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Natural first order Lagrangians for immersions

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Abstract. We define natural first order Lagrangians for immersions of Riemannian manifolds and we prove a bijective correspondence between such Lagrangians and the symmetric functions on an open subset of m-dimensional Euclidean space.

Introduction. Critical points of functionals on Riemannian manifolds are particular geometric objects and they obviously depend on Lagrangians chosen. *Naturally* defined Lagrangians are expected to provide the most interesting maps or structures on the manifolds. For instance, we may look for Riemannian metrics as critical points. If we normalize the volume and take the Lagrangian to be the scalar curvature then the Einstein metrics are critical points of the variational problem.

Here we are interested in variations of immersions with a fixed Riemannian metric. For instance, minimal submanifolds are Riemannian immersions which are critical points of the volume functional as well as of the energy functional. These functionals are in a certain sense *natural* with respect to the Riemannian structures on the considered manifolds. We mean, roughly speaking, that for isometric manifolds such Lagrangians are equal and that if we vary the Riemannian structures continuously in the C^1-topology then the Lagrangians vary continuously. Such a definition was given by Palais (cf. [P_2]) for all maps between Riemannian manifolds. The classification of such Lagrangians is given in [P_2] and [K]. Essentially such Lagrangians are determined by the value of a certain function on the eigenvalues of the diagonal matrix arising from the polar decomposition of the differential of the map. The naturality of our Lagrangians is connected with Riemannian metrics on the manifolds. This naturality is a very particular case of the one considered in the theory of natural bundles (cf. [KMS]).

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In the present paper we are interested in Lagrangians which are defined only for immersions. This restriction makes the space of Lagrangians larger. It is clear that all natural first order Lagrangians, as defined by Palais, are Lagrangians for immersions. In this paper we show that any smooth symmetric function on the positive orthant of $\mathbb{R}^m$ uniquely determines a natural first order Lagrangian for immersions.

1. Lagrangians for immersions. Let M and N be smooth (C^∞) manifolds. Then on the space $C^\infty(M,N)$ of smooth maps between M and N there is defined the weak C^1-topology; a generic neighbourhood of a map belonging to $C^\infty(M,N)$ is a set of smooth maps which on a given compact subset of M have their values and the values of the first partial derivatives sufficiently close to the given map (cf. [H] or [M] for the precise definitions).

If M, N are two smooth manifolds then we denote by $J^1(M,N)$ the set of 1-jets of smooth maps. In other words, $J^1(M,N)$ consists of the equivalence classes, called jets, of local smooth maps from M to N. The equivalence is given by the requirement of equality of the Taylor series up to the first order at the source of a jet (cf. [P_1], [H] or [M]). We denote by $\widetilde{J}^1(M,N)$ a subset of $J^1(M,N)$ which consists of 1-jets of local immersions, i.e. immersions defined on open subsets of M. In other words, the differentials of such maps are monomorphisms. It is clear that the dimension of M has to be less than or equal to the dimension of N. Moreover, it can be proved that $\widetilde{J}^1(M,N)$ is an open subset of $J^1(M,N)$.

Let M_1, M_2, N_1 and N_2 be smooth manifolds. Suppose that we are given a diffeomorphism $\phi : M_1 \to M_2$ and a smooth map $\psi : N_1 \to N_2$. Then the couple of maps (ϕ,ψ) determines, in a natural way, a map

$$\Phi(\phi,\psi) : J^1(M_1,N_1) \to J^1(M_2,N_2)$$

such that if $j^1_{x_0} f \in J^1(M_1,N_1)$ then

$$\Phi(\phi,\psi) j^1_{x_0} f = j^1_{y_0}[\psi \circ f \circ \phi^{-1}]$$

where $y_0 = \phi(x_0)$. It can be shown that $\Phi(\phi,\psi)$ is a smooth map between $J^1(M_1,N_1)$ and $J^1(M_2,N_2)$.

