Dini continuity of the first derivatives of generalized solutions to the Dirichlet problem for linear elliptic second order equations in nonsmooth domains

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Abstract. We consider generalized solutions to the Dirichlet problem for linear elliptic second order equations in a domain bounded by a Dini–Lyapunov surface and containing a conical point. For such solutions we derive Dini estimates for the first order generalized derivatives.

1. Introduction. We consider generalized solutions to the Dirichlet problem for a linear uniformly elliptic second order equation in divergence form

\[
\begin{cases}
\frac{\partial}{\partial x_i} (a^{ij}(x) u_{x_j} + a^i(x) u) + b^i(x) u_{x_i} + c(x) u \\
= g(x) + \frac{\partial f^j(x)}{\partial x_j}, & x \in G, \\
u(x) = \varphi(x), & x \in \partial G
\end{cases}
\]

(summation over repeated indices from 1 to \(n\) is understood), where \(G \subset \mathbb{R}^n\) is a bounded domain with boundary \(\partial G\) and \(\partial G\) is a Dini–Lyapunov surface containing the origin \(O\) as a conical point. This last means that \(\partial G \setminus O\) is a smooth manifold but near \(O\) the domain \(G\) is diffeomorphic to a cone.

Hölder estimates for the first derivatives of generalized solutions to the problem (DL) are well known in the case where the leading coefficients \(a^{ij}(x)\) are Hölder continuous (see e.g. [5, 8.11] for smooth domains and [1] for domains with angular points). Here we derive Dini estimates for the first derivatives of generalized solutions of the problem (DL) in a domain with a conical boundary point under minimal smoothness conditions on the leading coefficients (Dini continuity). It should be noted that interior Dini continuity

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of the first and second derivatives of generalized solutions to the problem (DL) was investigated in [3, 7] under the condition of Dini continuity of the first derivatives of the leading coefficients.

We introduce the following notations and definitions:

- \([l]:\) the integral part of \(l\) (if \(l\) is not an integer);
- \(r = |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}\);
- \(G' \in G: G'\) has compact closure contained in \(G\);
- \(\text{mes} G: \) volume of \(G\);
- \(S^{n-1}: \) the unit sphere in \(\mathbb{R}^n\);
- \(\Omega: \) a domain in \(S^{n-1}\) with smooth \((n-2)\)-dimensional boundary \(\partial\Omega\);
- \(G_a^b = G \cap \{(r, \omega) | 0 \leq a < r < b, \omega \in \Omega\}: \) a layer in \(\mathbb{R}^n\);
- \(\Gamma_a^b = \partial G \cap \{(r, \omega) | 0 \leq a < r < b, \omega \in \partial\Omega\}: \) the lateral surface of the layer \(G_a^b\);
- \(D_{ij}u = u_{x_i}u_{x_j} = \partial^2 u/\partial x_i \partial x_j;\)
- \(\nabla u = (u_{x_1}, \ldots, u_{x_n}): \) the gradient of \(u(x)\);
- \(\mathbf{n} = \mathbf{n}(x) = \{\nu_1, \ldots, \nu_n\}: \) the unit outward normal to \(\partial G\) at the point \(x;\)
- \(d\Omega: \) the \((n-1)\)-dimensional area element of the unit sphere;
- \(d\sigma: \) the \((n-1)\)-dimensional area element of \(\partial G;\)
- \(\Delta: \) the Laplacian in \(\mathbb{R}^n;\)
- \(\Delta_n: \) the Laplace–Beltrami operator on the unit sphere \(S^{n-1};\)
- \(d(x) = \text{dist}(x, \partial G \setminus O);\)
- \(\Phi(x): \) any possible extension into \(G\) of a boundary function \(\varphi(x), i.e., \Phi(x) = \varphi(x)\) for \(x \in \partial G;\)
- \(A(t): \) a function defined for \(t \geq 0,\) nonnegative, increasing, continuous at zero, with \(\lim_{t \to +0} A(t) = 0.\)

**Definition 1.1.** The function \(A\) is called *Dini continuous at zero* if
\[\int_0^d t^{-1} A(t) \, dt < \infty \text{ for some } d > 0.\]

**Definition 1.2.** The function \(A\) is called an \(\alpha\)-function, \(0 < \alpha < 1,\) on \((0, d]\) if \(t^{-\alpha} A(t)\) is decreasing on \((0, d],\) i.e.
\[A(t) \leq t^\alpha \tau^{-\alpha} A(\tau), \quad 0 < \tau \leq t \leq d.\]

