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A discrepancy principle for Tikhonov regularization with approximately specified data

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Abstract. Many discrepancy principles are known for choosing the parameter α in the regularized operator equation $(T^*T+\alpha I)x_\alpha^\delta = T^*y^\delta$, $\|y-y^\delta\| \le \delta$, in order to approximate the minimal norm least-squares solution of the operator equation $Tx = y$. We consider a class of discrepancy principles for choosing the regularization parameter when T^*T and T^*y^δ are approximated by A_n and z_n^δ respectively with A_n not necessarily self-adjoint. This procedure generalizes the work of Engl and Neubauer (1985), and particular cases of the results are applicable to the regularized projection method as well as to a degenerate kernel method considered by Groetsch (1990).

1. Introduction. We are concerned with the problem of finding approximations to the minimal norm least-squares solution $\widehat{x}$ of the operator equation

$$Tx = y, \tag{1.1}$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y, and y belongs to $D(T^\dagger) := R(T)+R(T)^\perp$, the domain of the Moore–Penrose inverse $T^\dagger$ of T. It is well known [8] that if the range $R(T)$ of T is not closed, then the operator $T^\dagger$ which associates $y \in D(T^\dagger)$ to $\widehat{x} := T^\dagger y$, the unique least-squares solution of minimal norm, is not continuous, and consequently the problem of solving (1.1) for $\widehat{x}$ is ill-posed. A prototype of an ill-posed problem is the Fredholm integral equation of the first kind

$$\int_0^1 k(s,t)x(t)\,dt = y(s), \quad 0 \le s \le 1, \tag{1.2}$$

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with nondegenerate kernel $k(\cdot,\cdot) \in L^2([0,1] \times [0,1])$, where $X = Y = L^2[0,1]$. Regularization methods are employed to find approximations to $\widehat{x}$. In Tikhonov regularization one looks for the unique x_α, $\alpha > 0$, which minimizes the functional

$$x \to \|Tx - y\|^2 + \alpha\|x\|^2, \qquad x \in X.$$

Equivalently, one solves the well-posed equation

$$(T^*T + \alpha I)x_\alpha = T^*y \tag{1.3}$$

for each $\alpha > 0$. Since $T^*T\widehat{x} = T^*y$, it follows that

$$\|\widehat{x} - x_\alpha\| = \|\alpha(T^*T + \alpha I)^{-1}\widehat{x}\| \le \|\widehat{x}\|. \tag{1.4}$$

It is known ([8], [16]) that

$$\|\widehat{x} - x_\alpha\| \to 0 \quad \text{as } \alpha \to 0 \tag{1.5}$$

and

$$\widehat{x} \in R((T^*T)^\nu),\ 0 \le \nu \le 1, \quad \text{implies} \quad \|\widehat{x} - x_\alpha\| = O(\alpha^\nu). \tag{1.6}$$

In practical applications the data y may not be available exactly, instead one may have an approximation y^δ with say $\|y - y^\delta\| \le \delta$, $\delta > 0$. Then one solves the equation

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta \tag{1.7}$$

instead of (1.3) and requires $\|\widehat{x} - x_\alpha^\delta\| \to 0$ as $\alpha \to 0$ and $\delta \to 0$. It follows from (1.3) and (1.7) that

$$\begin{aligned}
\|x_\alpha - x_\alpha^\delta\|^2 &= \|(T^*T + \alpha I)^{-1}T^*(y - y^\delta)\|^2 \\
&= \langle (T^*T + \alpha I)^{-1}T^*(y - y^\delta), (T^*T + \alpha I)^{-1}T^*(y - y^\delta)\rangle \\
&= \langle (TT^* + \alpha I)^{-2}TT^*(y - y^\delta), (y - y^\delta)\rangle \\
&\le \|(TT^* + \alpha I)^{-2}TT^*\| \cdot \|(y - y^\delta\|^2 \le \delta^2/\alpha,
\end{aligned}$$

so that

$$\|\widehat{x} - x_\alpha^\delta\| \le \|\widehat{x} - x_\alpha\| + \delta/\sqrt{\alpha}. \tag{1.8}$$

Now let $R_\alpha = (T^*T + \alpha I)^{-1}T^*$ for $\alpha > 0$. Then by (1.5) we have

$$\|R_\alpha y - T^\dagger y\| \to 0 \quad \text{as } \alpha \to 0$$

for $y \in D(T^\dagger)$. Therefore, if $R(T)$ is not closed, then the family $\{R_\alpha\}_{\alpha>0}$ is not uniformly bounded so that, as a consequence of the Uniform Boundedness Principle, there exists $v \in Y$ such that $\{R_\alpha v\}_{\alpha>0}$ is not bounded in Y. In particular, if $y^\delta = y + \delta v/\|v\|$, then $\|y - y^\delta\| \le \delta$ and $\{R_\alpha y^\delta\}_{\alpha>0}$ is unbounded in Y. Therefore, the problem of choosing the regularization parameter α depending on y^δ is important. Many works in the literature are devoted to this aspect (cf. [7], [17], [1], [2], [3], [6], [14], [4]).