Let $\mathcal{K}_m$ denote the category of pairs $X = ((M,g),(N,h))$ of Riemannian manifolds with $\dim M = m$. A morphism in $\mathcal{K}_m$ from $X_1 = ((M_1,g_1),(N_1,h_1))$ to $X_2 = ((M_2,g_2),(N_2,h_2))$ is a pair $F = (\phi,\psi)$ where $\phi : M_1 \to M_2$ is an isometry and $\psi : N_1 \to N_2$ is an isometric immersion. The composition of morphisms is defined componentwise. A morphism $F \in \mathrm{Mor}(X_1,X_2)$ induces a map

$$\Phi(F) : J^1(M_1,N_1) \to J^1(M_2,N_2)$$

of jet spaces which preserves the subspaces of jets of immersions and thus we have the restriction map

$$\Phi(F) : \widetilde{J}^1(M_1, N_1) \to \widetilde{J}^1(M_2, N_2).$$

Our aim is to define a natural first order Lagrangian for immersions. We consider maps $\widetilde{\mathcal{L}}$ on $\mathcal{K}_m$ with values in a class of functions. We suppose that if $X \in \mathcal{K}_m$ and $X = ((M, g), (N, h))$ then $\widetilde{\mathcal{L}}(X) \in C^\infty(\widetilde{J}^1(M, N), \mathbb{R})$. We mean that a pair of Riemannian manifolds determines, via $\widetilde{\mathcal{L}}$, a smooth real-valued function on the space of jets of functions between these two manifolds. Suppose that U, V are open subsets of M and N, respectively. Then the *restricted pair* $Y = ((U, g|_U), (V, h|_V))$ clearly belongs to $\mathcal{K}_m$. We assume that $\widetilde{\mathcal{L}}$ has the following property: for each X and for each restriction Y of X the restriction of $\widetilde{\mathcal{L}}(X)$ to $\widetilde{J}^1(U, V)$ is equal to $\widetilde{\mathcal{L}}(Y)$. A map $\widetilde{\mathcal{L}}$ with this property is called *local*. We consider here only $\widetilde{\mathcal{L}}$ of this type.

DEFINITION 1. A map $\widetilde{\mathcal{L}}$ is called a *natural first order Lagrangian for immersions* iff the following two conditions hold:

(i) if $X_1, X_2 \in \mathcal{K}_m$ and $F \in \mathrm{Mor}(X_1, X_2)$ then $\widetilde{\mathcal{L}}(X_1) = \widetilde{\mathcal{L}}(X_2) \circ \Phi(F)$;

(ii) if $j \in J^1(M, N)$ and $X = ((M, g), (N, h))$ then $\widetilde{\mathcal{L}}(X)j$ depends continuously on the Riemannian metrics on M and N in the C^1-topology.

We observe that if $X = ((M, g), (N, h)) \in \mathcal{K}_m$ has $\dim N < \dim M$ then $\widetilde{\mathcal{L}}(X) = \emptyset$.

This definition is a particular case of Palais' natural first order Lagrangian (cf. [P$_2$] and [K]). In our case, for each $X \in \mathcal{K}_m$, $\widetilde{\mathcal{L}}(X)$ is a function on $\widetilde{J}^1(M, N)$ which is the set of jets of local immersions. Since $\widetilde{J}^1(M, N)$ is a subset of $J^1(M, N)$, the set $C^\infty(\widetilde{J}^1(M, N), \mathbb{R})$ contains all the restrictions of functions from $C^\infty(J^1(M, N), \mathbb{R})$ and we have the following observation.

OBSERVATION 2. *Let $\mathcal{L}$ be a natural first order Lagrangian* (cf. [P$_2$]). *Then the restriction*

$$\mathcal{K}_m \ni X \to \mathcal{L}(X)|_{\widetilde{J}^1(M,N)}$$

is a natural first order Lagrangian for immersions.

Hence all natural first order Lagrangians are natural first order Lagrangians for immersions. We shall see that the converse is not true (cf. Observation 5).

EXAMPLE 3 (cf. [P$_1$], [K]). Let $X = \pi \in \mathcal{K}_m$ and let $j^1_{x_0} f \in J^1(M, N)$ be a jet of a smooth map. Then $f^* h_{x_0}$ is a symmetric bilinear form on $T_{x_0} M$. We associate with this form a linear map $A_{x_0} : T_{x_0} M \to T_{x_0} M$ such that $f^* h_{x_0}(u, v) = g(A_{x_0} u, v)$ for all $u, v \in T_{x_0} M$. It is clear that A_{x_0} is a non-negative self-adjoint endomorphism of $T_{x_0} M$. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of A_{x_0}, which may be multiple. Suppose that $\sigma : \Delta^+_m \to \mathbb{R}$ is a symmetric map on the positive orthant

$$\Delta_m^+ := \{(x_1, \ldots, x_m) \mid x_i \geq 0 \ \forall i = 1, \ldots, m\}.$$

Then we put $e_\sigma(X) j^1_{x_0} f := \sigma(\lambda_1, \ldots, \lambda_m)$. It is easy to prove that e_σ is a well-defined natural first order Lagrangian in the sense of Palais. In [K] it is shown that all natural first order Lagrangians are of this type.