In particular, setting \(t = ct, \quad c > 1,\) we have
\[A(ct) \leq c^\alpha A(\tau), \quad 0 < \tau \leq c^{-1}d.\]
If an \( \alpha \)-function \( A \) is Dini continuous at zero, then we say that \( A \) is an \( \alpha \)-Dini function. In that case we also define the function \( B(t) = \int_0^t (A(\tau)/\tau) d\tau \). It is obvious that \( B \) is increasing and continuous on \([0,d]\), and \( B(0) = 0 \). We integrate the inequality (1.1) over \( \tau \) from 0 to \( t \):

\[
A(t) \leq \alpha B(t).
\]

Similarly from (1.1) we derive

\[
\delta \int_0^d (A(t)/t^2) dt \leq \delta - \alpha \delta A(\delta) \int_0^d t^{\alpha-2} dt \leq (1 - \alpha)^{-1} A(\delta)/\delta,
\]

whence by (1.3),

\[
\delta \int_0^d (A(t)/t^2) dt \leq (1 - \alpha)^{-1} A(\delta), \quad \forall \alpha \in (0,1), \; 0 < \delta < d.
\]

**Definition 1.3.** The function \( B \) is called equivalent to \( A \), written \( A \sim B \), if there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 A(t) \leq B(t) \leq C_2 A(t) \quad \text{for all } t \geq 0.
\]

An equivalence test is known [4, theorem of Sec. 1]: \( A \sim B \) if and only if

\[
\lim_{t \to 0} A(2t)/A(t) > 1.
\]

In some cases we shall consider functions \( A \) such that also

\[
\sup_{0 < \tau \leq 1} A(\tau t)/A(\tau) \leq c A(t), \quad \forall t \in (0,d],
\]

with some constant \( c \) independent of \( t \). Examples of \( \alpha \)-Dini functions \( A \) which satisfy (1.5), (1.6) with \( c = 1 \) are:

\[
t^\alpha, \quad 0 \leq t < \infty; \\
t^\alpha \ln(1/t), \quad t \in (0,d], \quad d = \min(e^{-\alpha}, e^{-1/\alpha}), \quad e^{-1} < \alpha < 1.
\]

We will consider the following function spaces:

- \( C^l(G) \): the Banach space of functions having all the derivatives of order at most \( l \) (if \( l \) = integer \( \geq 0 \)) and of order \( |l| \) (if \( l \) is noninteger) continuous in \( G \) and whose \( |l| \)-th order partial derivatives are uniformly Hölder continuous with exponent \( l - |l| \) in \( G \); \( |u|_{l;G} \) is the norm of the element \( u \in C^l(G) \); if \( l \neq |l| \) then

\[
|u|_{l;G} = \sum_{j=0}^{|l|} \sup_G |D^j u| + \sup_{|\alpha|=|l|} \sup_{x \neq y \in G} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^{l-|l|}}.
\]

- \( C^k_0(G) \): the set of functions in \( C^k(G) \) with compact support in \( G \).
• $C^{0,A}(G)$: the set of bounded and continuous functions $f$ on $G$ with

$$[f]_{A;G} = \sup_{x,y \in G, x \neq y} \frac{|f(x) - f(y)|}{A(|x - y|)} < \infty;$$

equipped with the norm

$$\|f\|_{A;G} = |f|_{0;G} + [f]_{A;G},$$

this set is a Banach space. We also define the quantity

$$[f]_{A,x} = \sup_{y \in G \setminus \{x\}} \frac{|f(x) - f(y)|}{A(|x - y|)}.$$

It is not difficult to see that if $A \sim B$ then $[f]_{A} \sim [f]_{B}$.

If $k \geq 1$ is an integer then we denote by $C^{k,A}(G)$ the subspace of $C^{k}(G)$ consisting of functions whose $(k-1)$th order partial derivatives are uniformly Lipschitz continuous and each $k$th order derivative belongs to $C^{0,A}(G)$; it is a Banach space with the norm

$$\|f\|_{k,A;G} = |f|_{k;G} + \sum_{|\beta| = k} [D^\beta f]_{A;G}.$$

The interpolation inequality (see [8, (10.1)]) will be needed: if the domain has a Lipschitz boundary, then for any $\varepsilon > 0$ there exists a constant $c(\varepsilon, G)$ such that for every $f \in C^{1,A}(G)$,

$$(1.7) \quad \sum_{i=1}^{n} |D_i f|_{0;G} \leq \varepsilon \sum_{i=1}^{n} [D_i f]_{A;G} + c(\varepsilon, G) |f|_{0;G}.$$  

• $L_p(G)$: the Banach space of $p$-integrable functions $u$ on $G$ $(p \geq 1)$ with norm $\|u\|_{p;G}$.

