y_1 + \ldots + y_l)\Psi_l(y_1, \ldots, y_l)\, dy_1 \ldots dy_l$$

for $\Phi_p \in L_{(-\omega,\ldots,-\omega)}((\overline{\mathbb{R}}^2_+)^p)$, $\Psi_l \in L_{(-\omega,\ldots,-\omega)}((\overline{\mathbb{R}}^2_+)^l)$, $1 \le p, l \le j-1$, then

$$\begin{aligned} S_p * R_{j-p}[\phi] &= \int_{Z^j} \phi(z_1 + \ldots + z_j)\Phi_p(z_1, \ldots, z_p)\Psi_{j-p}(z_{p+1}, \ldots, z_j)\, dz_1 \ldots dz_j \\ &= \int_{Z^j} \phi(z_1 + \ldots + z_j)\Theta_{j,p}(z_1, \ldots, z_j)\, dz_1 \ldots dz_j, \end{aligned}$$

and $\Theta_{j,p} \in L_{(-\omega,\ldots,-\omega)}((\overline{\mathbb{R}}^2_+)^j)$.

Observe that for $\phi \in L_a$, $a < \omega$,

$$T_0^{*j}[\phi] = \int_{Z^j} \phi(x_1 + \ldots + x_j)\Phi(x_1) \ldots \Phi(x_j)\, dx_1 \ldots dx_j,$$

hence

$$\begin{aligned} |T_0^{*j}[\phi]| &\le \|\phi\|_{a,0} \int_{Z^j} |\Phi(x_1)| \ldots |\Phi(x_j)| e^{a(x_1 + \ldots + x_j)}\, dx_1 \ldots dx_j \\ &= \|\phi\|_{a,0} \Big(\int_Z |\Phi(x)| e^{ax}\, dx \Big)^j = \|\phi\|_{a,0} (K_a)^j, \end{aligned}$$

and $\operatorname{supp} T_0^{*j} \subset jZ$. Therefore we see clearly that $g_*^{(\nu)}(T_0) \in L'_{(\omega)}(Z)$ for $\nu = 0, 1, \ldots,$

$$|g_*(T_0)[\phi]| \le \|\phi\|_{a,0} \sum_{j=0}^{\infty} |c_j| (K_a)^j = \|\phi\|_{a,0} |g_*|(K_a)$$

and

$$|g_*^{(\nu)}(T_0)[\phi]| \le \|\phi\|_{a,0} \sum_{j=0}^{\infty} |c_{j+\nu}|(j+\nu) \ldots (j+1)(K_a)^j = \|\phi\|_{a,0} |g_*^{(\nu)}|(K_a),$$

where

$$|g_*|(x) = \sum_{j=0}^{\infty} |c_j| x^j, \qquad |g_*^{(\nu)}|(x) = \sum_{j=0}^{\infty} |c_{j+\nu}|(j+\nu) \ldots (j+1) x^j.$$

5. Problem of division

LEMMA 5. *Let T_0 be a Laplace distribution defined by* (10) *and suppose a polynomial P satisfies* (i)–(iv). *If $P_{x_1} > 0$ on $\overline{\mathbb{R}}^2_+$, then the distribution S_0*

defined by

$$S_0[\phi] = \int_Z (\phi/P_{x_1})_{x_1}(x)\Phi(x)\,dx \tag{11}$$

is a solution of the equation $PS = T_0$, $S_0 \in L'_{(\omega)}$ *and* $\operatorname{supp} S_0 \subset Z$.