If $f : M \to N$ is a smooth map, M is compact and ν_g is the density on M induced by g then $E_\sigma(f) := \frac{1}{2} \int_M e_\sigma(X) j^1 f \nu_g$ is called the *σ-energy* of f (cf. [ES]). There is a particular interest in the first elementary symmetric function $\sigma_1(\lambda_1, \ldots, \lambda_m) = \lambda_1 + \ldots + \lambda_m$ because σ_1-energy is just the energy of f and thus the critical points of E_{σ_1} are the harmonic maps between M and N.

Example 4. We put $\operatorname{int} \Delta_m^+ := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i > 0 \ \forall i = 1, \ldots, m\}$ and we consider a smooth function σ on $\operatorname{int} \Delta_m^+$. Then σ determines a natural first order Lagrangian $\widetilde{e}_\sigma$ for immersions in a similar way to Example 3. In fact, if $X \in \mathcal{K}_m$ and $X = ((M, g), (N, h))$ then for $j^1_{x_0} f \in \widetilde{J}^1(M, N)$ we put

$$\widetilde{e}_\sigma(X) j^1_{x_0} f := \sigma(\lambda_1, \ldots, \lambda_m)$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of the linear map A_{x_0} which is determined by $f^* h_{x_0}$; it is clear that the eigenvalues $\lambda_1, \ldots, \lambda_m$ of A_{x_0} are positive. Then it is easy to observe that $\widetilde{e}_\sigma$ is a natural first order Lagrangian for immersions.

Observation 5. It is clear that not all symmetric smooth functions on $\operatorname{int} \Delta_m^+$ are restrictions of functions from Δ_m^+. For instance, the map $(x_1, \ldots, x_m) \to (x_1 + \ldots + x_m)^{-1}$ cannot be even continuously extended to Δ_m^+.

Natural first order Lagrangians for immersions are defined on pairs of Riemannian manifolds belonging to $\mathcal{K}_m$. However, we shall show that they are determined by their value on the pair $((\mathbb{R}^m, \mathbf{st}), (\mathbb{R}^m, \mathbf{st}))$ where $\mathbf{st}$ denotes the standard metric structure on $\mathbb{R}^m$.

Lemma 6 (cf. [P_2]). *Let (M, g) be a Riemannian manifold of dimension m and let $x_0 \in M$. Then there exists an infinite sequence of Riemannian metrics on M converging to g in the C^1-topology such that each element of the sequence is flat in some neighbourhood of x_0.*

Proof. 1° We suppose that: M is the unit ball of $\mathbb{R}^m$ centred at zero and the canonical coordinates of $\mathbb{R}^m$ are normal for the Riemannian metric g. We denote by $\overline{g}$ the standard flat metric on M induced from $\mathbb{R}^m$. Then, in the canonical coordinates of $\mathbb{R}^m$, we have

$$\overline{g}_x = \sum_{i=1}^{m} dx_i \otimes dx_i, \qquad g_x = \sum_{i,j=1}^{m} g_{ij}(x) dx_i \otimes dx_j$$

for each $x \in M$, where $(g_{ij}(x))_{i,j=1,\ldots,m}$ is a symmetric positive definite matrix of C^∞ functions on M. We develop each g_{ij} in a Maclaurin series up to the second order. The crucial observation is that the first derivatives of g_{ij} vanish at zero. In fact, let ∇ be the Levi-Civita connection of g. Then for each $\gamma = 1, \ldots, m$ we have

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x_\gamma}(0) &= \nabla_{\partial/\partial x_\gamma} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\ &= (\nabla_{\partial/\partial x_\gamma} g)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) + g\left(\nabla_{\partial/\partial x_\gamma}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\ &\quad + g\left(\frac{\partial}{\partial x_i}, \nabla_{\partial/\partial x_\gamma}\frac{\partial}{\partial x_j}\right) = 0\end{aligned}$$

because all three terms vanish at zero. Hence for each $i, j = 1, \ldots, m$ we have

$$g_{ij}(x) = \delta_{ij} + \sum_{\alpha,\beta=1}^{m} f_{ij}^{\alpha\beta}(x) x_\alpha x_\beta$$

where δ_{ij} is the Kronecker delta and $f_{ij}^{\alpha\beta}(x)$ are C^∞ functions on M. Then we introduce an auxiliary C^∞ function $\mu : \mathbb{R} \to [0,1]$ such that

$$\mu(t) = \begin{cases} 1 & \text{if } t \le 1/2, \\ 0 & \text{if } t \ge 1. \end{cases}$$