Moreover, $\lambda = \lambda(\Omega)$ will stand for the smallest positive eigenvalue of the problem

$$(\text{EVP}) \quad \left\{ \begin{array}{l}
\Delta \omega \psi + \lambda(\lambda + n - 2)\psi = 0, \quad \omega \in \Omega \subset S^{n-1}, \\
\psi(\omega) = 0, \quad \omega \in \partial \Omega,
\end{array} \right.$$  

and $c(...)$ will be different constants depending only on the quantities appearing in parentheses.

Let $T \subset \partial G$ be a nonempty set. Following [5, Sec. 6.2] and [8, Sec. 3] we shall say that the boundary portion $T$ is of class $C^{1,A}$ if for each point $x_0 \in T$ there are a ball $B = B(x_0)$, a one-to-one mapping $\psi$ of $B$ onto a ball $B'$ and a constant $K > 0$ such that:

1. $B \cap \partial G \subset T$, $\psi(B \cap G) \subset \mathbb{R}^n_+$;
2. $\psi(B \cap \partial G) \subset \Sigma$;
3. $\psi \in C^{1,A}(B)$, $\psi^{-1} \in C^{1,A}(B')$.
Newtonian potentials in \( B(2.3) \)

It is not difficult to see that for \( y = \psi(x) \) one has

\[
K^{-1}|y - y'| \leq |x - x'| \leq K|y - y'|, \quad \forall x, x' \in B.
\]

**Lemma** [8, Sec. 7, (iv)]. Let \( A \) be an \( \alpha \)-function and \( f \in C^{0,A}(B) \), \( \psi^{-1} \in C^{1,A}(B') \). Then \( f \circ \psi^{-1} \in C^{1,A}(B) \) and

\[
[f \circ \psi^{-1}]_{A;B} \leq K^\alpha [f]_{A;B}.
\]

2. Dini estimates of the first derivatives for the generalized Newtonian potential (cf. [5, Ch. 4]). We shall consider the Dirichlet problem for the Poisson equation

\[
(PE) \begin{cases}
\Delta v = g + \sum_{j=1}^n D_j F_j, & x \in G, \\
v(x) = 0, & x \in \partial G.
\end{cases}
\]

Let \( \Gamma(x - y) \) be the normalized fundamental solution of Laplace’s equation. The following estimates are known (see e.g. [5, (2.12), (2.14)]):

\[
|\Gamma(x - y)| = |x - y|^{2-n}/(n(n - 2)\omega_n), \quad n \geq 3,
\]

\[
|D_i \Gamma(x - y)| \leq |x - y|^{1-n}/(n\omega_n),
\]

\[
|D_{ij} \Gamma(x - y)| \leq |x - y|^{-n}/\omega_n,
\]

\[
|D^\beta \Gamma(x - y)| \leq C(n, \beta)|x - y|^{2-n-|\beta|}.
\]

We define the functions

\[
z(x) = \int_G \Gamma(x - y) g(y) dy, \quad w(x) = D_j \int_G \Gamma(x - y) F_j(y) dy,
\]

assuming that the functions \( g(x) \) and \( F_j(x) \), \( j = 1, \ldots, n \), are integrable on \( G \). The function \( z \) is called the Newtonian potential with density function \( g \), and \( w \) is called the generalized Newtonian potential with density function \( F \). We now give estimates for these potentials.

Let \( B_1 = B_R(x_0) \), \( B_2 = B_{2R}(x_0) \) be concentric balls in \( \mathbb{R}^n \) and \( z, w \) be Newtonian potentials in \( B_2 \).

**Lemma 1.** Suppose \( g \in L_p(B_2) \), \( p > n/2 \), and \( F_j \in L_{\infty}(B_2) \), \( j = 1, \ldots, n \). Then

\[
|z|_{0,B_1} \leq c(p) R^{2/p'} \ln^{1/p'} (1/(2R)) \| g \|_{p;B_2}, \quad n = 2,
\]

\[
|z|_{0,B_1} \leq c(p, n) R^{2-n+n/p'} \| g \|_{p;B_2}, \quad n \geq 3,
\]
where $1/p + 1/p' = 1$.

Proof. The estimates follow from inequalities (2.1), Hölder’s inequality and [5, Lemma 4.1].

In the following the $D$ operator is always taken with respect to the $x$ variable.

**Lemma 2** [5, Lemmas 4.1, 4.2]. Let $\partial G \in C^{1,A}$, $G \in L_p(G)$, $p > n$, $F^j \in C^{0,A}(G)$, where $A$ is an $\alpha$-function Dini continuous at zero. Then for any $x \in G$,

\begin{align}
D_i z(x) &= \int_D D_i \Gamma(x - y) G(y) \, dy, \\
D_i w(x) &= \int_{G_0} D_{ij} \Gamma(x - y)(F^j(y) - F^j(x)) \, dy \\
&\quad - F^j(x) \int_{\partial G_0} D_i \Gamma(x - y) \nu_j \, dy \sigma
\end{align}

\begin{enumerate}
(i = 1, \ldots, n); here $G_0$ is any domain containing $G$ for which the Gauss divergence theorem holds and $F^j$ are extended to vanish outside $G$.

