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Effective formulas for invariant functions —case of elementary Reinhardt domains

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Abstract. We find effective formulas for the invariant functions, appearing in the theory of several complex variables, of the elementary Reinhardt domains. This gives us the first example of a large family of domains for which the functions are calculated explicitly.

0. Introduction. Holomorphically invariant functions and pseudometrics have proved to be very useful in the theory of several complex variables. Nevertheless, the problem of finding effective formulas for them has turned out to be very difficult. So far there have been very few examples of domains for which explicit formulas for these functions are known.

Among many different invariant functions and pseudometrics let us mention the Lempert and Green functions, the Kobayashi and Carathéodory pseudodistances as well as their infinitesimal versions, i.e. the Kobayashi–Royden, Carathéodory and Azukawa pseudometrics.

Due to Lempert's theorem (see [L1,2]) all holomorphically invariant functions and pseudometrics coincide in the class of convex domains. But even in the convex case it is difficult to find explicit formulas. Among the few existing results in this direction let us mention here the special case of convex (see [BFKKMP], [JP2]) and non-convex (see [PZ]) ellipsoids. Another class of non-convex domains for which some of the functions were calculated is the class of elementary Reinhardt domains (see [JP1,2]). In our paper we extend the results obtained in the latter domains to all invariant functions and pseudometrics mentioned above. The formulas obtained enable us to understand better the mutual relations between the invariant objects and give surprising solutions to some problems.

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1. Definitions, notations and main results. By E we will always denote the unit disc in $\mathbb{C}$. We put

$$m(\lambda_1,\lambda_2):=\frac{|\lambda_1-\lambda_2|}{|1-\overline{\lambda}_1\lambda_2|},\quad \lambda_1,\lambda_2\in E,$$
$$\gamma_E(\lambda;\alpha):=\frac{|\alpha|}{1-|\lambda|^2},\quad \lambda\in E,\ \alpha\in\mathbb{C}.$$

Let D be a domain in $\mathbb{C}^n$. Following [L1], [Ko1], [Kl1,2], [C], [A], [R] and [JP2] for $(w,z)\in D\times D$ and $(w,X)\in D\times\mathbb{C}^n$ we define the following functions:

$$\widetilde{k}_D^*(w,z):=\inf\{m(\lambda_1,\lambda_2):\exists\varphi\in\mathcal{O}(E,D),\ \varphi(\lambda_1)=w,\ \varphi(\lambda_2)=z\},$$
$$k_D^*(w,z):=\tanh k_D(w,z),$$

where k_D is the largest pseudodistance smaller than or equal to $\widetilde{k}_D:=\tanh^{-1}\widetilde{k}_D^*$,

$$g_D(w,z):=\sup\{u(z):\log u\in\mathrm{PSH}(D,[-\infty,0)),$$
$$\exists M,R>0:u(\zeta)\le M\|\zeta-w\|\text{ for }\zeta\in D,\ \|\zeta-w\|<R\},$$
$$c_D^*(w,z):=\sup\{m(\varphi(w),\varphi(z)):\varphi\in\mathcal{O}(D,E)\};$$

and also their infinitesimal versions:

$$\kappa_D(w;X):=\inf\{\gamma_E(\lambda;\alpha):\exists\varphi\in\mathcal{O}(E,D),\ \varphi(\lambda)=w,\ \alpha\varphi'(\lambda)=X\},$$
$$A_D(w;X):=\limsup_{\lambda\to0,\,\lambda\ne0}\frac{g_D(w,w+\lambda X)}{|\lambda|},$$
$$\gamma_D(w;X):=\sup\{\gamma_E(\varphi(w);\varphi'(w)X):\varphi\in\mathcal{O}(D,E)\}.$$

The function $\widetilde{k}_D^*$ (resp. g_D, k_D^*, c_D^*) is called the *Lempert function* (resp. the *Green function*, the *Kobayashi* and *Carathéodory pseudodistance*). The function κ_D (resp. A_D and γ_D) is called the *Kobayashi–Royden* (resp. *Azukawa* and *Carathéodory–Reiffen*) pseudometric.

Note that the functions $\widetilde{k}_D^*$, k_D^* and c_D^* are always symmetric, whereas g_D need not have this property. For the basic properties of the functions defined we refer the interested reader to [JP2]. Let us mention here only some basic relations:

$$\widetilde{k}_D^*\ge k_D^*\ge c_D^*,\quad \widetilde{k}_D^*\ge g_D\ge c_D^*,\quad \kappa_D\ge A_D\ge\gamma_D.$$

A mapping $\varphi\in\mathcal{O}(E,D)$ is called a $\widetilde{k}_D$*-geodesic* for (w,z), $w\ne z$, if $\varphi(\lambda_1)=w$, $\varphi(\lambda_2)=z$ and $m(\lambda_1,\lambda_2)=\widetilde{k}_D^*(w,z)$ for some $\lambda_1,\lambda_2\in E$.

The class of domains we are interested in is defined below.

For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}_+^n$, $n>1$, ($\mathbb{R}_+:=(0,\infty)$) define

$$D_\alpha:=\{z\in\mathbb{C}^n:|z_1|^{\alpha_1}\dots|z_n|^{\alpha_n}<1\}.$$

We say that α is of *rational type* if there are $t > 0$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_*^n$ such that $\alpha = t\beta$; otherwise α is of *irrational type*. Note that if α is of rational type we may without loss of generality assume that all α_j's are relatively prime natural numbers. We also define

$$\widetilde{D}_\alpha := \{z \in D_\alpha : z_1 \ldots z_n \neq 0\}.$$

For $\alpha \in \mathbb{N}_*^n$, we set

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \qquad F^\alpha(z) := z^\alpha,$$
$$F_{(r)}^\alpha(z)X := \sum_{\beta_1 + \ldots + \beta_n = r} \frac{1}{\beta_1! \ldots \beta_n!} \frac{\partial^{\beta_1 + \ldots + \beta_n} F^\alpha(z)}{\partial z_1^{\beta_1} \ldots \partial z_n^{\beta_n}} X^\beta, \qquad z, X \in \mathbb{C}^n.$$

Note that the domain D_α is always unbounded, Reinhardt, complete, and pseudoconvex but not convex.

As mentioned in the introduction, some of the invariant functions for the domains D_α are explicitly known. We gather the results known so far in the following theorem.

Theorem 1 (see [JP2]). *If $\alpha \in \mathbb{N}_*^n$, where α_j's are relatively prime, then*

$$\begin{aligned} c_{D_\alpha}^*(w,z) &= m(w^\alpha, z^\alpha), \\ g_{D_\alpha}(w,z) &= (m(w^\alpha, z^\alpha))^{1/r}, \\ \gamma_{D_\alpha}(w;X) &= \gamma_E(w^\alpha; (F^\alpha)'(w)X), \\ A_{D_\alpha}(w;X) &= (\gamma_E(w^\alpha; F_{(r)}^\alpha(w)X))^{1/r}, \qquad w, z \in D_\alpha,\ X \in \mathbb{C}^n, \end{aligned}$$

where r is the order of the zero of $F^\alpha(\cdot) - F^\alpha(w)$ at w.

If α is of irrational type, then

$$\begin{aligned} c_{D_\alpha}^*(w,z) &= 0, \\ \gamma_{D_\alpha}(w;X) &= 0, \qquad w, z \in D_\alpha,\ X \in \mathbb{C}^n. \end{aligned}$$

We extend the results of Theorem 1 to other invariant functions and pseudometrics and we find the remaining formulas for the Green function (and the Azukawa pseudometric) in the irrational case. The results are presented in two theorems: for rational and irrational α. In both theorems the formulas for the Lempert function may seem incomplete (not all the cases are covered); nevertheless, because of the symmetry of both functions one easily obtains the formulas in the remaining cases.

Theorem 2. *Assume that $\alpha \in \mathbb{N}_*^n$ with α_j's relatively prime. Let $(w,z) \in D_\alpha \times D_\alpha$ and $(w,X) \in D_\alpha \times \mathbb{C}^n$. Set $\mathcal{J} := \{j \in \{1, \ldots, n\} : w_j = 0\} = \{j_1, \ldots, j_k\}$. Then*

$$\widetilde{k}^*_{D_\alpha}(w,z) = \begin{cases} \min\{m(\lambda_1,\lambda_2) : \lambda_1,\lambda_2 \in E, \\ \quad \lambda_1^{\min\{\alpha_j\}} = w^\alpha,\ \lambda_2^{\min\{\alpha_j\}} = z^\alpha\} & \text{if } w,z \in \widetilde{D}_\alpha, \\ |z^\alpha|^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset, \end{cases}$$

$$k^*_{D_\alpha}(w,z) = \min\{m((w^\alpha)^{1/\min\{\alpha_j\}}, (z^\alpha)^{1/\min\{\alpha_j\}})\},$$

where the minimum is taken over all possible roots; *and in the infinitesimal case we have*

$$\kappa_{D_\alpha}(w;X) = \begin{cases} \gamma_E\Big((w^\alpha)^{1/\min\{\alpha_k\}}; (w^\alpha)^{1/\min\{\alpha_k\}} \dfrac{1}{\min\{\alpha_k\}} \displaystyle\sum_{j=1}^n \dfrac{\alpha_j X_j}{w_j}\Big) & \text{if } \mathcal{J} = \emptyset, \\ (|w_1|^{\alpha_1} \ldots |X_{j_1}|^{\alpha_{j_1}} \ldots |X_{j_k}|^{\alpha_{j_k}} \ldots |w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset. \end{cases}$$

Observe that if $\min\{\alpha_j\} = 1$, then $\widetilde{k}^*_{D_\alpha}(w,z) = g_{D_\alpha}(w,z)$ for $w,z \in \widetilde{D}_\alpha$; otherwise, if $w^\alpha \neq z^\alpha$, then the Green function is strictly less than the Lempert function.