We denote by $B(r)$ the closed ball in $\mathbb{R}^m$ centred at zero and of radius r. For later estimates we introduce a constant C defined as follows:

$$C := \max\left\{ \sup_{t\in\mathbb{R}} |\mu'(t)|, \sup_{x\in B(1/2)} |f_{ij}^{\alpha\beta}(x)|, \sup_{x\in B(1/2)} \left|\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_\gamma}\right| : i, j, \alpha, \beta, \gamma = 1, \ldots, m \right\}.$$

2° Define $g_n := \mu(2^n\|x\|)\overline{g} + [1 - \mu(2^n\|x\|)]g$. It is easy to observe that for each n:

(i) g_n is a Riemannian metric on M;
(ii) g_n and $\overline{g}$ coincide on $B(1/2^{n+1})$;
(iii) g_n and g coincide on $M \setminus B(1/2^n)$.

We are going to prove that g_n tends to g in the C^1-topology, i.e. for each compact subset K of M, g_n tends to g uniformly on K and $\partial g_n/\partial x_\gamma$ tends to $\partial g/\partial x_\gamma$ uniformly on K. Equivalently, we are to show that $g_n - g$ and $\partial(g_n - g)/\partial x_\gamma$ tend to zero uniformly on K.

3° We have

$$g_n - g = \mu(2^n\|x\|)\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{m} f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta dx_i \otimes dx_j.$$

Hence to show that $g_n - g$ tends to zero in the C^1-topology it is enough to show that $f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta$ tends to zero for all i, j, α, β. Since the support of $\mu(2^n\|x\|)$ is contained in $B(1/2^n)$, for each compact subset K of M we have

$$\sup_{x\in K}|\mu(2^n\|x\|)f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta| \le \sup_{x\in B(1/2^n)}|\mu(2^n\|x\|)f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta|, \tag{1.1}$$

$$\sup_{x\in K}\left|\frac{\partial[\mu(2^n\|x\|)f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta]}{\partial x_\gamma}\right| \le \sup_{x\in B(1/2^n)}\left|\frac{\partial[\mu(2^n\|x\|)f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta]}{\partial x_\gamma}\right|. \tag{1.2}$$

Hence to prove that $g_n - g$ tends to zero in the C^1-topology it is enough to show that the right hand sides of (1.1) and (1.2) tend to zero for all $i, j, \alpha, \beta, \gamma = 1, \ldots, m$.

4° We have

$$\begin{aligned}\sup_{x\in B(1/2^n)}|\mu(2^n\|x\|)f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta| &\le \sup_{x\in B(1/2^n)} C|x_\alpha||x_\beta| \\ &\le \sup_{x\in B(1/2^n)} C\|x\|^2 = C(1/2^n)^2.\end{aligned}$$

Hence the right hand side of (1.1) tends to zero as n tends to infinity.

5° We have

$$\begin{aligned}\frac{\partial}{\partial x_\gamma}[\mu(2^n\|x\|)f_{ij}^{\alpha\beta}(x)x_\alpha x_\beta] = {} & \overbrace{\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_\gamma}\mu(2^n\|x\|)x_\alpha x_\beta}^{A_1(x)} \\ & + \overbrace{f_{ij}^{\alpha\beta}(x)2^n\frac{x_\gamma}{\|x\|}\mu'(2^n\|x\|)x_\alpha x_\beta}^{A_2(x)} \\ & + \overbrace{f_{ij}^{\alpha\beta}(x)\mu(2^n\|x\|)(\delta_{\alpha\gamma}x_\beta + x_\alpha\delta_{\beta\gamma})}^{A_3(x)}.\end{aligned}$$

Then we have the following estimates:

$$\sup_{x\in B(1/2^n)}|A_1(x)| \le \sup_{x\in B(1/2^n)} C\|x\|^2 = \frac{C}{2^{2n}},$$

$$\sup_{x\in B(1/2^n)}|A_2(x)| \le \sup_{x\in B(1/2^n)} C2^n\frac{|x_\gamma|}{\|x\|}C\|x\|^2 \le \frac{C^2}{2^n},$$

$$\sup_{x\in B(1/2^n)} |A_3(x)| \le \sup_{x\in B(1/2^n)} C2\|x\| = \frac{C}{2^{n-1}}.$$

Hence the right hand side of (1.2) tends to zero. As g_n coincides with g on $M \setminus B(1/2^n)$, we do not lose generality by restricting our proof to M being the unit ball in $\mathbb{R}^m$.