**Lemma 3** (cf. [5, Lemma 4.4]). Let $G \in L_p(B_2)$, $p > n$, $F^j \in C^{0,A}(B_2)$, where $A$ is an $\alpha$-function Dini continuous at zero. Then $z, w \in C^{1,B}(B_1)$ and

\begin{align}
\|z\|_{1,B;B_1} &\leq c(n, p, A^{-1}(2R)) \|G\|_{p;B_2}, \\
\|w\|_{1,B;B_1} &\leq c(n, p, \alpha, R, A^{-1}(2R), B(2R)) \sum_{j=1}^n \|F^j\|_{0,A;B_2}.
\end{align}

Proof. Let $x, y \in B_1$ and $G = B_2$. By formulas (2.5), (2.6), taking into account (2.1) and Hölder’s inequality and setting $|x - y| = t$, $y - x = t\omega$, $dy = t^{n-1}dt d\Omega$, we have

\begin{align}
|D_i z| &\leq (n\omega_n)^{-1} \int_{B_2} |x - y|^{1-n} |G(y)| \, dy \\
&\leq (n\omega_n)^{-1} \|G\|_{p;B_2} \left\{ \int_{B_2} |x - y|^{(1-n)p'} \, dy \right\}^{1/p'} \\
&= \frac{p-1}{p-n} (2R)^{(p-n)/(p-1)} \|G\|_{p;B_2}, \\
|D_i w(x)| &\leq (n\omega_n)^{-1} R^{1-n} \sum_{j=1}^n |F^j(x)| \int_{\partial B_2} d\nu \sigma
\end{align}

Proof. The estimates follow from inequalities (2.1), Hölder’s inequality and [5, Lemma 4.1].
\[ + \omega_n^{-1} \sum_{j=1}^{n} [F^j]_{A,x} \int_{B_2} \frac{A(x-y)}{|x-y|^{n}} dy \]
\[ \leq 2^n \sum_{j=1}^{n} |F^j(x)| + n \sum_{j=1}^{n} [F^j]_{A,x} \int_{0}^{2R} \frac{A(t)}{t} dt \]
\[ \leq c(n)B(2R) \sum_{j=1}^{n} (|F^j(x)| + [F^j]_{A,x}). \]

Taking into account (2.5) we obtain by subtraction
\[ |D_z(x) - D_z(x)| \leq \int_{B_2} |D_z \Gamma(x-y) - D_z \Gamma(x-y) \cdot |G(y)| dy. \]

We set \( \delta = |x-v| \), \( \xi = \frac{1}{2}(x-v) \) and write \( B_2 = B_\delta (\xi) \cup \{B_2 \setminus B_\delta (\xi) \} \). Then
\[
(2.11) \int_{B_\delta (\xi)} |D_z \Gamma(x-y) - D_z \Gamma(x) \cdot |G(y)| dy \\
\leq \int_{B_\delta (\xi)} |D_z \Gamma(x-y)| \cdot |G(y)| dy + \int_{B_\delta (\xi)} |D_z \Gamma(x-y) \cdot |G(y)| dy \\
\leq (n \omega_n)^{-1} \left\{ \int_{B_\delta (\xi)} |x-y|^{1-n} |G(y)| dy + \int_{B_\delta (\xi)} |x-y|^{1-n} |G(y)| dy \right\} \\
\leq 2(n \omega_n)^{-1} \int_{B_{3/2}(x)} |x-y|^{1-n} |G(y)| dy \\
\leq 2(n \omega_n)^{-1} |G|_{p;B_2} \left( \int_{B_{3/2}(x)} |x-y|^{(1-n)p'} dy \right)^{1/p'} \\
\leq 2(n \omega_n)^{-1/p} |G|_{p;B_2} \left( \frac{3 \delta^p}{2} \right) \left\{ n + (1-n)p' \right\}^{-1/p'} \\
\leq 2(n \omega_n)^{-1/p} (2R)^{-1-n/p} \frac{A(|x-v|)}{A(2R)} |G|_{p;B_2}, \quad 1/p + 1/p' = 1
\]