Proof. It is obvious that if $\phi \in L_{(\omega)}$ then also $P\phi \in L_{(\omega)}$. Therefore (by the definition of multiplication of distributions by regular functions) an easy computation shows that

$$\begin{aligned}(PS_0)[\phi] = S_0[P\phi] &= \int_Z \left(\frac{P\phi}{P_{x_1}}\right)_{x_1}(x)\Phi(x)\,dx\\ &= \int_Z \phi(x)\Phi(x)\,dx + \int_Z P(x)\left(\frac{\phi}{P_{x_1}}\right)_{x_1}(x)\Phi(x)\,dx = T_0[\phi].\end{aligned}$$

It is seen immediately that $S_0 \in L'_{(\omega)}$ and $\operatorname{supp} S_0 \subset Z$. We also have

$$\begin{aligned}|S_0[\phi]| &= \left|\int_Z \phi_{x_1}(x)\frac{\Phi(x)}{P_{x_1}(x)}\,dx - \int_Z \phi(x)\frac{P_{x_1x_1}(x)}{P_{x_1}(x)^2}\Phi(x)\,dx\right|\\ &\le K'_a\|\phi\|_{a,(1,0)} + K''_a\|\phi\|_{a,0} \le \widehat{K}_a\|\phi\|_{a,0}\end{aligned}$$

where

$$K'_a = \int_Z \left|\frac{\Phi(x)}{P_{x_1}(x)}\right| e^{ax}\,dx, \qquad K''_a = \int_Z \left|\frac{P_{x_1x_1}(x)\Phi(x)}{[P_{x_1}(x)]^2}\right| e^{ax}\,dx.$$

The distribution S_0 defined by (11) will be denoted by $\frac{T_0}{P}$.

Now we are in a position to solve the equation (8).

Lemma 6. *Under the assumptions of Lemma* 5 *the distribution* T_1 *defined by*

$$T_1 = \frac{g_*(T_0)}{P} = \sum_{j=0}^{\infty} c_j \frac{T_0^{*j}}{P} \tag{12}$$

is a solution of (8) *and* $T_1 \in L'_{(\omega)}(Z)$.

Proof. The distribution T_1 is well defined. Indeed, for $j = 0$,

$$\frac{\delta_0}{P}[\phi] = \delta_0\left[\frac{\phi}{P}\right] = \frac{1}{P(0)}\delta_0[\phi].$$

For $j = 1$, $\frac{T_0}{P}$ is defined by (11), and for $j \ge 2$, $\frac{T_0^{*j}}{P}$ makes sense because $P \ne 0$ on $\operatorname{supp} T_0^{*j} \subset jZ$. It is also clear that $T_1 \in L'_{(\omega)}(Z)$. Moreover, we can choose constants N_a and M_a such that

$$|T_1[\phi]| \le N_a|g_*|(M_a)\|\phi\|_{a,0}$$

for every $\phi \in L_a$, $a < \omega$.

LEMMA 7. *For every $k \ge 2$, there exists a solution T_k of* (9) *such that* $T_k \in L'_{(\omega)}(\overline{\mathbb{R}}^2_+)$.

Proof. We write (9) as a sum of a finite number of summands:

$$\begin{aligned} PT_k &= \sum_{p=1}^{k-1} \sum_{\substack{k_1+\ldots+k_p=k-1 \\ 1\le k_i\le k-1}} T_{k_1} * \ldots * T_{k_p} * \sum_{j=p}^{\infty} c_j \binom{j}{p} T_0^{*(j-p)} \\ &= \sum_{p=1}^{k-1} \sum_{\substack{k_1+\ldots+k_p=k-1 \\ 1\le k_i\le k-1}} T_{k_1} * \ldots * T_{k_p} * \frac{1}{p!} g_*^{(p)}(T_0). \end{aligned}$$

From Lemma 4 and from remarks of Section 3 it follows that the right-hand side of the expression above is a Laplace distribution belonging to $L'_{(\omega)}(Z)$, hence by Lemma 5 it can be divided by P. Therefore

$$T_k = \sum_{p=1}^{k-1} \sum_{\substack{k_1+\ldots+k_p=k-1 \\ 1\le k_i\le k-1}} \frac{1}{P}\left(T_{k_1} * \ldots * T_{k_p} * \frac{1}{p!} g_*^{(p)}(T_0)\right) \tag{13}$$

is a Laplace distribution, $T_k \in L'_{(\omega)}(Z)$.

