ANNALES
POLONICI MATHEMATICI
LXIX.3 (1998)

Multiple positive solutions of a nonlinear fourth order periodic boundary value problem

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Abstract. The fourth order periodic boundary value problem $u^{(4)}-m^4u+F(t,u)=0$, $0<t<2\pi$, with $u^{(i)}(0)=u^{(i)}(2\pi)$, $i=0,1,2,3$, is studied by using the fixed point index of mappings in cones, where F is a nonnegative continuous function and $0<m<1$. Under suitable conditions on F, it is proved that the problem has at least two positive solutions if $m\in(0,M)$, where M is the smallest positive root of the equation $\tan m\pi=-\tanh m\pi$, which takes the value 0.7528094 with an error of $\pm 10^{-7}$.

1. Introduction. This paper deals with the fourth order periodic boundary value problem

$$
\begin{cases} u^{(4)}-m^4u+F(t,u)=0, & 0<t<2\pi, \\ u^{(i)}(0)=u^{(i)}(2\pi), & i=0,1,2,3, \end{cases} \tag{1.1}
$$

where $0<m<1$ and $F:[0,2\pi]\times[0,\infty)\to[0,\infty)$ is a nonnegative continuous function.

Recently, the periodic boundary value problems have been studied extensively (see [1–2, 4–7] and references therein). In [1], A. Cabada studied a fourth order periodic boundary value problem similar to (1.1), using a generalized method of upper and lower solutions and developing the monotone iterative technique in the presence of upper and lower solutions, but he did not study the multiplicity of the solutions.

The purpose of this paper is to study the existence of multiple positive solutions to the problem (1.1) by using the fixed point index of mappings in cones. Our method is different from [1] and yields a multiplicity result for positive solutions.

The following hypotheses are adopted in this paper, depending on various circumstances:

1991 *Mathematics Subject Classification*: 34B10, 34B15, 34C25.

Key words and phrases: nonlinear periodic boundary value problem, multiple positive solutions, cone, fixed point index.

(H1) There exists a $p > 0$ such that $0 \le u \le p$ implies $F(t,u) < \lambda p$, where

$$\lambda = \frac{1}{2\pi G(\pi,m)}, \quad G(\pi,m) = \frac{1}{4m^3}\left(\frac{1}{\sinh m\pi} + \frac{1}{\sin m\pi}\right). \tag{1.2}$$

(H2) There exists a $p > 0$ such that $\sigma p \le u \le p$ implies $F(t,u) > (\lambda/\sigma)p$, where

$$\sigma = \frac{(e^{2m\pi} - 1)\cos m\pi + (e^{2m\pi} + 1)\sin m\pi}{e^{2m\pi} + 2e^{m\pi}\sin m\pi - 1}. \tag{1.3}$$

We call a function $u(t)$ a *positive solution* of (1.1) if it satisfies:

(1) $u \in C^3[0,2\pi] \cap C^4(0,2\pi)$, $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0,1,2,3$, and $u(t) > 0$ for all $t \in (0,2\pi)$, and

(2) the equality $u^{(4)} - m^4 u = -F(t,u)$ holds for all $t \in (0,2\pi)$.

The main result of this paper is as follows.

THEOREM 1. *If $m \in (0,M)$, then the problem* (1.1) *has at least two positive solutions u_1 and u_2 satisfying $0 < \|u_1\| < p < \|u_2\|$ provided that*

(I) *the condition* (H1) *holds and*

$$\lim_{u\to 0}\min_{t\in[0,2\pi]}\frac{F(t,u)}{u} > \frac{\lambda}{\sigma^2}, \qquad \lim_{u\to \infty}\min_{t\in[0,2\pi]}\frac{F(t,u)}{u} > \frac{\lambda}{\sigma^2}, \qquad \textit{or}$$

(II) *the condition* (H2) *holds and*

$$\lim_{u\to 0}\max_{t\in[0,2\pi]}\frac{F(t,u)}{u} < \lambda, \qquad \lim_{u\to \infty}\max_{t\in[0,2\pi]}\frac{F(t,u)}{u} < \lambda,$$

where M is the smallest positive root of the equation $\tan m\pi = -\tanh m\pi$, and λ is given by (1.2).

The following theorem will be used in our proof (see [3]).