As opposed to the rational case, in the irrational case not only the Lempert function, Kobayashi pseudodistance and Kobayashi–Royden pseudometric, but also the Green function and the Azukawa pseudometric have not been calculated so far.

Theorem 3. *Assume that α is of irrational type. Let $(w,z) \in D_\alpha \times D_\alpha$ and $(w,X) \in D_\alpha \times \mathbb{C}^n$. Set $\mathcal{J} := \{j \in \{1,\ldots,n\} : w_j = 0\} = \{j_1,\ldots,j_k\}$. Then*

$$\widetilde{k}^*_{D_\alpha}(w,z) = \begin{cases} m((|w_1|^{\alpha_1} \ldots |w_n|^{\alpha_n})^{1/\min\{\alpha_j\}}, \\ \quad (|z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n})^{1/\min\{\alpha_j\}}) & \text{if } w,z \in \widetilde{D}_\alpha, \\ (|z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset, \end{cases}$$

$$k^*_{D_\alpha}(w,z) = m\Big(\Big(\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\min\{\alpha_j\}}, \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{1/\min\{\alpha_j\}}\Big),$$

$$g_{D_\alpha}(w,z) = \begin{cases} 0 & \text{if } \mathcal{J} = \emptyset, \\ (|z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset; \end{cases}$$

and in the infinitesimal case we have

$$\kappa_{D_\alpha}(w;X) = \begin{cases} \gamma_E\Big(\Big(\displaystyle\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\min\{\alpha_k\}}; \\ \quad \Big(\displaystyle\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\min\{\alpha_k\}} \dfrac{1}{\min\{\alpha_k\}} \displaystyle\sum_{j=1}^n \dfrac{\alpha_j X_j}{w_j}\Big) & \text{if } \mathcal{J} = \emptyset, \\ (|w_1|^{\alpha_1} \ldots |X_{j_1}|^{\alpha_{j_1}} \ldots |X_{j_k}|^{\alpha_{j_k}} \ldots |w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset, \end{cases}$$

$$A_{D_\alpha}(w;X) = \begin{cases} 0 & \textit{if } \mathcal{J}=\emptyset, \\ (|w_1|^{\alpha_1}\ldots|X_{j_1}|^{\alpha_{j_1}}\ldots|X_{j_k}|^{\alpha_{j_k}}\ldots|w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \textit{if } \mathcal{J}\neq\emptyset. \end{cases}$$

Observe that for an arbitrary balanced pseudoconvex domain D we always have $\widetilde{k}_D^*(0,z)=h_D(z)$ for $z\in D$, where h_D denotes the Minkowski function for D. In the above formula, $k_{D_\alpha}^*(0,z)<h_{D_\alpha}(z)$ for $0\neq z\in D_\alpha$. It would be interesting to find the general form of $k_D^*(0,\cdot)$ for an arbitrary balanced pseudoconvex domain D.

2. Auxiliary results. For $z\in\mathbb{C}^n$ put

$$T_z:=\{(e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n):\theta_j\in\mathbb{R}\}.$$

Note that T_z is a group with multiplication defined as follows:

$$(e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n)\circ(e^{i\widetilde{\theta}_1}z_1,\ldots,e^{i\widetilde{\theta}_n}z_n):=(e^{i(\theta_1+\widetilde{\theta}_1)}z_1,\ldots,e^{i(\theta_n+\widetilde{\theta}_n)}z_n).$$

Define $T_{z,\alpha}$ to be the subgroup of T_z generated by the set

$$\{(e^{i\frac{\alpha_{j_1}}{\alpha_1}2k_1\pi}z_1,\ldots,e^{i\frac{\alpha_{j_n}}{\alpha_n}2k_n\pi}z_n):j_1,\ldots,j_n\in\{1,\ldots,n\},\ k_1,\ldots,k_n\in\mathbb{Z}\}.$$

Note that if α is of rational type, then $T_{z,\alpha}$ is finite; more precisely, if we assume that $\alpha\in\mathbb{N}_*^n$ and α_j's are relatively prime, then

$$T_{z,\alpha}=\{(\varepsilon_1z_1,\ldots,\varepsilon_nz_n):\varepsilon_j^{\alpha_j}=1 \text{ for all } j\}.$$

However, if α is of irrational type, then a well-known theorem of Kronecker (see [HW], Theorem 439) gives

$$\overline{T}_{z,\alpha}=T_z. \tag{1}$$

For $\mu\in E_*$ we define

$$\Phi_\mu:\mathbb{C}^{n-1}\ni(\lambda_1,\ldots,\lambda_{n-1})\to (e^{\alpha_n\lambda_1},\ldots,e^{\alpha_n\lambda_{n-1}},\mu e^{-\alpha_1\lambda_1}\ldots e^{-\alpha_{n-1}\lambda_{n-1}})\in D_\alpha.$$

Put

$$V_\mu:=\Phi_\mu(\mathbb{C}^{n-1}),\quad \mu\in E_*,\qquad V_0:=\{z\in\mathbb{C}^n:z_1\ldots z_n=0\}.$$

Note that

$$\bigcup_{\mu\in E}V_\mu=D_\alpha.$$

Remark 4. Let $\mu\in E_*$. Assume that $w,z\in V_\mu$, and $X\in\mathbb{C}^n$ satisfies $\sum_{j=1}^n\alpha_jX_j/w_j=0$. Then

$$\widetilde{k}_{D_\alpha}^*(w,z)=0,\quad \kappa_{D_\alpha}(w;X)=0.$$

In fact, $w=\Phi_\mu(\lambda)$ and $z=\Phi_\mu(\gamma)$ for some $\lambda,\gamma\in\mathbb{C}^{n-1}$, so

$$\widetilde{k}_{D_\alpha}^*(w,z)=\widetilde{k}_{D_\alpha}^*(\Phi_\mu(\lambda),\Phi_\mu(\gamma))\le\widetilde{k}_{\mathbb{C}^{n-1}}^*(\lambda,\gamma)=0.$$

For the second equality note that assuming $\Phi_\mu(\lambda) = w$ we have

$$\Phi'_\mu(\lambda)(Y) = \Big(\alpha_n w_1 Y_1, \ldots, \alpha_n w_{n-1} Y_{n-1}, -\sum_{j=1}^{n-1} \alpha_j w_n Y_j\Big), \quad Y \in \mathbb{C}^{n-1}.$$

One may easily verify that

$$\Phi'_\mu(\lambda)(\mathbb{C}^{n-1}) = \Big\{X \in \mathbb{C}^n : \sum_{j=1}^{n} \alpha_j X_j / w_j = 0\Big\}.$$

Note that

$$0 = \kappa_{\mathbb{C}^{n-1}}(\lambda; Y) \ge \kappa_{D_\alpha}(\Phi_\mu(\lambda); \Phi'_\mu(\lambda)Y), \quad Y \in \mathbb{C}^{n-1},$$

which finishes the proof.

In the proof of Lemma 5 below we shall replace E in the definition of the Lempert function with $H := \{x + iy : 1 > x > -1\}$.

LEMMA 5. *Fix $w, z \in D_\alpha$. Take any $\widetilde{z} \in T_{z,\alpha}$. Then for any $\varphi \in \mathcal{O}(E, D_\alpha)$ such that $\varphi(\lambda_1) = w$, $\varphi(\lambda_2) = z$ for some $\lambda_1 \ne \lambda_2$, there is $\widetilde{\varphi} \in \mathcal{O}(E, D_\alpha)$ such that $\widetilde{\varphi}(\lambda_1) = w$ and $\widetilde{\varphi}(\lambda_2) = \widetilde{z}$. Consequently,*

$$\widetilde{k}^*_{D_\alpha}(w, z) = \widetilde{k}^*_{D_\alpha}(w, \widetilde{z}) \quad \textit{for any } \widetilde{z} \in T_{z,\alpha}.$$

Proof. Take any mapping $\varphi \in \mathcal{O}(H, D_\alpha)$ with $\varphi(0) = w$ and $\varphi(it) = z$ for some $t > 0$. Define (for $k_n \in \mathbb{Z}$ fixed)

$$\widetilde{\varphi} : H \ni \lambda \to \\ (\varphi_1(\lambda), \ldots, \varphi_{n-2}(\lambda), e^{-2k_n\pi\lambda/t}\varphi_{n-1}(\lambda), e^{\alpha_{n-1}2k_n\pi\lambda/(\alpha_n t)}\varphi_n(\lambda)) \in D_\alpha.$$

We have

$$\widetilde{\varphi}(0) = w, \quad \widetilde{\varphi}(it) = (z_1, \ldots, z_{n-1}, e^{i(\alpha_{n-1}/\alpha_n)2k_n\pi} z_n).$$