(here we take into account that \( \delta^\alpha \leq (2R)^\alpha A(\delta)/A(2R) \) for all \( \alpha > 0 \) by (1.1), since \( \delta \leq 2R \)). Similarly,
\[
(2.12) \int_{B_2 \setminus B_\delta (\xi)} |D_z \Gamma(x-y) - D_z \Gamma(x-y) \cdot |G(y)| dy \\
\leq |x-v| \int_{B_2 \setminus B_\delta (\xi)} |D \Gamma(x-y)| \cdot |G(y)| dy \\
\quad \text{(for some } \tilde{x} \text{ between } x \text{ and } v)\]
\[ \leq \delta \omega_n^{-1} \int_{|y-\xi|\geq \delta} |y - \tilde{x}|^{-n} |\mathcal{G}(y)| \, dy \]

\[ \leq 2^n \delta \omega_n^{-1} \int_{|y - \xi|\geq \delta} |y - \xi|^{-n} |\mathcal{G}(y)| \, dy \quad \text{since } |y - \xi| \leq 2 |y - \tilde{x}| \]

\[ \leq 2^n \delta \omega_n^{-1} |\mathcal{G}|_{p:B_2} \left( \int_{|y - \xi|\geq \delta} |y - \xi|^{-np'} \, dy \right)^{1/p'} \]

\[ \leq 2^n \omega_n^{-1/p}(p - 1)^{1/p'} \delta^{1-n/p} |\mathcal{G}|_{p:B_2} \]

\[ \leq 2^n \omega_n^{-1/p}(p - 1)^{1/p'} (2R)^{1-n/p} \frac{A(|x - \bar{x}|)}{A(2R)} |\mathcal{G}|_{p:B_2}. \]

From (2.11) and (2.12), taking into account (1.3), we obtain

\[ |D_i z(x) - D_i z(\bar{x})| \]

\[ \leq c(n, p, R) A^{-1}(2R) |\mathcal{G}|_{p:B_2} A(|x - \bar{x}|) \]

\[ \leq c(n, p, R) A^{-1}(2R) |\mathcal{G}|_{p:B_2} B(|x - \bar{x}|), \quad \forall x, \bar{x} \in B_1. \]

The first of the required estimates, (2.7), follows from (2.3) and (2.13).

Now we derive the estimate (2.8).

By (2.6) for all \( x, \bar{x} \in B_1 \) we have

\[ D_i w(\bar{x}) - D_i w(x) \]

\[ = \sum_{j=1}^{n} \left( (\mathcal{F}^j(x) J_{1j} + (\mathcal{F}^j(x) - \mathcal{F}^j(\bar{x})) J_{2j} + J_3 + J_4 + \sum_{j=1}^{n} (\mathcal{F}^j(x) - \mathcal{F}^j(\bar{x})) J_{5j} + J_6, \]

where

\[ J_{1j} = \int_{\partial B_2} (D_i \Gamma(x - y) - D_i \Gamma(\bar{x} - y)) \nu_j(y) \, d_y \sigma, \]

\[ J_{2j} = \int_{\partial B_2} D_i \Gamma(\bar{x} - y) \nu_j(y) \, d_y \sigma, \]

\[ J_3 = \int_{B_1(x)} D_{ij} \Gamma(x - y) (\mathcal{F}^j(x) - \mathcal{F}^j(\bar{x})) \, dy, \]

\[ J_4 = \int_{B_1(\bar{x})} D_{ij} \Gamma(\bar{x} - y) (\mathcal{F}^j(y) - \mathcal{F}^j(\bar{x})) \, dy, \]

\[ J_{5j} = \int_{B_2 \setminus B_1(x)} D_{ij} \Gamma(x - y) \, dy, \]
\[ J_6 = \int_{B_2 \setminus B_\delta(\xi)} (D_{ij} \Gamma(x - y) - D_{ij} \Gamma(x - \overline{\pi})) (\mathcal{F}^j(x) - \mathcal{F}^j(y)) \, dy. \]

(Here we set again \( \delta = |x - \overline{\pi}|, \xi = \frac{1}{2}(x - \overline{\pi}) \) and write \( B_2 = B_\delta(\xi) \cup \{B_2 \setminus B_\delta(\xi)\} \).)

We estimate these integrals by analogy with [5, pp. 58–59]:

\[ |J_{1j}| \leq |x - \overline{\pi}| \int_{\partial B_2} |DD_{ij} \Gamma(x - y)| \, dy. \]

(for some point \( \bar{x} \) between \( x \) and \( \overline{\pi} \))

\[ \leq |x - \overline{\pi}| n \omega^{-1}_n \int_{\partial B_2} |x - y|^{-n} \, dy \]

\[ \leq n^2 2^{n-1} |x - \overline{\pi}| R^{-1} \quad \text{(since } |x - y| \geq R \text{ for } y \in \partial B_2 \text{)} \]

\[ \leq n^2 2^{n-1} A(|x - \overline{\pi}|) R^{-1} \delta / A(\delta) \]

\[ \leq n^2 2^n \alpha B(\delta) / A(2R) \quad \text{(since } \delta = |x - \overline{\pi}| \leq 2R \text{ and } \delta / A(\delta) \leq 2R / A(2R) \text{ by (1.1)} \]

\[ \leq n^2 2^{n-1} A(|x - \overline{\pi}|) R^{-1} \delta / A(2R) \quad \text{(by (1.3))}. \]