2. Main theorem. Let $\widetilde{\mathcal{L}}$ be a natural first order Lagrangian for immersions and let $X = ((M,g),(N,h)) \in \mathcal{K}_m$. Suppose that $x_0 \in M$, $y_0 \in N$, and let g_n and h_n be sequences of Riemannian metrics tending to g and h, respectively, in the C^1-topology. Moreover, assume that g_n and h_n are flat in some open neighbourhoods of x_0 and y_0. Hence we have a sequence of objects $X_n = ((M,g_n),(N,h_n)) \in \mathcal{K}_m$. Let $j^1_{x_0}f \in \widetilde{J}^1(M,N)$. Then from condition (ii) of Definition 1 we have

$$\widetilde{\mathcal{L}}(X)j^1_{x_0}f = \lim_{n\to\infty} \widetilde{\mathcal{L}}(X_n)j^1_{x_0}f. \tag{2.1}$$

Since $\widetilde{\mathcal{L}}$ is local, $\widetilde{\mathcal{L}}(X_n)j^1_{x_0}f$ for each n is determined by a pair which is a restriction of X to an open flat neighbourhood of x_0 and y_0. Such flat sufficiently small neighbourhoods are isometric, via the inverses of exponential maps, with open neighbourhoods of $T_{x_0}M$ and $T_{y_0}N$, respectively. Since $\widetilde{\mathcal{L}}$ is local and invariant by isometries we get

$$\widetilde{\mathcal{L}}(X_n)j^1_{x_0}f = \widetilde{\mathcal{L}}((T_{x_0}M,\mathbf{st}),(T_{y_0}N,\mathbf{st}))j^1_0B \tag{2.2}$$

where $B = d_{x_0}f$. We observe that we apply here property (i) of Definition 1. We also notice that the jet j^1_0B corresponds to $j^1_{x_0}f$ via exponential maps. From (2.2) it follows that the sequence considered in (2.1) is constant. Hence

$$\widetilde{\mathcal{L}}(X)j^1_{x_0}f = \widetilde{\mathcal{L}}((T_{x_0}M,\mathbf{st}),(T_{y_0}N,\mathbf{st}))j^1_0B. \tag{2.3}$$

Then we have a polar decomposition $B = P\circ O$ such that

$$O : T_{x_0}M \to T_{x_0}M \quad \text{and} \quad P : T_{x_0}M \to T_{y_0}N$$

where O is a self-adjoint positive isomorphism and P is an isometric monomorphism. Let $\lambda_1,\ldots,\lambda_m$ be the eigenvalues of O. There exists a linear isometry $T : \mathbb{R}^m \to T_{x_0}M$ such that

$$T\circ B\circ T^{-1} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_m \end{pmatrix} \tag{2.4}$$

in the canonical basis of $\mathbb{R}^m$. Since $\widetilde{\mathcal{L}}$ is invariant under isometries of pairs of Riemannian manifolds, from (2.3) and (2.4) we get

$$\widetilde{\mathcal{L}}(X)j^1_{x_0}f = \widetilde{\mathcal{L}}((\mathbb{R}^m,\mathbf{st}),(\mathbb{R}^m,\mathbf{st}))j^1_0[\mathrm{diag}(\lambda_1,\ldots,\lambda_m)]$$

where $\operatorname{diag}[\lambda_1, \ldots, \lambda_m]$ is an endomorphism of $\mathbb{R}^m$ whose matrix in the canonical basis is as in (2.4). We observe that the value of $\widetilde{\mathcal{L}}(X)$ at $j^1_{x_0} f$ depends only on the eigenvalues of the self-adjoint part of the polar decomposition of $d_{x_0} f$. Hence $\widetilde{\mathcal{L}}$ determines a smooth real-valued and symmetric function σ on $\operatorname{int} \Delta^+_m$ such that

$$\sigma(\lambda_1, \ldots, \lambda_m) := \widetilde{\mathcal{L}}((\mathbb{R}^m, \mathbf{st}), (\mathbb{R}^m, \mathbf{st})) j^1_0[\operatorname{diag}(\lambda_1, \ldots, \lambda_m)].$$