Note that we may replace α_{n-1} above with any other α_j, and z_n with $e^{i(\alpha_j/\alpha_n)2k_n\pi} z_n$, and also we may continue the procedure with the next components z_j varying, which finishes the proof. ■

REMARK 6. From the proof of Lemma 5 we also have the following property:

Fix $\alpha \in \mathbb{N}_*^n$ with α_j's relatively prime and $0 < \delta_1 \le m(\lambda_1, \lambda_2) \le \delta_2 < 1$. Take any $\psi \in \mathcal{O}(E, \mathbb{C}^n)$ with $\psi(E) \Subset (\mathbb{C}_*)^n$ and choose $z \in \mathbb{C}^n$ such that $z_j^{\alpha_j} = \psi_j^{\alpha_j}(\lambda_2)$ for $j = 1, \ldots, n$. Then there is a mapping $\widetilde{\psi} \in \mathcal{O}(E, \mathbb{C}^n)$ such that $\widetilde{\psi}(E) \Subset (\mathbb{C}_*)^n$, $\psi(\lambda_1) = \widetilde{\psi}(\lambda_1)$, $\widetilde{\psi}(\lambda_2) = z$ and

$$\psi_1^{\alpha_1}(\lambda) \ldots \psi_n^{\alpha_n}(\lambda) = \widetilde{\psi}_1^{\alpha_1}(\lambda) \ldots \widetilde{\psi}_n^{\alpha_n}(\lambda), \quad \lambda \in E,$$
$$m\|\psi_j\|_E \le \|\widetilde{\psi}_j\|_E \le M\|\psi_j\|_E, \quad j = 1, \ldots, n,$$

where $m, M > 0$ depend only on δ_1 and α.

LEMMA 7. *Fix* $L_1^1, L_1^2 \Subset E$, $L_2 \Subset \mathbb{C}_*$ *and* $\alpha \in (\mathbb{R}_+)^n$. *Assume that there is* $\delta > 0$ *such that for any* $\lambda_1 \in L_1^1$ *and* $\lambda_2 \in L_1^2$ *we have* $m(\lambda_1, \lambda_2) \geq \delta$. *Then there is* $L_2 \subset K \Subset \mathbb{C}_*$ *such that for any* $z_1, z_2 \in L_2$ *and for any* $\lambda_1 \in L_1^1$ *and* $\lambda_2 \in L_1^2$ *there is* $\psi \in \mathcal{O}(E, \mathbb{C}_*)$ *with* $\psi(\lambda_j) = z_j$ *for* $j = 1, 2$, *and* $\psi(E) \subset K$. *Moreover*, *there is* $\widetilde{K} \Subset \mathbb{C}_*$ *such that for any numbers* $z_1, \ldots, z_n \in L_2$ *and* $w_1, \ldots, w_k \in L_2$, $k < n$, *with*

$$|z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n} = 1$$

there are functions $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$ *with* $\psi_j(E) \subset \widetilde{K}$ *for* $j = 1, \ldots, n$ *and*

$$\psi_1^{\alpha_1}(\lambda) \ldots \psi_n^{\alpha_n}(\lambda) = e^{i\theta}, \quad \lambda \in E,$$
$$\psi_j(\lambda_1) = z_j,\ j = 1, \ldots, n, \quad \psi_j(\lambda_2) = w_j,\ j = 1, \ldots, k.$$

Proof. For the first part it is sufficient to consider $L_1^1 = \{\lambda_1\}$ and $L_1^2 = \{\lambda_2\}$ with $m(\lambda_1, \lambda_2) = \delta$. The general case is then obtained by composing the functions with automorphisms of E and the dilatation $R\lambda$, where $0 \leq R < 1$, since the images of new functions are contained in that of the original one.

Define

$$L := \exp^{-1}(L_2) \cap (\mathbb{R} \times [0, 2\pi)) \Subset \mathbb{C}.$$

Now put

$$\begin{aligned} K := \{&\exp(h(\lambda)) : \lambda \in E, \text{ and } h \text{ is of the type} \\ &h(\lambda) = a\lambda + b,\ a, b \in \mathbb{C},\ h(\lambda_1) = \widetilde{z}_1,\ h(\lambda_2) = \widetilde{z}_2,\ \widetilde{z}_1, \widetilde{z}_2 \in L\}. \end{aligned}$$

Observe that $K \Subset \mathbb{C}_*$. The mappings we are looking for are of the form $\exp \circ h$, where h is one of the functions appearing in the definition of K.

For the proof of the second part of the lemma we set w_j for $j = k + 1, \ldots, n - 1$ to be any number from L_2 and we take mappings $\psi_1, \ldots, \psi_{n-1}$ as in the first part of the lemma. Define

$$\psi_n(\lambda) := \frac{e^{i\widetilde{\theta}}}{(\psi_1^{\alpha_1}(\lambda) \ldots \psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}},$$

where the branches of powers are chosen arbitrarily and $\widetilde{\theta} \in \mathbb{R}$ is chosen so that $\psi_n(\lambda_1) = z_n$. ∎

LEMMA 8. *Let* $L_1^1, L_1^2, L_2, \delta$ *be as in Lemma* 7. *Fix* $\alpha \in \mathbb{N}_*^n$, *where* α_j*'s are relatively prime. Then there is* $K \Subset \mathbb{C}_*$ *such that for any mappings* $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$, $j = 1, \ldots, n$, *with*

$$\psi_1^{\alpha_1} \ldots \psi_n^{\alpha_n} = 1, \quad \lambda \in E,$$

and $\psi_j(\lambda_1), \psi_j(\lambda_2) \in L_2$, *where* $\lambda_1 \in L_1^1$ *and* $\lambda_2 \in L_1^2$, *there are functions* $\widetilde{\psi}_j \in \mathcal{O}(E, \mathbb{C}_*)$ *such that*

$$\widetilde{\psi}_1^{\alpha_1} \dots \widetilde{\psi}_n^{\alpha_n} = 1 \quad \text{on } E,$$
$$\widetilde{\psi}_j(\lambda_1) = \psi_j(\lambda_1), \quad \widetilde{\psi}_j(\lambda_2) = \psi_j(\lambda_2), \quad \widetilde{\psi}_j(E) \subset K, \quad j = 1, \dots, n.$$

Proof. Put $z_j := \psi_j(\lambda_1)$ and $w_j := \psi_j(\lambda_2)$, $j = 1, \dots, n$. From Lemma 7 there are $\widetilde{\psi}_j$, $j = 1, \dots, n-1$, as desired. Put

$$\widetilde{\psi}_n(\lambda) := \frac{1}{(\widetilde{\psi}_1^{\alpha_1}(\lambda) \dots \widetilde{\psi}_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}.$$

We choose the branch of the power $1/\alpha_n$ so that $\widetilde{\psi}_n(\lambda_1) = z_n$; note also that $\widetilde{\psi}_n^{\alpha_n}(\lambda_2) = w_n^{\alpha_n}$. From Remark 6 we may change $\widetilde{\psi} := (\widetilde{\psi}_1, \dots, \widetilde{\psi}_n)$ so that all the desired properties are preserved and, additionally, $\widetilde{\psi}_n(\lambda_2) = w_n$. ∎

Now we present a lemma which is a weaker infinitesimal version of Lemma 7.

LEMMA 9. *Let* $w \in \mathbb{C}_*$, $X \in \mathbb{C}$ *and* $\lambda_1 \in E$. *Then there is a mapping* $\psi \in \mathcal{O}(E, \mathbb{C}_*)$ *such that*

$$\psi(\lambda_1) = w, \quad \psi'(\lambda_1) = X.$$

Moreover, *for given numbers* $w_1, \dots, w_n \in \mathbb{C}_*$, $X_1, \dots, X_k \in \mathbb{C}$ $(k < n)$ *and* $\alpha \in (\mathbb{R}_+)^n$, *where* $|w_1|^{\alpha_1} \dots |w_n|^{\alpha_n} = 1$, *there are mappings* $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$, $j = 1, \dots, n$, such that

$$\psi_j(\lambda_1) = w_j, \ j = 1, \dots, n, \quad \psi_j'(\lambda_1) = X_j, \ j = 1, \dots, k,$$
$$\psi_1^{\alpha_1}(\lambda) \dots \psi_n^{\alpha_n}(\lambda) = e^{i\theta}, \quad \lambda \in E.$$

Proof. The first part goes as in the proof of Lemma 7 (note that we do not need to specify more, since we do not demand so much about the mapping ψ as in Lemma 7). The mapping we are looking for is of the form $\exp(a\lambda + b)$.

For the second part of the lemma let X_j $(j = k+1, \dots, n-1)$ be any complex number. Take ψ_j as given in the first part of the lemma (for $j = 1, \dots, n-1$) with w replaced with w_j and X replaced with X_j. Put

$$\psi_n(\lambda) := \frac{e^{i\widetilde{\theta}}}{(\psi_1^{\alpha_1}(\lambda) \dots \psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}},$$

where the branches of powers are chosen arbitrarily and $\widetilde{\theta} \in \mathbb{R}$ is chosen so that $\psi_n(\lambda_1) = w_n$. ∎

Now we are able to give formulas for the Lempert function and the Kobayashi–Royden metric for special points.