Next,

\[ |J_{2j}| \leq 2^{n-1}, \]

\[ |J_{3j}| \leq \omega^{-1}_n [\mathcal{F}^j]_{A, x} \int_{B_\delta(\xi)} |x - y|^{-n} A(|x - y|) \, dy \]

\[ \leq \omega^{-1}_n [\mathcal{F}^j]_{A, x} \int_{B_{3\delta/2}(x)} |x - y|^{-n} A(|x - y|) \, dy \]

\[ = n [\mathcal{F}^j]_{A, x} \int_0^{3\delta/2} t^{-1} A(t) \, dt \]

\[ \leq (3/2)^n n [\mathcal{F}^j]_{A, x} B(\delta) \quad \text{(by (1.2)).} \]

By analogy with the estimate for \( J_3 \) we obtain

\[ |J_4| \leq (3/2)^n n [\mathcal{F}^j]_{A, \overline{\pi}} B(\delta), \quad |J_5j| \leq 2^n \quad \text{(see [5, p. 59]),} \]

and

\[ |J_6| \leq |x - \overline{\pi}| \int_{B_2 \setminus B_\delta(\xi)} |DD_{ij} \Gamma(\bar{x} - y)| \cdot [\mathcal{F}^j(\overline{\pi}) - \mathcal{F}^j(y)] \, dy \]

\[ \leq |x - \overline{\pi}| c(n) \int_{|y - \overline{\pi}| \geq \delta} |\mathcal{F}^j(\overline{\pi}) - \mathcal{F}^j(y)| \cdot |x - y|^{-n-1} \, dy \]
\[ \leq c(n)\delta[F^j]_{A,\pi} \int_{|y-\xi|\geq \delta} A(|\pi - y|)|\pi - y|^{n-1} dy \]
\[ \leq 2^{n+1}c(n)\delta[F^j]_{A,\pi} \int_{|y-\xi|\geq \delta} A((3/2)|\xi - y|)|\xi - y|^{n-1} dy \]
\[ \quad \text{(since } |\pi - y| \leq (3/2)|\xi - y| \leq 3|x - \tilde{y}|) \]
\[ \leq 2^{n+1}n\omega_n c(n)\left(\frac{3}{2}\right)^\alpha \delta[F^j]_{A,\pi} B(|x - \pi|), \forall x, \pi \in B_1. \]

Finally, from (2.10) and (2.15) it follows that \( w \in C^1(B_1) \) and the estimate (2.8) holds. Lemma 3 is proved.

**Theorem 1.** Let \( v \) be a generalized solution of equation (PE) in \( B_2^+ \) with \( G \in L_{n/(1-\alpha)}(B_2^+) \), \( F^j \in C_{0,\partial G}^0(\overline{B_2^+}) \), where \( A \) is an \( \alpha \)-function satisfying the Dini condition at zero, and let \( v = 0 \) on \( B_2 \cap \Sigma \). Then \( v \in C^1(B_1^+) \) and

\[ \|v\|_{1,B_1^+} \leq c\left(\|v\|_{0,B_2^+} + \|G\|_{n/(1-\alpha);\partial G} + \sum_{j=1}^{n} \|F^j\|_{0,A;\partial G} \right), \]

where \( c = c(n, \alpha, R, A^{-1}(2R), B(2R)) \).

Theorem 1 follows from (2.7), (2.8), representation of solutions of (PE) by means of the fundamental solution and by the same argument as in [5, 4.4–4.5] (see also [5, 8.11]).

3. Dini continuity near a smooth portion of the boundary

**Theorem 2 (cf. [5, Corollary 8.36]).** Let \( A \) be an \( \alpha \)-Dini function \((0 < \alpha < 1)\) satisfying the condition (1.5). Let \( T \subset \partial G \) be of class \( C^{1,A} \). Let \( u \in W^1(G) \) be a weak solution of the problem (DL) with \( \varphi \in C^{1,A}(\partial G) \). Suppose the coefficients of the equation in (DL) satisfy the conditions
Dini continuity for equations in nonsmooth domains

\[ a^{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad \forall x \in \overline{G}, \ \xi \in \mathbb{R}^n, \]
\[ a^{ij}, a^i, f^i \in C^{0,\delta}(\overline{G}) \quad (i, j = 1, \ldots, n), \]
\[ b^i, c \in L_\infty(G), \quad g \in L_{n/(1-\alpha)}(G). \]