In order to solve (1.7) numerically, it is required to consider approximations of T^*T and of T^*y^δ. So the problem actually at hand would be of the form

$$(A_n + \alpha I)x^\delta_{\alpha,n} = z^\delta_n, \tag{1.9}$$

where (A_n) and (z^δ_n) are approximations of T^*T and of T^*y^δ respectively.

In the well known regularized projection methods (cf. [10], [2], [3]),

$$A_n = P_n T^*TP_n \quad \text{and} \quad z^\delta_n = P_n T^* y^\delta,$$

where (P_n) is a sequence of orthogonal projections on X such that $P_n \to I$ pointwise. In this case we have

$$\|T^*T - A_n\| \to 0 \quad \text{as } n \to \infty,$$

and discrepancy principles are known for choosing the regularization parameter α in (1.9) (see e.g. [2], [3], [13], [5]).

In the degenerate kernel methods for the integral equation (1.2) with $k(\cdot,\cdot) \in C([0,1]\times[0,1])$, A_n is obtained by approximating the kernel $\widetilde{k}(\cdot,\cdot)$ of the integral operator T^*T by a degenerate kernel $\widetilde{k}_n(\cdot,\cdot)$ so that $\|\widetilde{k}-\widetilde{k}_n\|_\infty \to 0$ as $n\to\infty$. Then it follows that

$$\|T^*T - A_n\| \le \|\widetilde{k}-\widetilde{k}_n\|_2 \le \|\widetilde{k}-\widetilde{k}_n\|_\infty \to 0 \quad \text{as } n\to\infty.$$

(See [11] and [12] for a discussion on degenerate kernel methods for integral equations.) In a degenerate kernel method considered by Groetsch [9] the approximation $\widetilde{k}_n(\cdot,\cdot)$ is obtained from

$$\widetilde{k}(s,t) := \int_0^1 k(\tau,s)k(\tau,t)\,dt, \qquad a \le s,t \le b.$$

by using a convergent quadrature rule. In this case one has $\|\widetilde{k}-\widetilde{k}_n\|_\infty \to 0$ as $n\to\infty$ for nice enough kernels $k(\cdot,\cdot)$.

Moreover, for the degenerate kernel method of Groetsch [9] as well as for the regularized projection methods, the operators A_n are non-negative and self-adjoint.

In this paper we consider the generalized form of a class of discrepancy principles in [1], namely,

$$\|A_n x^\delta_{\alpha,n} - z^\delta_n\| = \frac{\delta^p}{\alpha^q}, \qquad p>0,\ q>0, \tag{1.10}$$

for large enough n, to choose the regularization parameter $\alpha = \alpha(n,\delta)$ in (1.9), where (A_n) is a sequence of bounded linear operators on X and (z^δ_n) in X such that

$$\|T^*T - A_n\| \to 0 \quad \text{and} \quad \|T^*y^\delta - z^\delta_n\| \to 0 \quad \text{as } n\to\infty.$$

It has to be observed that we do not assume the operators A_n to be non-negative and self-adjoint. The consideration of a general A_n, as has been done recently by Nair [15], is important from the computational point of view, because even if one starts with a non-negative self-adjoint operator as approximation of T^*T, due to truncation errors etc., one actually may not be dealing with a non-negative self-adjoint operator.

With α chosen according to (1.10), we show the convergence of the solution $x^\delta_{\alpha,n}$ of (1.9) to $\widehat{x}$ as $\delta \to 0$, $n \to \infty$, and also obtain estimates for the error $\|\widehat{x} - x^\delta_{\alpha,n}\|$ whenever $\widehat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$. Our result on error estimates shows that if ν_0 is an estimate for the possibly unknown ν, with $0 < \nu \le \nu_0 \le 1$, then taking $p/(q+1) = 2/(2\nu_0+1)$ one obtains the rate $O(\delta^{2\nu/(2\nu_0+1)})$. In particular, prior knowledge of ν enables us to obtain the *optimal* rate $O(\delta^{2\nu/(2\nu+1)})$ (cf. Schock [16]).