LEMMA 10. *Fix $w \in V_0$. Let $z \in D_\alpha$ and $X \in \mathbb{C}^n$. Then*

$$\widetilde{k}^*_{D_\alpha}(w,z) = (|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\dots+\alpha_{j_k})},$$

$$\kappa_{D_\alpha}(w;X) = (|w_1|^{\alpha_1} \dots |X_{j_1}|^{\alpha_{j_1}} \dots |X_{j_k}|^{\alpha_{j_k}} \dots |w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\dots+\alpha_{j_k})},$$

where $\mathcal{J} := \{j \in \{1,\dots,n\} : w_j = 0\} = \{j_1,\dots,j_k\}$.

Proof. Without loss of generality we may assume that $w_1 = \dots = w_k = 0$, $w_{k+1},\dots,w_n \neq 0$, $n \geq k \geq 1$. We prove both equalities simultaneously.

First we consider the case $z \in \widetilde{D}_\alpha$ (resp. $X_j \neq 0$ for all $j = 1,\dots,k$). Take any $\varphi \in \mathcal{O}(\overline{E}, D_\alpha)$ such that $\varphi(0) = w$, $\varphi(t) = z$ (resp. $\varphi(0) = w$, $t\varphi'(0) = X$) for some $t > 0$. We have

$$\varphi(\lambda) = (\lambda\psi_1(\lambda),\dots,\lambda\psi_k(\lambda),\psi_{k+1}(\lambda),\dots,\psi_n(\lambda)), \qquad \psi_j \in \mathcal{O}(\overline{E},\mathbb{C}),\ j = 1,\dots,n.$$

Put

$$u(\lambda) := \prod_{j=1}^n |\psi_j(\lambda)|^{\alpha_j}.$$

We know that $\log u \in \mathrm{SH}(\overline{E})$ and $u \leq 1$ on ∂E, so the maximum principle for subharmonic functions implies that $u \leq 1$ on E. In particular, $u(t) \leq 1$ (resp. $u(0) \leq 1$), so

$$\frac{\prod_{j=1}^n |z_j|^{\alpha_j}}{t^{\alpha_1+\dots+\alpha_k}} \leq 1 \quad \Big(\text{resp. } \frac{\prod_{j=1}^k |X_j|^{\alpha_j} \prod_{j=k+1}^n |w_j|^{\alpha_j}}{t^{\alpha_1+\dots+\alpha_k}} \leq 1\Big),$$

which gives

$$t \geq \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{1/(\alpha_1+\dots+\alpha_k)}$$

$$\Big(\text{resp. } t \geq \Big(\prod_{j=1}^k |X_j|^{\alpha_j} \prod_{j=k+1}^n |w_j|^{\alpha_j}\Big)^{1/(\alpha_1+\dots+\alpha_k)}\Big).$$

Therefore,

$$\widetilde{k}^*_{D_\alpha}(w,z) \geq \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{1/(\alpha_1+\dots+\alpha_k)}$$

$$\Big(\text{resp. } \kappa_{D_\alpha}(w;X) \geq \Big(\prod_{j=1}^k |X_j|^{\alpha_j} \prod_{j=k+1}^n |w_j|^{\alpha_j}\Big)^{1/(\alpha_1+\dots+\alpha_k)}\Big).$$

To get equality put

$$t := \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{1/(\alpha_1+\dots+\alpha_k)}$$

$$(\text{resp. } t := \Big(\prod_{j=1}^{k}|X_j|^{\alpha_j}\prod_{j=k+1}^{n}|w_j|^{\alpha_j}\Big)^{1/(\alpha_1+\ldots+\alpha_k)})$$

and consider the mapping

$$\varphi(\lambda) := (\lambda\psi_1(\lambda),\ldots,\lambda\psi_k(\lambda),\psi_{k+1}(\lambda),\ldots,\psi_n(\lambda)),\quad \lambda\in E,$$

where $\psi_j\in\mathcal{O}(E,\mathbb{C}_*)$, $j=1,\ldots,n$, $\prod_{j=1}^{n}\psi_j^{\alpha_j}(\lambda)=e^{i\theta}$ on E and

$$\psi_j(t)=z_j/t,\quad j=1,\ldots,k,\qquad \psi_j(t)=z_j,\quad j=k+1,\ldots,n;$$
$$\psi_j(0)=w_j,\quad j=k+1,\ldots,n\quad (\text{see Lemma 7})$$
$$(\text{resp. } \psi_j(0)=X_j/t,\ j=1,\ldots,k,\ \psi_j(0)=w_j,\ j=k+1,\ldots,n,$$
$$\psi_j'(0)=X_j/t,\ j=k+1,\ldots,n;\ \text{see Lemma 9}).$$

Then $\varphi\in\mathcal{O}(E,D_\alpha)$, $\varphi(0)=w$, $\varphi(t)=z$ (resp. $t\varphi'(0)=X$), which finishes that case.

We are left with the case $z\in V_0$ (resp. $X_j=0$ for some $1\le j\le k$). If there is j such that $w_j=z_j=0$ (resp. $w_j=X_j=0$), then the mapping

$$\mathbb{C}^{n-1}\ni(z_1,\ldots,\check z_j,\ldots,z_n)\to(z_1,\ldots,0,\ldots,z_n)\in D_\alpha$$

gives

$$0=\widetilde{k}^*_{\mathbb{C}^{n-1}}((w_1,\ldots,\check w_j,\ldots,w_n),(z_1,\ldots,\check z_j,\ldots,z_n))\ge\widetilde{k}^*_{D_\alpha}(w,z),$$

(resp.

$$0=\kappa_{\mathbb{C}^{n-1}}((w_1,\ldots,\check w_j,\ldots,w_n);(X_1,\ldots,\check X_j,\ldots,X_n))\ge\kappa_{D_\alpha}(w;X)).$$

Now, there only remains the Lempert function and then we may assume that for all j we have $|w_j|+|z_j|>0$.

For fixed $1>\beta>0$ define the mapping $\varphi:=(\varphi_1,\ldots,\varphi_n)$ as follows:

$$\varphi_j(\lambda):=\begin{cases}\dfrac{\lambda-\beta}{1-\beta\lambda}\psi_j(\lambda) & \text{if } w_j=0,\\[2mm] \dfrac{\lambda+\beta}{1+\beta\lambda}\psi_j(\lambda) & \text{if } z_j=0,\\[2mm] \psi_j(\lambda) & \text{if } w_jz_j\ne0,\end{cases}$$

where $\psi_j\in\mathcal{O}(E,\mathbb{C}_*)$, $\prod_{j=1}^{n}\psi_j^{\alpha_j}(\lambda)=e^{i\theta}$ on E and $\varphi(\beta)=w$, $\varphi(-\beta)=z$ (the values of $\psi_j(\beta)$ and $\psi_j(-\beta)$ are prescribed if only $w_jz_j\ne0$; for $w_jz_j=0$ only one of them is prescribed; more precisely, take j_1 such that $z_{j_1}=0$, and define $\psi_{j_1}(-\beta)$ so that $|\psi_1(-\beta)|^{\alpha_1}\ldots|\psi_n(-\beta)|^{\alpha_n}=1$; note also that there is j_2 such that $w_{j_2}=0$, so $\psi_{j_2}(\beta)$ has no fixed value—it is the reason why we are allowed to use Lemma 7). Note also that $\varphi\in\mathcal{O}(E,D_\alpha)$. As $\beta>0$ may be chosen arbitrarily small this completes the proof of the lemma. ■

In the next step we prove a formula for the Lempert function in the special case of the domain $D_{(1,\ldots,1)}$. Following (to some extent) the ideas from [JPZ] and [PZ] we extend the formulas to the general case using a technique which could be called transport of geodesics. Roughly speaking, the idea is to transport the formulas from simpler domains to more complex ones with the help of "good" mappings. In [JPZ] and [PZ] it was the Euclidean ball that was a model domain. In the present paper it is the domain $D_{(1,\dots,1)}$.

Lemma 11. *If* $w, z \in V_0$, *then*

$$\widetilde{k}^*_{D_{(1,\dots,1)}}(w,z) = 0.$$

Assume that $w \in \widetilde{D}_{(1,\dots,1)}$. *Then*

$$\widetilde{k}^*_{D_{(1,\dots,1)}}(w,z) = (m(w_1\dots w_n, z_1\dots z_n))^{1/k},$$

where

$$k := \max\{\#\{j : z_j = 0\}, 1\}.$$

Proof. The first part is a consequence of Lemma 10. Moreover, also the case $z \in V_0$ is a consequence of Lemma 10.

Consider now the case $w, z \in \widetilde{D}_{(1,\dots,1)}$. We may assume that $w_1\dots w_n \neq z_1\dots z_n$ (the other case is covered by Remark 4).