THEOREM 2. *Let E be a Banach space, and $K \subseteq E$ a cone in E. For $p > 0$, define $K_p = \{u \in K : \|u\| \le p\}$. Assume that $\Phi : K_p \to K$ is a compact map such that $\Phi u \ne u$ for $u \in \partial K_p = \{u \in K : \|u\| = p\}$.*

(i) *If $\|u\| \le \|\Phi u\|$ for $u \in \partial K_p$, then $i(\Phi, K_p, K) = 0$.*

(ii) *If $\|u\| \ge \|\Phi u\|$ for $u \in \partial K_p$, then $i(\Phi, K_p, K) = 1$.*

2. Proof of Theorem 1. As shown in [1], problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds \tag{2.1}$$

where

$$G(t,s,m)=G(|t-s|,m) \tag{2.2}$$
$$=\begin{cases}\dfrac{f(t-s)+g(t-s)}{4m^3(e^{m\pi}-e^{-m\pi})^2(1-\cos 2m\pi)}, & 0\le s\le t\le 2\pi,\\ \dfrac{f(2\pi+t-s)+g(2\pi+t-s)}{4m^3(e^{m\pi}-e^{-m\pi})^2(1-\cos 2m\pi)}, & 0\le t\le s\le 2\pi,\end{cases}$$

and

$$f(t)=(e^{m\pi}-e^{-m\pi})^2(\sin mt+\sin m(2\pi-t)), \tag{2.3}$$

$$g(t)=(e^{mt}-e^{-mt}+e^{m(2\pi-t)}-e^{-m(2\pi-t)})(1-\cos 2m\pi). \tag{2.4}$$

LEMMA 1. *If $m\in(0,1)$, then the function $G(t,m)$ in the interval $[0,2\pi]$ attains its minimum for $t=0$ and its maximum for $t=\pi$.*

Proof. Let $w(t)=f(t)+g(t)$. Since $w(2\pi-t)=w(t)$, it suffices to consider the function w in the interval $[0,\pi]$.

If $m\in(0,1/2]$, then by a direct computation, we get $w^{(4)}(t)>0$ in $[0,\pi]$ and $w'''(\pi)=0$, and hence $w'''(t)\le 0$ in $[0,\pi]$. Thus, $w'(t)$ is a concave function in $[0,\pi]$. Moreover, since $w'(0)=0$ and $w'(\pi)=0$, we have $w'(t)\ge 0$ in $[0,\pi]$. Therefore, $w(t)$ is nondecreasing in $[0,\pi]$.

In [1] it is proved that, if $m\in(1/2,1)$, then the unique root of f in $[0,\pi]$ is $\frac{2m-1}{2m}\pi$, and $w(t)$ is nondecreasing in $\big[0,\frac{2m-1}{2m}\pi\big]$. We claim that, if $m\in(1/2,1)$, then $w(t)$ is also nondecreasing in $\big[\frac{2m-1}{2m}\pi,\pi\big]$. In fact, it is not difficult to show that $w^{(4)}(t)>0$ in $\big[\frac{2m-1}{2m}\pi,\pi\big]$ and $w'''(\pi)=0$, so $w'''(t)\le 0$ in $\big[\frac{2m-1}{2m}\pi,\pi\big]$, and hence $w'(t)$ is concave in $\big[\frac{2m-1}{2m}\pi,\pi\big]$. Moreover, since $w'\big(\frac{2m-1}{2m}\pi\big)>0$ and $w'(\pi)=0$, we have $w'(t)\ge 0$ in $\big[\frac{2m-1}{2m}\pi,\pi\big]$. This shows our claim.

To sum up, the function $w(t)$ attains its minimum in $[0,2\pi]$ at $t=0$ and its maximum at $t=\pi$, and so does $G(t,m)$. The proof is complete.

By Lemma 1, the greatest value of m for which $G(t,m)$ is positive in $[0,2\pi]$ will be the smallest positive zero of the expression

$$w(0)=(e^{2m\pi}-e^{-2m\pi})(1-\cos 2m\pi)+(e^{m\pi}-e^{-m\pi})^2\sin 2m\pi.$$

This expression is zero if and only if either $m\in\mathbb{N}$ or

$$\tan m\pi=-\tanh m\pi. \tag{2.5}$$

The smallest positive root of (2.5), which we denote by M, takes a value of 0.7528094 with an error of $\pm 10^{-7}$. This is the unique root in $(0,1)$ (see [1]).

Let $m\in(0,M)\subset(0,1)$. Then $G(0,m)>0$. Define the mapping $\Phi:C[0,2\pi]\to C[0,2\pi]$ by

$$(\Phi u)(t):=\int_0^{2\pi}G(t,s,m)F(s,u(s))\,ds. \tag{2.6}$$

It is obvious that Φ is completely continuous. We define a cone in the Banach space $C[0,2\pi]$ by

$$K := \{u \in C[0,2\pi] : u(t) \geq 0 \text{ for all } t \text{ and } \min_{t\in[0,2\pi]} u(t) \geq \sigma\|u\|\},$$

where $\|u\| = \sup_{t\in[0,2\pi]}|u(t)|$ and σ is given by (1.3).