If $A_n = P_nT^*TP_n$ and $z^\delta_n = P_nT^*y^\delta$ then (1.10) coincides with a discrepancy principle considered by Engl and Neubauer [2] and we recover the optimal result in [2] as a particular case. Thus this paper generalizes the type of results in [2] and [9] for projection methods and degenerate kernel method for integral equations respectively, providing also a parameter choice strategy in the latter case.

2. Approximate solution and convergence. Let X and Y be Hilbert spaces and $T : X \to Y$ be a bounded linear operator with its range $R(T)$ not necessarily closed in Y. Let $y \in D(T^\dagger) := R(T) + R(T)^\perp$, $y \ne 0$, so that there exists a unique $\widehat{x} \in X$ of minimal norm such that

$$\|T\widehat{x} - y\| = \inf\{\|Tx - y\| : x \in X\}.$$

Let (A_n) be a sequence of bounded linear operators on X and for $\delta > 0$, let $y^\delta \in Y$ and (z^δ_n) in X be such that

$$\|T^*T - A_n\| \le \varepsilon_n, \quad \|y - y^\delta\| \le \delta, \quad \|T^*y^\delta - z^\delta_n\| \le \eta^\delta_n,$$

where (ε_n) and (η^δ_n) are sequences of nonnegative real numbers such that

$$\varepsilon_n \to 0 \quad \text{as } n \to \infty$$

and

$$\eta^\delta_n \to 0 \quad \text{as } n \to \infty \text{ and } \delta \to 0. \tag{2.1}$$

Throughout the paper we denote the operator T^*T by A, and c, c', c_1, c_2, etc., denote positive constants which may assume different values in different contexts.

Theorem 2.1. *If* $\varepsilon_n \le c_0\alpha$ *with* $0 < c_0 < 1$, *then* $A_n + \alpha I$ *is bijective and*

$$\|(A_n + \alpha I)^{-1}\| \le 1/(\alpha(1 - c_0)).$$

Moreover, if x_α^δ *and* $x_{\alpha,n}^\delta$ *are the unique solutions of* (1.7) *and* (1.9) *respectively, then*

$$\|\widehat{x} - x_{\alpha,n}^\delta\| \le c\biggl(\|\widehat{x} - x_\alpha^\delta\| + \frac{\eta_n^\delta}{\alpha} + \frac{\varepsilon_n}{\alpha}\biggr). \tag{2.2}$$

In particular, if $\alpha := \alpha(\delta, n)$ *is chosen in such a way that*

$$\alpha(\delta,n) \to 0, \quad \frac{\delta}{\sqrt{\alpha(\delta,n)}} \to 0, \quad \frac{\varepsilon_n}{\alpha(\delta,n)} \to 0 \quad \textit{and} \quad \frac{\eta_n^\delta}{\alpha(\delta,n)} \to 0$$

as $\delta \to 0$ *and* $n \to \infty$, *then*

$$\|\widehat{x} - x_{\alpha,n}^\delta\| \to 0 \quad \textit{as } \delta \to 0 \textit{ and } n \to \infty.$$

P r o o f. Since A is non-negative and self-adjoint, it follows from spectral theory that for each $\alpha > 0$, $(A + \alpha I)^{-1}$ exists as a bounded linear operator on X and

$$\|(A + \alpha I)^{-1}\| \le 1/\alpha.$$

Therefore, if $\|A - A_n\| < 1/\|(A + \alpha I)^{-1}\|$ then, by results on perturbation of operators, $(A_n + \alpha I)^{-1}$ exists and is a bounded operator, and

$$\begin{aligned}\|(A_n + \alpha I)^{-1}\| &\le \frac{\|(A + \alpha I)^{-1}\|}{1 - \|A - A_n\| \cdot \|(A + \alpha I)^{-1}\|} \\ &\le \frac{1/\alpha}{1 - \varepsilon_n/\alpha} \le \frac{1}{\alpha(1 - c_0)}.\end{aligned}$$

Now let $w_{\alpha,n}^\delta$ be the unique solution of the equation (1.9) with $T^* y^\delta$ in place of z_n^δ, i.e.,

$$(A_n + \alpha I) w_{\alpha,n}^\delta = T^* y^\delta. \tag{2.3}$$

Then from (1.7), (1.9) and (2.3), we have

$$x_{\alpha,n}^\delta - w_{\alpha,n}^\delta = (A_n + \alpha I)^{-1}(z_n^\delta - T^* y^\delta)$$

and

$$w_{\alpha,n}^\delta - x_\alpha^\delta = (A_n + \alpha I)^{-1}(A - A_n) x_\alpha^\delta.$$