Then \( u \in C^{1,\delta}(G \cup T) \) and for every \( G' \subset G \cup T, \)

\[ \|u\|_{1, B; G'} \leq c(n, T, \nu, k, d') \left( \|u\|_{0, G} + \|\varphi\|_{1, A; \partial G} \right) + \sum_{i=1}^n \|f^i\|_{0, A; G}, \]

where \( d' = \text{dist}(G', \partial G \setminus T) \) and

\[ k = \max_{i,j=1,\ldots,n} \{ |a^{ij}, a^i|_{0, A; G}, |b^i, c|_{0, G} \}. \]

Proof. We use the perturbation method. We freeze the leading coefficients \( a^{ij}(x) \) at \( x_0 \in G \cup T \) by setting, without loss of generality, \( a^{ij}(x_0) = \delta_i^j \) (see [5, Lemma 6.1]), and rewrite the equation of (DL) in the form (PE) for the function \( v(x) = u(x) - \varphi(x) \) with

\[ G(x) = g(x) - b^i(x)(D_i v + D_i \varphi) - c(x)(v(x) + \varphi(x)), \]
\[ F^i(x) = (a^{ij}(x_0) - a^{ij}(x))D_j v - a^{ij}(x)D_j \varphi - a^i(x)(v(x) + \varphi(x)) + f^i(x) \quad (i = 1, \ldots, n). \]

It is not difficult to observe that the conditions on the coefficients of the equation and on \( T \) are invariant under maps of class \( C^{1,\delta} \). Therefore after a preliminary rectification of \( T \) by means of a diffeomorphism \( \psi \in C^{1,\delta} \) it is sufficient to prove the theorem in the case \( T \subset \Sigma \). This is carried out using Theorem 1 in a standard way (see [5, Chs. 6, 8]). In this connection we use the following estimates for the functions (3.2), (3.3):

\[ |g|_{n/(1-\alpha); B_2^+} \leq |g|_{n/(1-\alpha); B_2^+} + k \left( \sum_{i=1}^n |D_i v|_{0; B_2^+} + |v|_{0; B_2^+} \right) \]
\[ + \sum_{i=1}^n |D_i \varphi|_{0; B_2^+} + |\varphi|_{0, B_2^+} \]
\[ \leq |g|_{n/(1-\alpha); B_2^+} + k \left( \sum_{i=1}^n |D_i v|_{A; B_2^+} \right) \]
\[ + c_\varepsilon |v|_{0; B_2^+} + |\varphi|_{1, B_2^+} \quad (\text{by } (1.7)), \]
\begin{equation}
(3.5) \sum_{j=1}^{n} \|F^j\|_{0,A;B_2^+} \leq nkA(2R)\|\nabla v\|_{0,A;B_2^+} + k \sum_{i=1}^{n} |D_i v|_{0,B_2^+} \nonumber \\
+ c(k)(|v|_{0,B_2^+} + \|\varphi\|_{1,A;B_2^+}) + \sum_{j=1}^{n} \|f^j\|_{0,A;B_2^+}.
\end{equation}

Taking into account once more the inequality (1.7) and the condition (1.5) that ensures the equivalence \([ A \sim [ B, from (3.4)-(3.5) we finally obtain
\begin{equation}
(3.6) \|v\|_{n/(1-\alpha);B_2^+} + \sum_{j=1}^{n} \|F^j\|_{0,A;B_2^+} \nonumber \\
\leq k(\varepsilon + nA(2R))\|v\|_{1,B;B_2^+} + c\varepsilon(k)(|v|_{0,B_2^+} + \|\varphi\|_{1,A;B_2^+}) \nonumber \\
+ \sum_{j=1}^{n} \|f^j\|_{0,A;B_2^+} + \|g\|_{n/(1-\alpha);B_2^+} \quad \text{for all } \varepsilon > 0.
\end{equation}

Since \(A\) is continuous, choosing \(\varepsilon, R > 0\) sufficiently small we obtain the desired assertion and the estimate (3.1) in a standard way from (2.16) and (3.6).