Consider the following mapping (see Lemma 7):

$$\varphi(\lambda) := (\psi_1(\lambda), \dots, \psi_{n-1}(\lambda), e^{-i\theta}\lambda\psi_n(\lambda)),$$

where

$$\lambda_1 := w_1\dots w_n, \quad \lambda_2 := z_1\dots z_n,$$
$$\psi_j \in \mathcal{O}(E, \mathbb{C}_*), \quad j = 1,\dots,n, \qquad \psi_1(\lambda)\dots\psi_n(\lambda) = e^{i\theta}, \quad \lambda \in E,$$
$$\psi_j(\lambda_1) = w_j, \qquad \psi_j(\lambda_2) = z_j, \qquad j = 1,\dots,n-1$$

(using Lemma 7 we may even assume that $\psi_j(E) \subset K \Subset \mathbb{C}_*$, $j = 1,\dots,n$; compare Remark 12 below).

Note that $\varphi \in \mathcal{O}(E, D_{(1,\dots,1)})$, $\varphi(\lambda_1) = w$, $\varphi(\lambda_2) = z$. Therefore, combining this with the contractivity property of the Lempert function we have

$$m(w_1\dots w_n, z_1\dots z_n) \geq \widetilde{k}^*_{D(1,\dots,1)}(w,z) = m(w_1\dots w_n, z_1\dots z_n).$$

This completes the proof. ∎

Remark 12. From the proof of Lemma 11 we see that for any $w, z \in \widetilde{D}_{(1,\dots,1)}$ with $w_1\dots w_n \neq z_1\dots z_n$ there is a $\widetilde{k}_{D_{(1,\dots,1)}}$-geodesic for (w,z) of the form

$$\left(\psi_1(\lambda), \dots, \psi_{n-1}(\lambda), e^{i\theta}\frac{\lambda-\beta}{1-\overline{\beta}\lambda}\psi_n(\lambda)\right)$$

with $\psi_1(\lambda)\dots\psi_n(\lambda) = 1$ and $\psi_j(E) \Subset \mathbb{C}_*$.

The domains D_α, although very regular, do not have a property which is crucial in the theory of holomorphically invariant functions: they are not taut. Therefore, we have no guarantee that they admit $\widetilde{k}_{D_\alpha}$-geodesics. However, as Lemma 13 will show, they do admit them at least in the rational case and for points which are "separated" by the Lempert function. The existence of the geodesics will play a great role in the proof of the formula for the Lempert function in the rational case.

LEMMA 13. *Assume that $\alpha \in \mathbb{N}_*^n$ and α_j's are relatively prime. Let $w, z \in \widetilde{D}_\alpha$ with $w^\alpha \neq z^\alpha$. Then there is a bounded $\widetilde{k}_{D_\alpha}$-geodesic $\varphi \in \mathcal{O}(E, D_\alpha)$ for (w, z).*

Proof. We know that (use contractivity of the Lempert function)

$$t := \widetilde{k}^*_{D_\alpha}(w, z) \geq m(w^\alpha, z^\alpha) > 0;$$

consequently, there are mappings $\varphi^{(k)} = (\varphi_1^{(k)}, \ldots, \varphi_n^{(k)})$, $k = 1, 2, \ldots$, such that $\varphi^{(k)} \in \mathcal{O}(\overline{E}, D_\alpha)$, $\varphi^{(k)}(0) = w$ and $\varphi^{(k)}(t_k) = z$, where $t_k \geq t_{k+1} \to t > 0$. We have

$$\varphi_j^{(k)} = B_j^{(k)} \psi_j^{(k)}, \quad j = 1, \ldots, n,$$

where $B_j^{(k)}$ is a Blaschke product and $\psi_j^{(k)} \in \mathcal{O}(E, \mathbb{C}_*)$.

Put $\psi^{(k)} := (\psi_j^{(k)})_{j=1}^n$. There are two possibilities (due to the maximum principle for subharmonic functions—remember the pseudoconvexity of D_α):

$$\psi^{(k)}(E) \subset D_\alpha, \tag{2}$$

$$\psi^{(k)}(E) \subset \partial D_\alpha. \tag{3}$$

Below we prove that without loss of generality we may restrict our attention to a case which is some kind of generalization of (3).

Take any k such that (2) is satisfied. First, notice that the mapping

$$\widetilde{\psi}^{(k)} := ((\psi_1^{(k)})^{\alpha_1/(\alpha_1 \ldots \alpha_n)}, \ldots, (\psi_n^{(k)})^{\alpha_n/(\alpha_1 \ldots \alpha_n)})$$

is in $\mathcal{O}(E, D_{(1,\ldots,1)})$. From Remark 12 there is a $\widetilde{k}_{D_{(1,\ldots,1)}}$-geodesic for $(\widetilde{\psi}^{(k)}(0), \widetilde{\psi}^{(k)}(t_k))$ of the form

$$\mu^{(k)} := \left(\widehat{\psi}_1^{(k)}, \ldots, \widehat{\psi}_{n-1}^{(k)}, e^{i\theta_k} \frac{\lambda - \beta_k}{1 - \overline{\beta}_k \lambda} \widehat{\psi}_n^{(k)}\right),$$

where $\widehat{\psi}_1^{(k)} \ldots \widehat{\psi}_n^{(k)} = 1$ on E, such that $\mu^{(k)}(0) = \widetilde{\psi}^{(k)}(0)$ and $\mu^{(k)}(R_k t_k) = \widetilde{\psi}^{(k)}(t_k)$, $\beta_k \in E$, $R_k \leq 1$.

Coming back to the domain D_α we see that instead of considering $\varphi^{(k)}$ with the property (2) we may consider the mapping (note that $(\alpha_1 \ldots \alpha_n)/\alpha_j \in \mathbb{N}$)

$$\widetilde{\varphi}^{(k)}(\lambda) := (B_j^{(k)}(\lambda)(\mu_j^{(k)})^{(\alpha_1 \ldots \alpha_n)/\alpha_j}(R_k \lambda))_{j=1}^n,$$

because $\widetilde{\varphi}^{(k)} \in \mathcal{O}(E, D_\alpha)$, $\widetilde{\varphi}^{(k)}(0) = w$ and $\widetilde{\varphi}^{(k)}(t_k) = z$.

Therefore we may assume that (irrespective of which case we consider, (2) or (3))

$$\varphi_j^{(k)} = B_j^{(k)} \psi_j^{(k)}, \quad j = 1, \ldots, n,$$

where $(\psi_1^{(k)})^{\alpha_1} \ldots (\psi_n^{(k)})^{\alpha_n} = 1$ and $|B_j^{(k)}| \le 1$ (although $B_j^{(k)}$'s need not longer be Blaschke products).

Choosing a subsequence if necessary, we may assume that for all $j = 1, \ldots, n$, $\{B_j^{(k)}\}_{k=1}^{\infty}$ converges locally uniformly on E. Keeping in mind that $\varphi^{(k)}(0) = w$ and $\varphi^{(k)}(t_k) = z$ we see that in view of Lemma 8 there is $K \Subset \mathbb{C}_*$ such that we may assume that $\psi_j^{(k)}(E) \subset K$ for any j, k (we may apply Lemma 8 because $L_2 := \{\psi_j^{(k)}(t_k), \psi_j^{(k)}(0)\}_{j,k} \Subset \mathbb{C}_*$, which follows from the convergence and boundedness of $\{B_j^{(k)}\}_{k=1}^{\infty}$, the fact that $w_j z_j \neq 0$ for $j = 1, \ldots, n$, and the equality $(\psi_1^{(k)})^{\alpha_1} \ldots (\psi_n^{(k)})^{\alpha_n} = 1$), and then choosing a subsequence if necessary we deduce that $\varphi^{(k)}$ converges to a mapping $\varphi \in \mathcal{O}(E, \overline{D}_\alpha)$ with $\varphi(E) \Subset (\mathbb{C}_*)^n$ such that $\varphi(0) = w$ and $\varphi(t) = z$. The maximum principle for subharmonic functions implies, however, that $\varphi(E) \subset D_\alpha$. This completes the proof of the lemma. ∎

3. The rational case—proof of Theorem 2. In the present section we provide the proof of Theorem 2. Since the theorem consists of a number of formulas, we prove them below one by one. We start with the Lempert function, which is basic in the calculation of other functions.

We begin with a formula for the Möbius function which seems to be known, but we have not been able to find any references in the literature. Its proof is elementary but it needs tedious calculations, so we skip it.

LEMMA 14. *Fix $0 < s \le 1$. Then for any $\lambda_1 \in (0,1)$ and $\lambda_2 \in E$ we have*

$$m(\lambda_1^s, \lambda_2^s) \ge m(\lambda_1, \lambda_2),$$

where $\lambda_1^s \in (0,1)$ and the power λ_2^s is chosen so that the left-hand side of the formula is smallest possible.