Since $\varepsilon_n \le c_0 \alpha$, it follows that

$$\|x_{\alpha,n}^\delta - w_{\alpha,n}^\delta\| \le c_1 \eta_n^\delta / \alpha$$

and

$$\|w_{\alpha,n}^\delta - x_\alpha^\delta\| \le c_2(\|\widehat{x} - x_\alpha^\delta\| + \varepsilon_n/\alpha),$$

so that

$$\|\widehat{x} - x_{\alpha,n}^\delta\| \le c(\|\widehat{x} - x_\alpha^\delta\| + \eta_n^\delta/\alpha + \varepsilon_n/\alpha).$$

Now the assumptions on $\alpha := \alpha(\delta, n)$ together with (1.6) and (1.8) imply the convergence $\|\widehat{x} - x_{\alpha,n}^\delta\| \to 0$ as $\delta \to 0$ and $n \to \infty$. ∎

3. The discrepancy principle. By our assumption (2.1) on (η_n^δ) and the fact that $0 \neq y \in D(T^\dagger)$, we have $c_1 \le \|z_n^\delta\| \le c_2$ for all large enough n, say $n \ge n_0(\delta)$ and for each $\delta \in (0, \delta_0]$ for some δ_0. Therefore by Theorem 2.1,

$$\|A_n x_{\alpha,n}^\delta - z_n^\delta\| = \|\alpha x_{\alpha,n}^\delta\| = \|\alpha(A_n + \alpha I)^{-1} z_n^\delta\| \le \gamma_1 \tag{3.1}$$

for some constant γ_1 and for all $\alpha \ge \varepsilon_n/c_0$. Moreover, if

$$\alpha \ge \gamma_0 := \max\{\varepsilon_n/c_0 : n = 1, 2, \ldots\} \quad \text{and} \quad \delta \le \delta_0,$$

then

$$\|A_n x_{\alpha,n}^\delta - z_n^\delta\| \ge \|\alpha(A_n + \alpha I)^{-1} z_n^\delta\| \ge \frac{\gamma_0 \|z_n^\delta\|}{\|A_n\| + \alpha} \ge \gamma_2 \tag{3.2}$$

for some $\gamma_2 > 0$, since (A_n) is uniformly bounded.

Now to choose the regularization parameter α in (1.9), we consider the discrepancy principle (1.10).

For simplicity of presentation we assume that

$$\eta_n^\delta \le c_3 \delta^r \quad \text{and} \quad \varepsilon_n \le c_4 \delta^k \tag{3.3}$$

for some positive reals r and k, and for all $n \ge n_0(\delta)$.

THEOREM 3.1. *Let p and q be positive integers. Then for each $\delta \in (0, \delta_0]$, there exists a positive integer $n_1(\delta)$ and for each $n \ge n_1(\delta)$, there exists $\alpha := \alpha(\delta, n)$ such that* (1.10) *is satisfied. Moreover*,

$$\alpha \le c_1 \delta^{p/(q+1)} \quad \textit{and} \quad \delta^p/\alpha^q \le c_2 \delta^\mu, \quad n \ge n_1(\delta), \tag{3.4}$$

where

$$\mu = \min\left\{r, \frac{p}{(q+1)}, 1 + \frac{p}{2(q+1)}\right\}.$$

Proof. Let $\delta \in (0, \delta_0]$. For $\alpha \ge \varepsilon_n/c_0$ and $n = 1, 2, \ldots$, define

$$f_n(\alpha) = \alpha^q \|A_n x_{\alpha,n}^\delta - z_n^\delta\|.$$

Then from (3.1) it follows that $f_n(\alpha) \le \gamma_1 \alpha^q$ so that

$$f_n(\varepsilon_n/c_0) \to 0 \quad \text{as } n \to \infty.$$

Let $n_1(\delta) \ge n_0(\delta)$ be the smallest positive integer such that for all $n \ge n_1(\delta)$,

$$\varepsilon_n \le c_0 \min\{(\delta^p/\gamma_2)^{1/q}, (\delta^p/\gamma_1)^{1/q}\}.$$