Example 7. Let $\sigma : \operatorname{int} \Delta^+_m \to \mathbb{R}$ be a smooth symmetric real-valued function on $\operatorname{int} \Delta^+_m$. Then we denote by $\widetilde{\mathcal{L}}_\sigma$ a Lagrangian such that if $X = ((M, g), (N, h)) \in \mathcal{K}_m$ and $j^1_{x_0} f \in \widetilde{J}^1(M, N)$ then $\widetilde{\mathcal{L}}_\sigma(X) j^1_{x_0} f := \sigma(\lambda_1, \ldots \ldots, \lambda_m)$ where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of the self-adjoint part of the polar decomposition of $d_{x_0} f$. It is easy to prove that $\widetilde{\mathcal{L}}_\sigma$ is a well-defined first order natural Lagrangian for immersions.

Then we have the following version of the Palais classification theorem for Lagrangians (cf. [P_2], [K]).

Theorem 8. *There is one-to-one correspondence between the first order natural Lagrangians for immersions and the symmetric smooth functions on* $\operatorname{int} \Delta^+_m$. *The correspondence is defined in the following way: with a smooth symmetric map* $\sigma : \operatorname{int} \Delta^+_m \to \mathbb{R}$ *we associate the Lagrangian* $\widetilde{\mathcal{L}}_\sigma$.

Proof. It is clear that $\widetilde{\mathcal{L}}_\sigma$ is a natural first order Lagrangian for immersions. On the other hand, from previous considerations it follows that each such Lagrangian has to be of the type $\widetilde{\mathcal{L}}_\sigma$.

To end this paper we give an explicit description of the set of smooth symmetric functions on $\operatorname{int} \Delta^+_m$, denoted by $C^\infty_{\mathrm{s}}(\operatorname{int} \Delta^+_m, \mathbb{R})$. By the theorem of Glaeser (cf. [G]) such functions may be expressed as compositions of standard symmetric functions on Δ^+_m and any smooth functions on $\operatorname{int} \Delta^+_m$. More precisely, let $\sigma_r : \operatorname{int} \Delta^+_m \to \mathbb{R}$ be the rth elementary symmetric function defined as

$$\sigma_r(t_1, \ldots, t_m) = \sum_{1 \le i_1 < \ldots < i_r \le m} t_{i_1} \ldots\ t_{i_r}$$

where $r = 1, \ldots, m$. Then the map

$$\Xi = (\sigma_1, \ldots, \sigma_m) : \operatorname{int} \Delta^+_m \to \mathbb{R}^m$$

consists of m elementary symmetric functions. Clearly the set $\operatorname{int} \Delta^+_m$ is invariant under Ξ. Then from Theorem II of [G] we get

$$C^\infty_{\mathrm{s}}(\operatorname{int} \Delta^+_m, \mathbb{R}) = \{\phi \circ \Xi \mid \phi \in C^\infty(\operatorname{int} \Delta^+_m, \mathbb{R})\}.$$

We observe that such a factorisation is valid for all Ξ-invariant subsets of $\mathbb{R}^m$.

REMARK 9. Let $\sigma \in C^\infty_s(\operatorname{int} \Delta^+_m, \mathbb{R})$. It determines the maps $\widetilde{e}_\sigma$ and $\widetilde{\mathcal{L}}_\sigma$ which are natural first order Lagrangians for immersions. Let $X = ((M,g),(N,h)) \in \mathcal{K}_m$ and $j^1_{x_0} f \in \widetilde{J}^1(M,N)$. Then the bilinear symmetric form $(f^*h)_{x_0}$ determines the positive eigenvalues $\lambda_1, \ldots, \lambda_m$ and we have $\widetilde{e}_\sigma(X) j^1_{x_0} f = \sigma(\lambda_1, \ldots, \lambda_m)$ (cf. Example 4). We also have the polar decomposition $d_{x_0} f = P \circ O$ and the eigenvalues of the self-adjoint map O. Then the eigenvalues of O are, up to permutation, $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}$. Hence

$$\widetilde{\mathcal{L}}_\sigma(X) j^1_{x_0} f = \sigma(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}).$$

This indicates a relationship between $\widetilde{e}_\sigma$ and $\widetilde{\mathcal{L}}_\sigma$.

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