Proof of the formula for $\widetilde{k}_{D_\alpha}^*$ in the rational case. The case $w_1 \ldots w_n = 0$ is a consequence of Lemma 10. The case $w, z \in \widetilde{D}_\alpha$, $w^\alpha = z^\alpha$ follows from Remark 4. We are left with the case $w, z \in \widetilde{D}_\alpha$, $w^\alpha \neq z^\alpha$. By Lemma 13 there is a bounded $\widetilde{k}_{D_\alpha}$-geodesic $\varphi \in \mathcal{O}(E, D_\alpha)$ for $(w, z) = (\varphi(\lambda_1), \varphi(\lambda_2))$. Proceeding as in the proof of Lemma 13 we may assume that

$$\varphi_j = B_j \psi_j, \quad j = 1, \ldots, n,$$

where B_j is a Blaschke product (up to a constant $|c_j| = 1$), $\psi_j(E) \subset K \Subset \mathbb{C}_*$ and $\psi_1^{\alpha_1} \ldots \psi_n^{\alpha_n} = 1$. In fact, consider the decomposition of φ_j as above with

Blaschke product B_j. Put

$$\widetilde{\psi} := (\psi_j^{\alpha_j/(\alpha_1\ldots\alpha_n)})_{j=1}^n.$$

Consider two cases. If $\psi_1^{\alpha_1}\ldots\psi_n^{\alpha_n}$ is not constant on E, then $\widetilde{\psi} \in \mathcal{O}(E, D_{(1,\ldots,1)})$ and it is a $\widetilde{k}_{D_{(1,\ldots,1)}}$-geodesic for $(\widetilde{\psi}(\lambda_1), \widetilde{\psi}(\lambda_2))$: otherwise, there would be $\widehat{\psi} \in \mathcal{O}(E, D_{(1,\ldots,1)})$ such that $\widehat{\psi}(\lambda_1) = \widetilde{\psi}(\lambda_1)$, $\widehat{\psi}(\lambda_2) = \widetilde{\psi}(\lambda_2)$ and $\widehat{\psi}(E) \Subset D_{(1,\ldots,1)}$ and taking $\widehat{\varphi}(\lambda) := (B_j(\lambda)\widehat{\psi}_j^{(\alpha_1\ldots\alpha_n)/\alpha_j}(\lambda))_{j=1}^n$ we get a mapping such that $\widehat{\varphi}(\lambda_1) = \varphi(\lambda_1)$, $\widehat{\varphi}(\lambda_2) = \varphi(\lambda_2)$ and $\widehat{\varphi}(E) \Subset D_\alpha$—contradiction. By Remark 12 there is a $\widetilde{k}_{D_{(1,\ldots,1)}}$-geodesic μ for $(\widetilde{\psi}(\lambda_1), \widetilde{\psi}(\lambda_2)) = (\mu(\lambda_1), \mu(\lambda_2))$, where $\widehat{\psi}_1\ldots\widehat{\psi}_n = 1$ and $\widehat{\psi}_j(E)$'s are relatively compact in $\mathbb{C}_*$. Taking now $(B_j(\lambda)\mu_j(\lambda)^{(\alpha_1\ldots\alpha_n)/\alpha_j})_{j=1}^n$ instead of φ we get the desired property.

In case $\psi_1^{\alpha_1}\ldots\psi_n^{\alpha_n} = e^{i\theta}$, we may assume that $\psi_j(E) \subset K \Subset \mathbb{C}_*$ for some K because of Lemma 8 (and then without loss of generality we may assume that $e^{i\theta} = 1$).

Therefore, $\varphi(E)$ is contained in some polydisk. Consequently, $\varphi(E)$ is contained in some smooth bounded pseudoconvex complete Reinhardt domain $G \subset D_\alpha$, which arises from the domain D_α by "cutting the ends" and "smoothing the corners". Therefore, φ is a $\widetilde{k}_G$-geodesic for (w, z). Using the results of [E], [Pa] we find that there are mappings $h_j \in H^\infty(E, \mathbb{C})$, $j = 1, \ldots, n$, and $\varrho : \partial E \to (0, \infty)$ such that

$$\frac{1}{\lambda} h_j^*(\lambda)\varphi_j^*(\lambda) = \varrho(\lambda)\alpha_j|(\varphi^*(\lambda))^\alpha|, \quad j = 1, \ldots, n, \text{ for almost all } \lambda \in \partial E$$

(we easily exclude the case $(\varphi^*(\lambda))^\alpha = 0$ for λ from some subset of ∂E with non-zero Lebesgue measure). Using the result of Gentili (see [Ge]) we deduce that for some $b_j \in \mathbb{C}_*$, $j = 1, \ldots, n$, $\beta \in E$,

$$\varphi_j(\lambda)h_j(\lambda) = b_j(1 - \overline{\beta}\lambda)(\lambda - \beta), \quad j = 1, \ldots, n, \ \lambda \in E,$$

where $b_j/\alpha_j = b_k/\alpha_k$, $j, k = 1, \ldots, n$. Consequently, we may take

$$B_j(\lambda) = c_j\left(\frac{\lambda - \beta}{1 - \overline{\beta}\lambda}\right)^{r_j}, \quad |c_j| = 1,$$

where $r_j \in \{0, 1\}$ and not all r_j's are 0. Without loss of generality we may assume that $\beta = 0$ (we then change only λ_1 and λ_2).

Now we come back to the domain D_α. We may assume that $r_1 = \ldots = r_k = 1$ and $r_{k+1} = \ldots = r_n = 0$ ($1 \le k \le n$). We want to have for some $\lambda_1, \lambda_2 \in E$ (without loss of generality we may assume that $c_j = 1$—if necessary we change w and z with the help of rotations of suitable components, so the Lempert function does not change)

$$\lambda_1\psi_j(\lambda_1) = w_j,\ \ j = 1,\dots,k,\quad \psi_j(\lambda_1) = w_j,\ \ j = k+1,\dots,n,$$
$$\lambda_2\psi_j(\lambda_2) = z_j,\ \ j = 1,\dots,k,\quad \psi_j(\lambda_2) = z_j,\ \ j = k+1,\dots,n.$$

Taking the α_jth power and multiplying the equalities we get

$$\lambda_1^{\alpha_1+\dots+\alpha_k} = w^\alpha,\qquad \lambda_2^{\alpha_1+\dots+\alpha_k} = z^\alpha.$$

The formulas above describe all possibilities which may yield candidates for the realization of the Lempert function. Now for all possible λ_1, λ_2 as above we find mappings which map λ_1 and λ_2 to w and z. Note that there are mappings $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$, $j = 2,\dots,n$, such that (see Lemma 7)

$$\begin{aligned}
\psi_j(\lambda_1) &= \frac{w_j}{(w^\alpha)^{1/(\alpha_1+\dots+\alpha_k)}} = \frac{w_j}{\lambda_1}, && j = 2,\dots,k,\\
\psi_j(\lambda_2) &= \frac{z_j}{(z^\alpha)^{1/(\alpha_1+\dots+\alpha_k)}} = \frac{z_j}{\lambda_2}, && j = 2,\dots,k,\\
\psi_j(\lambda_1) &= w_j,\quad \psi_j(\lambda_2) = z_j, && j = k+1,\dots,n,
\end{aligned}$$

Define also

$$\psi_1(\lambda) := \frac{1}{(\psi_2^{\alpha_2}(\lambda)\dots\psi_n^{\alpha_n}(\lambda))^{1/\alpha_1}},\qquad \lambda \in E.$$

Put

$$\varphi(\lambda) := (\lambda\psi_1(\lambda),\dots,\lambda\psi_k(\lambda),\psi_{k+1}(\lambda),\dots,\psi_n(\lambda)).$$

The α_1st root in the definition of ψ_1 is chosen so that $\varphi_1(\lambda_1) = w_1$, and we know that $\varphi_1^{\alpha_1}(\lambda_2) = z_1^{\alpha_1}$. One may also easily verify that $\varphi(\lambda_1) = w$ and $\varphi_j(\lambda_2) = z_j$ for $j = 2,\dots,n$, which, however, in view of Lemma 5 shows that there is also a mapping $\widetilde{\varphi} \in \mathcal{O}(E, D_\alpha)$ such that $\widetilde{\varphi}(\lambda_1) = w$ and $\widetilde{\varphi}(\lambda_2) = z$. Therefore we have proved that

$$\begin{aligned}
\widetilde{k}^*_{D_\alpha}(w,z) = \min\{m(\lambda_1,\lambda_2) : \lambda_1,\lambda_2 \in E,&\\
\lambda_1^{\alpha_{j_1}+\dots+\alpha_{j_k}} = w^\alpha,\ \lambda_2^{\alpha_{j_1}+\dots+\alpha_{j_k}} = z^\alpha\},&
\end{aligned}$$

where the minimum is taken over all possible subsets $\{j_1,\dots,j_k\} \subset \{1,\dots,n\}$. Now Lemma 14 finishes the proof (remark that without loss of generality we may assume that $w_j > 0$ for $j = 1,\dots,n$). ∎

Proof of the formula for $k^*_{D_\alpha}$ in the rational case. Note that $\tanh^{-1}$ of the desired formula is equal to $\tanh^{-1}$ of the Lempert function off the axis, satisfies the triangle inequality and is continuous. The definition of the Kobayashi pseudodistance and its continuity (see [JP2]) finish the proof. ∎

To finish the proof it remains to compute the Kobayashi–Royden pseudometric κ_{D_α}. We do that by defining an operator which connects κ_{D_α} to the Kobayashi pseudodistance.