Then taking $\alpha_0 = \max\{\gamma_0, (\delta^p/\gamma_2)^{1/q}\}$, we obtain $\varepsilon_n \le c_0 \alpha_0$ and $\alpha_0 \ge \gamma_0$ so that by (3.1) and (3.2), we have

$$f_n(\varepsilon_n/c_0) \le \delta^p \le f_n(\alpha_0).$$

Therefore by the Intermediate Value Theorem, there exists $\alpha := \alpha(\delta, n)$ such that

$$\varepsilon_n/c_0 \le \alpha \le \alpha_0 \quad \text{and} \quad \|A_n x_{\alpha,n}^\delta - z_n^\delta\| = \delta^p/\alpha^q$$

for all $n \geq n_1(\delta)$. We also note that

$$x_{\alpha,n}^\delta = \frac{1}{\alpha}(z_n^\delta - A_n x_{\alpha,n}^\delta)$$

so that for all $n \geq n_1(\delta)$ and $\alpha = \alpha(\delta, n)$,

$$\|z_n^\delta\| - \delta^p/\alpha^q = \|z_n^\delta\| - \|A_n x_{\alpha,n}^\delta - z_n^\delta\| \leq \|A_n x_{\alpha,n}^\delta\| \leq \|A_n\|\delta^p/\alpha^{q+1}.$$

Therefore $\alpha^{q+1} \leq \delta^p(\alpha + \|A_n\|)/\|z_n^\delta\| \leq c\delta^p$ and consequently

$$\alpha(\delta, n) \leq c_1 \delta^{p/(q+1)}, \qquad n \geq n_1(\delta).$$

Now, using the estimates in (1.4), (1.8) and (2.2), we have

$$\begin{aligned}\delta^p/\alpha^q &= \|A_n x_{\alpha,n}^\delta - z_n^\delta\| = \alpha\|x_{\alpha,n}^\delta\| \leq \alpha(\|\widehat{x}\| + \|\widehat{x} - x_{\alpha,n}^\delta\|)\\ &\leq c\alpha(\|\widehat{x}\| + \|\widehat{x} - x_\alpha^\delta\| + \eta_n^\delta/\alpha + \varepsilon_n/\alpha)\\ &\leq c'(\alpha + \delta\sqrt{\alpha} + \eta_n^\delta) \leq c_2\delta^\mu,\end{aligned}$$

where $\mu = \min\{r, p/(q+1), 1 + p/2(q+1)\}$. ■

4. Error estimates under the discrepancy principle. In order to prove the convergence of $x_{\alpha,n}^\delta$ to $\widehat{x}$ and to obtain the estimates for the error $\|\widehat{x} - x_{\alpha,n}^\delta\|$ under the discrepancy principle (1.10), we impose certain restrictions on the parameters p and q appearing in (1.10) in terms of the error levels η_n^δ and ε_n of the data A_n and z_n^δ respectively. More precisely, we assume that

$$\frac{p}{q+1} \leq \min\{2, r, k\}, \tag{4.1}$$

where r and k are as in (3.3).

THEOREM 4.1 *Let* $\alpha := \alpha(\delta, n)$ *be chosen according to* (1.10). *Then*:

(i) $\|\widehat{x} - x_{\alpha,n}^\delta\| \to 0$ *as* $n \to \infty$ *and* $\delta \to 0$.

(ii) *If* $\widehat{x} \in R(A^\nu)$, $0 < \nu \leq 1$, *then for all large enough* n *and small enough* δ,

$$\|\widehat{x} - x_{\alpha,n}^\delta\| \leq c\delta^s,$$

where

$$s = \min\left\{\frac{p\nu}{q+1}, 1 - \frac{p}{2(q+1)}, r - \frac{p}{q+1}, k - \frac{p}{q+1}\right\}.$$

(iii) *In particular*, *if*

$$\min\{r, k\} \geq \frac{2\nu + 2}{2\nu + 1} \quad \text{and} \quad \frac{p}{q+1} = \frac{2}{2\nu + 1},$$

then

$$\|\widehat{x} - x_{\alpha,n}^\delta\| \leq c\delta^{2\nu/(2\nu+1)}.$$

Proof. Using (3.4), we have

$$\delta^l/\alpha^m = \delta^{l-mp/q}(\delta^p/\alpha^q)^{m/q} \le c\delta^{l-m(p-\mu)/q}$$

for every $l \ge 0$ and $m \ge 0$, where μ is as in Theorem 3.1. But by the assumption (4.1), $\mu = p/(q+1)$, so that

$$\delta^l/\alpha^m \le c\delta^{l-mp/(q+1)}.$$

Therefore

$$\delta/\sqrt{\alpha} \le c_1\delta^{1-p/2(q+1)}, \quad \eta_n^\delta/\alpha \le c_2\delta^{r-p/(q+1)} \quad \text{and} \quad \varepsilon_n/\alpha \le c_3\delta^{k-p/(q+1)}.$$

Using this, the result in (i) follows from (1.5), (1.8) and (2.2), the estimate in (ii) follows from (1.6), (1.8) and (2.2), and (iii) is a consequence of (ii). ■

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