6. Solution of the main problem. Given $\omega \in \mathbb{R}^2$ and $\Phi \in L_{(-\omega)}$, by Lemmas 3, 6 and 7 we obtain a formal series (6) as a solution of (5) with T_0 given by (10), T_1 given by (12) and T_k given by (13), for $k \ge 2$. Putting $\varepsilon = 1$ in (5) and (6) we have a formal solution of (3) in the form (4).

Thus we have proved

THEOREM. *Let assumptions* (i)–(iv) *hold. Then for every $\omega \in \mathbb{R}^2$ and every $\Phi \in L_{(-\omega)}$ there exists a formal solution u of equation* (1) *of the form*

$$u(x) = \sum_{k=0}^{\infty} u_k(x)$$

where $u_k \in \mathcal{C}^\infty([-a,\infty)) = \mathcal{C}^\infty([-a_1,\infty) \times [-a_2,\infty))$ for all $a < \omega$ and k, and $u_k(x) = T_k[e^{-zx}]$ for some $T_k \in L_{(-\omega)}$.

REMARK. By a "formal solution" here we understand that the series $\sum_{k=0}^{\infty} u_k(x)$ only formally solves the equation (1), and it does not necessarily converge. Convergence results in some cases will appear in a forthcoming publication.

7. Example. The problem considered in this paper was motivated by attempts to solve at infinity the well known equation

$$\Delta u = e^u, \tag{14}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The problem has a long history, beginning with early results of Bieberbach [B] and of many other authors. Most of them have been solving (14) in bounded domains in $\mathbb{R}^n$, with some assumptions on the boundaries. Recently, using various methods, many people have been constructing global solutions; for example Popov [Po] has constructed global exact solutions of (14) from solutions of the Laplace equation. Here we find formal solutions represented at infinity as sums of the Laplace transforms of Laplace distributions.

If we write $P(D) = \Delta - 1$, then (14) is a particular case of (1) with $g(u) = 1 + \sum_{j=2}^{\infty} \frac{1}{j!} u^j$. Here $P(z) = z_1^2 + z_2^2 - 1$.

We now show that P satisfies conditions (i)–(iv) of Section 2. Indeed, $0 \notin \operatorname{Char} P$ and the set

$$\widetilde{Z} = \{(z_1, z_2) : z_1 = ik,\ z_2 = \sqrt{1 + k^2},\ k \in \mathbb{R}\}$$

is an unbounded curve in $\operatorname{Char} P$. Set $A = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$. Then $\det A = -2i \neq 0$ and

$$Z = A(\widetilde{Z}) = \{(x, y) \in \mathbb{R}_+^2 : y = 1/x\},$$

hence (iii) holds with $f(x) = 1/x$. An easy calculation shows that $\widetilde{P}(z) = P(A^{-1}(z)) = z_1 z_2 - 1$, $\widetilde{P}_{z_1}(z) = z_2$, $\widetilde{P}_{z_2}(z) = z_1$, so (iv) holds, too.

Now, putting $u(x) = T[e^{-vx}]$ and $\widetilde{T} = T \circ A^{-1}$ we have

$$\begin{aligned} P(D)u(x) &= (P(v)T)[e^{-vx}] = (\widetilde{P}(z)\widetilde{T})[e^{-A^{-1}(z)x} \cdot 1/2] \\ &= \widetilde{P}(z)\widetilde{T}[e^{-(i/2)(z_1 - z_2)x_1 - (1/2)(z_1 + z_2)x_2} \cdot 1/2] \\ &= \tfrac{1}{2}\widetilde{P}(z)\widetilde{T}[e^{-(1/2)(ix_1 + x_2)z_1 - (1/2)(-ix_1 + x_2)z_2}] \\ &= \widetilde{P}(D)\widetilde{u}\big(\tfrac{1}{2}(ix_1 + x_2), \tfrac{1}{2}(-ix_1 + x_2)\big) \end{aligned}$$

where $\widetilde{u}(y) = \frac{1}{2}\widetilde{T}[e^{-yz}]$. Therefore, by the method described in the previous sections we solve the equation