Following M. Jarnicki and P. Pflug (see [JP2]), for a domain $D\subset\mathbb{C}^n$ we define

$$\mathfrak{D}k_D(w;X):=\limsup_{\lambda\to0,\,\lambda\neq0}\frac{k^*_{D_\alpha}(w,w+\lambda X)}{|\lambda|},\qquad w\in D,\ X\in\mathbb{C}^n.$$

This function differs from that in [JP2], but it is no larger, so the inequality below, which is crucial for our considerations, remains true:

$$\mathfrak{D}k_D(w;X)\le\kappa_D(w;X),\qquad w\in D,\ X\in\mathbb{C}^n. \tag{4}$$

LEMMA 15. *Let $\alpha\in\mathbb{N}^n_*$, where α_j's are relatively prime. Then*

$$\mathfrak{D}k_{D_\alpha}(w;X)$$
$$=\gamma_E\biggl(\Bigl(\prod_{j=1}^n|w_j|^{\alpha_j}\Bigr)^{1/\min\{\alpha_k\}};\Bigl(\prod_{j=1}^n|w_j|^{\alpha_j}\Bigr)^{1/\min\{\alpha_k\}}\frac{1}{\min\{\alpha_k\}}\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\biggr)$$

for $w\in\widetilde{D}_\alpha$ and $X\in\mathbb{C}^n$.

Proof. Without loss of generality we may assume that $w_j>0$ for $j=1,\dots,n$, and $\alpha_n=\min\{\alpha_k\}$. Using the formula for $k^*_{D_\alpha}$ we get

$$\mathfrak{D}k_{D_\alpha}(w;X)=\limsup_{\lambda\to0,\,\lambda\neq0}\frac{|\prod_{j=1}^n(w_j+\lambda X_j)^{\alpha_j/\alpha_n}-\prod_{j=1}^n w_j^{\alpha_j/\alpha_n}|}{|1-\prod_{j=1}^n(w_j+\lambda X_j)^{\alpha_j/\alpha_n}\prod w_j^{\alpha_j/\alpha_n}|\cdot|\lambda|}. \tag{5}$$

Applying the Taylor formula we get, for λ close to 0,

$$(w_j+\lambda X_j)^{\alpha_j/\alpha_n}=w_j^{\alpha_j/\alpha_n}+\frac{\alpha_j}{\alpha_n}w_j^{\alpha_j/\alpha_n}\frac{\lambda X_j}{w_j}+\varepsilon_j(\lambda),\qquad j=1,\dots,n,$$

where $\varepsilon_j(\lambda)/\lambda\to0$ as $\lambda\to0$. Substituting the last equalities in (5) we get

$$\mathfrak{D}k_{D_\alpha}(w;X)=\limsup_{\lambda\to0,\,\lambda\neq0}\frac{(\prod_{j=1}^n|w_j^{\alpha_j}|^{1/\alpha_n})|\lambda||\sum_{j=1}^n\frac{\alpha_jX_j}{\alpha_nw_j}|}{(1-\prod_{j=1}^n|w_j|^{2\alpha_j/\alpha_n})|\lambda|},$$

which equals the desired value. ∎

Proof of the formula for κ_{D_α} in the rational case. If $\mathcal{J}\neq\emptyset$, then in view of Lemma 10 we are done. The case $\sum_{j=1}^n\alpha_jX_j/w_j=0$ follows from Remark 4.

Take $w\in\widetilde{D}_\alpha$. Without loss of generality we may assume that $w_j\in\mathbb{R}_+$ for $j=1,\dots,n$, and $\alpha_n=\min\{\alpha_j\}$. Below, for $X\in\mathbb{C}^n$ with $\sum_{j=1}^n\alpha_jX_j/w_j\neq0$ we shall construct a mapping $\varphi\in\mathcal{O}(E,D_\alpha)$ such that

$$\varphi(\lambda_1)=w,\qquad t\varphi'(\lambda_1)=X,$$

where

$$\lambda_1:=(w_1^{\alpha_1}\dots w_n^{\alpha_n})^{1/\alpha_n}>0,\quad t:=(w_1^{\alpha_1}\dots w_n^{\alpha_n})^{1/\alpha_n}\sum_{j=1}^n\frac{\alpha_jX_j}{\alpha_nw_j}.$$

The existence of such a φ finishes the proof by Lemma 15 and (4).

Define

$$\varphi(\lambda) := \left(\psi_1(\lambda), \ldots, \psi_{n-1}(\lambda), \frac{\lambda}{(\psi_1^{\alpha_1}(\lambda) \ldots \psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}\right),$$

where (see Lemma 9)

$$\psi_j(\lambda_1) = w_j, \quad t\psi_j'(\lambda_1) = X_j, \quad j = 1, \ldots, n-1.$$

We choose the $(1/\alpha_n)$th power so that $\varphi_n(\lambda_1) = w_n$. After some elementary transformation we get $t\varphi_n'(\lambda_1) = X_n$, which finishes the proof. ■

4. The irrational case—proof of Theorem 3. As in the rational case we start with the proof of the formula for the Lempert function. First, we make use of the special properties of the domains of irrational type to get:

LEMMA 16. *Let α be of irrational type. Then for any $w, z \in D_\alpha$,*

$$\widetilde{k}^*_{D_\alpha}(w, z) = \widetilde{k}^*_{D_\alpha}(\widetilde{w}, \widetilde{z}), \qquad \widetilde{w} \in T_w,\ \widetilde{z} \in T_z.$$

P r o o f. Certainly it is enough to prove that

$$\widetilde{k}^*_{D_\alpha}(w, z) = \widetilde{k}^*_{D_\alpha}(w, \widetilde{z}) \quad \text{whenever } \widetilde{z} \in T_z.$$

Assume that

$$\widetilde{k}^*_{D_\alpha}(w, \widetilde{z}_1) < \widetilde{k}^*_{D_\alpha}(w, \widetilde{z}_2) =: \varepsilon \tag{6}$$

for some $\widetilde{z}_1, \widetilde{z}_2 \in T_z$. Then in view of Lemma 5,

$$\widetilde{k}^*_{D_\alpha}(w, \widetilde{z}) = \varepsilon \tag{7}$$

for all $\widetilde{z} \in T_{\widetilde{z}_2,\alpha}$. Because of (1) we have $\widetilde{z}_1 \in T_z = T_{\widetilde{z}_2} = \overline{T}_{\widetilde{z}_2,\alpha}$. Together with (6) and (7), the last statement contradicts, however, the upper-semicontinuity of the Lempert function. ■

As an immediate corollary of Lemma 16 we get

COROLLARY 17. *Let α be of irrational type. Then for any $z \in D_\alpha$,*

$$\widetilde{k}^*_{D_\alpha}(z, \widetilde{z}) = 0 \quad \textit{for any } \widetilde{z} \in T_z.$$

Proof of the formula for $\widetilde{k}^*_{D_\alpha}$ in the irrational case. The case $\mathcal{J} \neq \emptyset$ is covered by Lemma 10. Consider now the remaining case. In view of Lemma 16 we have

$$\widetilde{k}^*_{D_\alpha}(w, z) = \widetilde{k}^*_{D_\alpha}((|w_1|, \ldots, |w_n|), (|z_1|, \ldots, |z_n|)).$$

Choose a sequence $\{\alpha^{(k)}\}_{k=1}^\infty \subset (\mathbb{Q}_+)^n$ such that $\alpha^{(k)} \to \alpha$. First notice that in view of Theorem 2, if $x, y \in (\mathbb{R}_+)^n \cap D_{\alpha^{(k)}}$, then

$$\begin{aligned} &\widetilde{k}^*_{D_{\alpha^{(k)}}}(x, y) \\ &\quad = m((x_1^{\alpha_1^{(k)}} \ldots x_n^{\alpha_n^{(k)}})^{1/\min\{\alpha_j^{(k)}\}}, (y_1^{\alpha_1^{(k)}} \ldots y_n^{\alpha_n^{(k)}})^{1/\min\{\alpha_j^{(k)}\}}). \end{aligned} \tag{8}$$

We may assume that $\min\{\alpha_j\} = \alpha_n$ and $\min\{\alpha_j^{(k)}\} = \alpha_n^{(k)}$.

First we prove that

$$\widetilde{k}^*_{D_\alpha}(w,z) \geq m((|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n})^{1/\alpha_n}, (|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\alpha_n}).$$

Indeed, otherwise there is a mapping $\varphi \in \mathcal{O}(\overline{E}, D_\alpha)$ such that $\varphi(\lambda_1) = (|w_1|,\ldots,|w_n|)$, $\varphi(\lambda_2) = (|z_1|,\ldots,|z_n|)$ and

$$m(\lambda_1,\lambda_2) < m((|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n})^{1/\alpha_n}, (|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\alpha_n}).$$

Then we may choose k so large that $\varphi(E) \subset D_{\alpha^{(k)}}$ and

$$m(\lambda_1,\lambda_2) < m((|w_1|^{\alpha_1^{(k)}}\ldots|w_n|^{\alpha_n^{(k)}})^{1/\alpha_n^{(k)}}, (|z_1|^{\alpha_1^{(k)}}\ldots|z_n|^{\alpha_n^{(k)}})^{1/\alpha_n^{(k)}}),$$

which, however, contradicts (8).