LEMMA 2. *$\Phi(K) \subset K$.*

Proof. Lemma 1 implies

$$\sigma = \frac{G(0,m)}{G(\pi,m)} \leq \frac{G(t,s,m)}{G(\pi,m)} \leq 1,$$

and hence for $u \in K$ we have

$$\begin{aligned}\min_{t\in[0,2\pi]}(\Phi u)(t) &= \min_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds \\ &\geq \sigma \int_0^{2\pi} G(\pi,m)F(s,u(s))\,ds \\ &\geq \sigma \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds = \sigma\|\Phi u\|.\end{aligned}$$

This shows that $\Phi(K) \subset K$.

Now we prove the first part of Theorem 1. Since

$$\lim_{u\to 0} \min_{t\in[0,2\pi]} \frac{F(t,u)}{u} > \frac{\lambda}{\sigma^2},$$

there exists a $0 < r < p$ such that $F(t,u) > (\lambda/\sigma^2)u$ for $0 \leq u \leq r$. For $u \in \partial K_r = \{u \in K : \|u\| = r\}$, we have

$$\begin{aligned}\|\Phi u\| &= \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds > \frac{\lambda}{\sigma}G(\pi,m)\int_0^{2\pi} u(s)\,ds \\ &\geq 2\pi\lambda G(\pi,m)\|u\| = \|u\|,\end{aligned}$$

i.e. $\|\Phi u\| > \|u\|$ for $u \in \partial K_r$, and hence Theorem 2 implies

$$i(\Phi, K_r, K) = 0. \tag{2.7}$$

In much the same way, we may prove that there exists an $R > p$ such that $\|\Phi u\| > \|u\|$ for $u \in \partial K_R$ by using $\lim_{u\to\infty} \min_{t\in[0,2\pi]} F(t,u)/u > \lambda/\sigma^2$. Hence Theorem 2 again implies

$$i(\Phi, K_R, K) = 0. \tag{2.8}$$

On the other hand, by (H1), for $u \in \partial K_p$ we have

$$\begin{aligned}\|\Phi u\| &= \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds \le G(\pi,m)\int_0^{2\pi} F(s,u(s))\,ds \\ &< 2\pi\lambda p G(\pi,m) = \|u\|,\end{aligned}$$

i.e. $\|\Phi u\| < \|u\|$ for $u \in \partial K_p$. It follows from Theorem 2 that

$$i(\Phi, K_p, K) = 1. \tag{2.9}$$

Now, the additivity of the fixed point index and (2.7)–(2.9) together imply

$$i(\Phi, K_p \setminus \mathring{K}_r, K) = 1, \qquad i(\Phi, K_R \setminus \mathring{K}_p, K) = -1.$$

Consequently, Φ has a fixed point u_1 in $K_p \setminus \mathring{K}_r$, and a fixed point u_2 in $K_R \setminus \mathring{K}_p$. Both are positive solutions of the problem (1.1). It is obvious that $0 < \|u_1\| < p < \|u_2\|$. This completes the proof of the first part.

We now prove the second part of Theorem 1. Since

$$\lim_{u\to 0} \max_{t\in[0,2\pi]} \frac{F(t,u)}{u} < \lambda,$$

there exists a $0 < r < p$ such that $F(t,u) < \lambda u$ for $0 \le u \le r$. For $u \in \partial K_r$, we have

$$\begin{aligned}\|\Phi u\| &= \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds \le G(\pi,m)\int_0^{2\pi} F(s,u(s))\,ds \\ &< 2\pi\lambda G(\pi,m)\|u\| = \|u\|.\end{aligned}$$

This shows that $\|\Phi u\| < \|u\|$ for $u \in \partial K_r$, and hence Theorem 2 implies

$$i(\Phi, K_r, K) = 1. \tag{2.10}$$

Similarly, we may prove that there exists $R > p$ such that $\|\Phi u\| < \|u\|$ for $u \in \partial K_R$ by using $\lim_{u\to\infty} \max_{t\in[0,2\pi]} F(t,u)/u < \lambda$. Hence Theorem 2 again implies

$$i(\Phi, K_R, K) = 1. \tag{2.11}$$

In addition, since $\min_{t\in[0,2\pi]} u(t) \ge \sigma\|u\| = \sigma p$ for $u \in \partial K_p$, using (H2) we have, for such u,

$$\begin{aligned}\|\Phi u\| &= \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s,m)F(s,u(s))\,ds \ge \sigma G(\pi,m)\int_0^{2\pi} F(s,u(s))\,ds \\ &> 2\pi\lambda p G(\pi,m) = \|u\|,\end{aligned}$$

i.e. $\|\Phi u\| > \|u\|$ for $u \in \partial K_p$. Thus, Theorem 2 implies

$$i(\Phi, K_p, K) = 0. \tag{2.12}$$

As before, (2.10)–(2.12) show that Φ has two positive fixed points, which means that the problem (1.1) has two positive solutions. The proof is complete.

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Reçu par la Rédaction le 9.10.1997
Révisé le 21.4.1997