$$(z_1 z_2 - 1)T = \delta_0 + \sum_{j=2}^{\infty} \frac{1}{j!} T^{*j}.$$

Namely, for $\omega \in \overline{\mathbb{R}}_+^2$, $\Phi \in L_{(-\omega)}$ we put, according to (10),

$$T_0[\phi] = \int_Z \phi(z)\Phi(z)\, dz = \int_0^{\infty} \phi(t, 1/t)\widehat{\Phi}(t)\, dt$$

where $\widehat{\Phi}(t) = \Phi(t, 1/t)\sqrt{1 + t^{-4}}$. Then

$$u_0(x) = T_0[e^{-zx}] = \int_0^{\infty} e^{-tx_1 - (1/t)x_2} \widehat{\Phi}(t)\, dt.$$

Now, according to (11), we have

$$\frac{T_0}{P}[\phi] = \int_0^\infty \phi_{z_1}(t, 1/t) t \widehat{\Phi}(t)\, dt,$$

so

$$\frac{T_0}{P}[e^{-zx}] = x_1 \int_0^\infty e^{-tx_1-(1/t)x_2} t\widehat{\Phi}(t)\, dt.$$

We observe that

$$P\left(t_1+\ldots+t_j, \frac{1}{t_1}+\ldots+\frac{1}{t_j}\right) = (t_1+\ldots+t_j)\left(\frac{1}{t_1}+\ldots+\frac{1}{t_j}\right) - 1$$
$$= j-1+\sum_{1\le k<l\le j}\left(\frac{t_k}{t_l}+\frac{t_l}{t_k}\right).$$

Hence, if we define $P_j(t_1,\ldots,t_j) = P(t_1+\ldots+t_j, 1/t_1+\ldots+1/t_j)$, then

$$T_0^{*j}[\phi] = \int_{\mathbb{R}_+^j} \phi\left(t_1+\ldots+t_j, \frac{1}{t_1}+\ldots+\frac{1}{t_j}\right)\widehat{\Phi}(t_1)\ldots\widehat{\Phi}(t_j)\, dt_1\ldots dt_j,$$

and for $j \ge 2$,

$$\frac{T_0^{*j}}{P}[\phi] = T_0^{*j}\left[\frac{\phi}{P}\right]$$
$$= \int_{\mathbb{R}_+^j} \phi\left(t_1+\ldots+t_j, \frac{1}{t_1}+\ldots+\frac{1}{t_j}\right)\frac{\widehat{\Phi}(t_1)\ldots\widehat{\Phi}(t_j)}{P_j(t_1,\ldots,t_j)}\, dt_1\ldots dt_j.$$

Therefore, by (12) we get

$$u_1(x) = T_1[e^{-zx}]$$
$$= -1+\sum_{j=2}^\infty \frac{1}{j!}\int_{\mathbb{R}_+^j} e^{-(t_1+\ldots+t_j)x_1-(1/t_1+\ldots+1/t_j)x_2}\frac{\widehat{\Phi}(t_1)\ldots\widehat{\Phi}(t_j)}{P_j(t_1,\ldots,t_j)}\, dt_1\ldots dt_j.$$

Of course $g'(u) = \sum_{j=1}^\infty (1/j!)u^j$, and $g^{(k)}(u) = g''(u) = \sum_{j=0}^\infty (1/j!)u^j$ for $k \ge 2$, hence, applying (13), e.g. for $k = 2$, we obtain

$$T_2 = \frac{1}{P}\left(\left(-\delta_0+\sum_{j=2}^\infty \frac{1}{j!}\cdot\frac{T_0^{*j}}{P}\right) * \sum_{j=0}^\infty \frac{1}{j!}T_0^{*j}\right)$$
$$= -\frac{T_0}{P} - \sum_{j=2}^\infty \frac{1}{j!}\cdot\frac{T_0^{*j}}{P} + \sum_{p=3}^\infty \sum_{\substack{k+l=p\\ k\ge 2,\, l\ge 1}} \frac{1}{k!l!}\cdot\frac{1}{P}\left(\frac{T_0^{*k}}{P} * T_0^{*l}\right).$$