To get equality consider the mapping $\varphi(\lambda) := (\psi_1(\lambda),\ldots,\psi_{n-1}(\lambda), \lambda\psi_n(\lambda))$, where (see Lemma 7) $\varphi_j \in \mathcal{O}(E,\mathbb{C}_*)$ for $j=1,\ldots,n-1$,

$$\lambda_1 := (|w_1|^{\alpha_1}\cdot\ldots\cdot|w_n|^{\alpha_n})^{1/\alpha_n} > 0, \quad \lambda_2 := (|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\alpha_n} > 0;$$
$$\psi_j(\lambda_1) = |w_j|, \quad \psi_j(\lambda_2) = |z_j|, \quad j = 1,\ldots,n-1.$$

Define also

$$\psi_n(\lambda) := \frac{1}{(\psi_1^{\alpha_1}(\lambda)\ldots\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}, \qquad \lambda \in E.$$

The α_nth root is chosen so that $\varphi_n(\lambda_1) = |w_n|$. One may also easily check from the form of ψ_j's in the proof of Lemma 7 that then $\varphi_n(\lambda_2) > 0$, so $\varphi_n(\lambda_2) = |z_n|$. This completes the proof. ∎

Just as in the rational case we have:

Proof of the formula for $k^*_{D_\alpha}$ in the irrational case. Note that $\tanh^{-1}$ of the desired formula satisfies the triangle inequality and coincides with the $\tanh^{-1}$ of the Lempert function off the axis. The definition of the Kobayashi pseudodistance and its continuity finish the proof. ∎

Having the formula for the Lempert function we get

Proof of the formula for g_{D_α} in the irrational case.

Case I: $\mathcal{J} = \emptyset$. Corollary 17 implies that $g_{D_\alpha}(w,z) = 0$ for any $z \in T_w$. The maximum principle for plurisubharmonic functions (applied to $g_{D_\alpha}(w,\cdot)$) implies that $g_{D_\alpha}(w,z) = 0$ for any z with $|z_j| \leq |w_j|$, which means that $g_{D_\alpha}(w,\cdot)$ vanishes on a set with non-empty interior (remember that $w_1\ldots w_n \neq 0$); but $g_{D_\alpha}(w,\cdot)$ is logarithmically plurisubharmonic, so it must vanish on D_α.

Case II: $\mathcal{J} \neq \emptyset$. This case is a simple consequence of Lemma 10, the inequality $g \leq \widetilde{k}^*$, the definition of the Green function and the fact that the function $(|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})}$ is logarithmically plurisubharmonic on D_α. ∎

Proof of the formula for A_{D_α} in the irrational case. The result follows from the formula for the Green function and the definition of the Azukawa pseudometric. ■

Now, as in the rational case, we finish up the proof by showing the formula for κ_{D_α}.

LEMMA 18. *Let α be of irrational type. Then*

$$\mathfrak{D}k_{D_\alpha}(w;X) = \gamma_E\Big(\Big(\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\min\{\alpha_k\}}; \Big(\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\min\{\alpha_k\}} \frac{1}{\min\{\alpha_k\}} \sum_{j=1}^n \frac{\alpha_j X_j}{w_j}\Big)$$

for $w \in \widetilde{D}_\alpha$ and $X \in \mathbb{C}^n$.

Proof. Without loss of generality we may assume that $\alpha_n = \min\{\alpha_k\}$. The formula for the Kobayashi pseudodistance gives us

$$(9)\quad \mathfrak{D}k_{D_\alpha}(w;X) = \limsup_{\lambda\to 0,\,\lambda\neq 0} \frac{|\prod_{j=1}^n |w_j+\lambda X_j|^{\alpha_j/\alpha_n} - \prod_{j=1}^n |w_j|^{\alpha_j/\alpha_n}|}{|1-\prod_{j=1}^n |w_j+\lambda X_j|^{\alpha_j/\alpha_n}\prod |w_j|^{\alpha_j/\alpha_n}|\cdot|\lambda|}.$$

Note that $\alpha_j/\alpha_n \ge 1$. Therefore, applying the Taylor formula we get, for λ close to 0,

$$|w_j+\lambda X_j|^{\alpha_j/\alpha_n} = |w_j|^{\alpha_j/\alpha_n} + \frac{\alpha_j}{\alpha_n}|w_j|^{\alpha_j/\alpha_n}\Big(\operatorname{Re}\Big(\frac{\lambda X_j}{w_j}\Big)\Big) + \varepsilon_j(\lambda), \qquad j=1,\ldots,n,$$

where $\varepsilon_j(\lambda)/\lambda \to 0$ as $\lambda\to 0$. Substituting the last equalities in (9) we get

$$\mathfrak{D}k_{D_\alpha}(w;X) = \limsup_{\lambda\to 0,\,\lambda\neq 0} \frac{\prod_{j=1}^n (|w_j|^{\alpha_j})^{1/\alpha_n} \operatorname{Re}\big(\lambda\big(\sum_{j=1}^n \frac{\alpha_j X_j}{\alpha_n w_j}\big)\big)}{\big(1-\prod_{j=1}^n |w_j|^{2\alpha_j/\alpha_n}\big)|\lambda|},$$

which equals the desired value. ■

Proof of the formula for κ_{D_α} in the irrational case. If $\mathcal{J}\neq\emptyset$, then in view of Lemma 10 we are done. Also the case $\sum_{j=1}^n \alpha_j X_j/w_j = 0$ follows from Remark 4. Below we deal with the remaining cases.

Take $w\in\widetilde{D}_\alpha$. Without loss of generality we may assume that $w_j\in\mathbb{R}_+$ for $j=1,\ldots,n$, and $\alpha_n=\min\{\alpha_j\}$. Below, for $X\in\mathbb{C}^n$ with $\sum_{j=1}^n \alpha_j X_j/w_j \neq 0$ we construct a mapping $\varphi\in\mathcal{O}(E,D_\alpha)$ such that $\varphi(\lambda_1)=w$ and $t\varphi'(\lambda_1) = X$, where

$$\lambda_1 := (w_1^{\alpha_1}\ldots w_n^{\alpha_n})^{1/\alpha_n} > 0, \qquad t := (w_1^{\alpha_1}\ldots w_n^{\alpha_n})^{1/\alpha_n}\sum_{j=1}^n \frac{\alpha_j X_j}{\alpha_n w_j}.$$

The existence of such a φ finishes the proof by Lemma 18 and (4).

Define

$$\varphi(\lambda) := \left(\psi_1(\lambda), \ldots, \psi_{n-1}(\lambda), \frac{\lambda}{(\psi_1^{\alpha_1}(\lambda) \ldots \psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}\right),$$

where (see Lemma 9)

$$\psi_j(\lambda_1) = w_j, \quad t\psi_j'(\lambda_1) = X_j, \quad j = 1, \ldots, n-1.$$

We choose the $(1/\alpha_n)$th power so that $\varphi_n(\lambda_1) = w_n$. After some elementary transformation we get $t\varphi_n'(\lambda_1) = X_n$, which finishes the proof. ∎

5. Some applications. We now formulate some corollaries which show how irregularly the invariant functions can behave although the domains considered are very regular.

For a given domain $D \subset \mathbb{C}^n$ we define a relation $\mathcal{R}$ on D as follows: $w\mathcal{R}z$ for $w, z \in D$ if $k_D^*(w, z) = 0$. In [Ko2], S. Kobayashi asked whether the quotient $D/\mathcal{R}$ always has a complex structure. The answer is "no", but the examples showing this are artificial (see [Ko1], p. 130; also [HD] and [Gi]). From Theorem 3 we know that if α is of irrational type, then $D_\alpha/\mathcal{R}$ is $[0, 1)$. This gives the first very simple example of a very regular domain for which the answer to the above question is "no".

One may consider some generalizations of the Carathéodory pseudodistance, called the kth Möbius function, denoted by m^k (for $k = 1, 2, \ldots$) (for definitions see [JP2]). S. Nivoche [N] has proved that if a domain is strictly hyperconvex, then the functions m^k tend to g. One may easily verify that if α is of irrational type, then all the m^k's vanish on $D_\alpha \times D_\alpha$. Therefore no such convergence holds in the domains D_α (α of irrational type), so one should not expect a result similar to [N] in the class of Reinhardt complete pseudoconvex domains.

In general, the Lempert function seems to be very distant from the Green function. The definition of the Kobayashi pseudodistance makes the impression that the Kobayashi pseudodistance should be larger than or equal to the Green function. Nevertheless, if $\alpha \in \mathbb{N}_*^n$ is such that all α_j's are relatively prime and $\min\{\alpha_j\} = 1$, then (see Theorem 2)

$$c_{D_\alpha}^* \equiv k_{D_\alpha}^* \le g_{D_\alpha} \le \widetilde{k}_{D_\alpha}^*, \qquad k_{D_\alpha}^* \ne g_{D_\alpha}.$$

In [Pa] and [L1] a notion of stationary maps was introduced and studied. In the class of strongly convex domains these mappings are exactly the $\widetilde{k}$-geodesics. In strongly pseudoconvex domains, geodesics are necessarily stationary maps. The converse implication does not hold in general (see [Pa] and [PZ]). From the proof of Theorem 2 we may construct also other examples disproving that implication.

6. Open problems. It would be interesting to find formulas for all the invariant functions discussed above for domains of the following more general type:

$$D_{\alpha^1} \cap \ldots \cap D_{\alpha^k} \cap ((R_1 E) \times \ldots \times (R_n E)),$$

where $\alpha^j \in (\mathbb{R}_+)^n$, $j = 1, \ldots, k$.

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