Putting $p=k+l$ for $k\ge 2$ and $l\ge 1$, we have

$$\frac{1}{P}\bigg(\frac{T_0^{*k}}{P}*T_0^{*l}\bigg)[e^{-zx}]=\bigg(\frac{T_0^{*k}}{P}*T_0^{*l}\bigg)\bigg[\frac{e^{-zx}}{P}\bigg]$$
$$=\int\limits_{\mathbb{R}_+^p} e^{-(t_1+\ldots+t_p)x_1-(1/t_1+\ldots+1/t_p)x_2}\,\frac{\widehat{\Phi}(t_1)\ldots\widehat{\Phi}(t_p)}{P_p(t_1,\ldots,t_p)P_k(t_1,\ldots,t_k)}\,dt_1\ldots dt_p,$$

hence

$$u_2(x)=\,-x_1\int\limits_0^\infty e^{-tx_1-(1/t)x_2}t\widehat{\Phi}(t)\,dt$$
$$-\sum_{j=2}^\infty\frac{1}{j!}\int\limits_{\mathbb{R}_+^j} e^{-(t_1+\ldots+t_j)x_1-(1/t_1+\ldots+1/t_j)x_2}\,\frac{\widehat{\Phi}(t_1)\ldots\widehat{\Phi}(t_j)}{P_j(t_1,\ldots,t_j)}\,dt_1\ldots dt_j$$
$$+\sum_{p=3}^\infty\int\limits_{\mathbb{R}_+^p} e^{-(t_1+\ldots+t_p)x_1-(1/t_1+\ldots+1/t_p)x_2}\,\frac{\widehat{\Phi}(t_1)\ldots\widehat{\Phi}(t_p)}{P_p(t_1,\ldots,t_p)}$$
$$\times\sum_{\substack{k+l=p\\ k\ge 2,\, l\ge 1}}\frac{1}{k!l!}\cdot\frac{1}{P_k(t_1,\ldots,t_k)}.$$

Now, for $k=3$ in (13), we have

$$T_3=\frac{1}{P}(T_2*g'(T_0))+\frac{1}{2!}\cdot\frac{1}{P}(T_1^{*2}*g''(T_0))$$
$$=-\sum_{p=2}^\infty\sum_{\substack{k+l=p\\ k,l\ge 1}}\frac{1}{k!l!}\cdot\frac{1}{P}\bigg(\frac{T_0^{*k}}{P}*T_0^{*l}\bigg)$$
$$+\sum_{p=4}^\infty\sum_{\substack{k_1+k_2+k_3=p\\ k_1\ge 2,\, k_2,k_3\ge 1}}\frac{1}{2!}\cdot\frac{1}{k_1!k_2!k_3!}\cdot\frac{1}{P}\bigg(\frac{1}{P}\bigg(\frac{T_0^{*k_1}}{P}*T_0^{*k_2}\bigg)*T_0^{*k_3}\bigg).$$

By analogy to the previous cases we can calculate $u_3(x)=T_3[e^{-zx}]$. Finally, by similar calculations we find u_k.

References

[B] L. Bieberbach, *$\Delta u=e^u$ und die automorphen Funktionen*, Math. Ann. 77 (1916), 173–212.

[P] M. E. Pliś, *Poincaré theorem and nonlinear PDE's*, Ann. Polon. Math. 69 (1998), 99–105.

[P-Z] M. E. Pliś and B. Ziemian, *Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables*, ibid. 67 (1997), 31–41.

[Po] A. G. Popov, *The non-euclidean geometry and differential equations*, in: Banach Center Publ. 33, Inst. Math., Polish Acad. Sci., 1996, 297–308.

[S-Z] Z. Szmydt and B. Ziemian, *Laplace distributions and hyperfunctions on* $\overline{\mathbb{R}}^n_+$, J. Math. Sci. Univ. Tokyo 5 (1998), 41–74.

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Reçu par la Rédaction le 27.10.1997
Révisé le 25.5.1998