4. Dini continuity near the conical point. We consider the problem (DL) under the following assumptions:

(i) \(\partial G\) is a Dini–Lyapunov surface with conical point \(O\);
(ii) the uniform ellipticity holds:
\[ \nu \xi^2 \leq a^{ij}(x)\xi_i \xi_j \leq \mu \xi^2, \quad \forall x \in G, \ \xi \in \mathbb{R}^n, \]
where \(\nu, \mu = \text{const} > 0\) and \(a^{ij}(0) = \delta^i_j\) \((i,j = 1, \ldots, n)\);
(iii) \(a^{ij}, a^i \in C^{0,A}(G)\) \((i,j = 1, \ldots, n)\) where \(A\) is an \(\alpha\)-Dini function on \((0, d], \ \alpha \in (0, 1)\), satisfying the conditions (1.5)-(1.6) and also
\begin{equation}
(4.1) \sup_{0 < \rho \leq 1} \rho^{\lambda-1}/A(\rho) \leq \text{const}, \nonumber \\
|x| \left( \sum (\dot{g}(x))^2 \right)^{1/2} + |x|^2 |c(x)| \leq A(|x|); \nonumber \\
(iv) g \in L_{n/(1-\alpha)}(G), \ \varphi \in C^{1-A}(\partial G), \ f^j \in C^{0,A}(\partial G), \ j = 1, \ldots, n; \nonumber \\
(v) \int_{G} r^{4-n-2\lambda} H^{-1}(r) g^2(x) \, dx < \infty, \nonumber \\
\int_{G} r^{2-n-2\lambda} H^{-1}(r) \left( \sum_{j=1}^{n} |f^j|^2 + |\nabla \phi|^2 + r^{-2} \phi^2 \right) \, dx < \infty, \nonumber 
\end{equation}

where \(\mathcal{H}\) is a continuous increasing function satisfying the Dini condition at \(t = 0\).
Theorem 3. Let $u$ be a generalized solution of (DL) and suppose that assumptions (i)–(v) are satisfied. Then there exist $d > 0$ and a constant $c > 0$ independent of $u$ and depending only on parameters and norms of the given functions appearing in assumptions (i)–(v), such that

\[
|u(x)| \leq c|x|A(|x|) \left( |g|_{n/(1-\alpha);G} + \sum_{i=1}^{n} |f_i|_{0,A;G} + \|\varphi\|_{1,A;\partial G} \right)
+ \left\{ \int_{G} \left( r^{4-n-2\lambda}H^{-1}(r)g^2(x) + r^{2-n-2\lambda}H^{-1}(r) \right)
\times \sum_{i=1}^{n} |f_i(x)|^2 + r^{2-n-2\lambda}H^{-1}(r)|\nabla\Psi|^2
+ |u|^2 + |\nabla u|^2 \right\}^{1/2}, \quad \forall x \in G^d_0,
\]

\[
|\nabla u(x)| \leq cA(|x|) \left( |g|_{n/(1-\alpha);G} + \sum_{i=1}^{n} |f_i|_{0,A;G} + \|\varphi\|_{1,A;\partial G} \right)
+ \left\{ \int_{G} \left( r^{4-n-2\lambda}H^{-1}(r)g^2(x) + r^{2-n-2\lambda}H^{-1}(r) \right)
\times \sum_{i=1}^{n} |f_i(x)|^2 + r^{2-n-2\lambda}H^{-1}(r)|\nabla\Psi|^2
+ |u|^2 + |\nabla u|^2 \right\}^{1/2}, \quad \forall x \in G^d_0.
\]

Proof. We use Kondrat’ev’s method of layers: we move away from the conical point of $\varrho > 0$ and work in $G^2_{\varrho/4}$; after the change of variables $x = \varrho x'$ the layer $G^2_{\varrho/4}$ takes the position of a fixed domain $G^2_{1/4}$ with smooth boundary.

1°. We consider a solution $u$ in the domain $G^2_{0}$ with some positive $d \ll 1$; then $u$ is a weak solution in $G^2_{0}$ of the problem

\[
(DL)_{0,2d}
\begin{align*}
\frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u \\
= g(x) + \frac{\partial f^j}{\partial x_j}, \quad & x \in G^2_0,
\\
u(x) = \varphi(x), \quad & x \in F^2_0 \subset \partial G^2_0.
\end{align*}
\]

We make the change of variables $x = \varrho x'$ and set $v(x') = g^{-1}A^{-1}(\varrho)u(\varrho x')$,
\( \varphi \in (0, d) \), \( 0 < d \ll 1 \). Then \( v \) satisfies in \( G^{2}_{1/4} \) the problem

\[
\begin{align*}
\frac{\partial}{\partial x_i^j}(a^{ij}(\varphi^t)v_{x^t_j}) + \varphi a^i(\varphi^t)v + \varphi b^i(\varphi^t)v_{x^t_i} + \varphi^2 c(\varphi^t)v \\
= A^{-1}(\varphi) \sum_{j=1}^{n} \frac{\partial f^j(\varphi^t)}{\partial x^t_j} + \varphi A^{-1}(\varphi)g(\varphi^t), \quad x' \in G^{2}_{1/4},
\end{align*}
\]

\[v(x') = \varphi^{-1}A^{-1}(\varphi)\varphi(x'), \quad x' \in \Gamma^{2}_{1/4}.
\]

To solve this problem we use Theorem 2. We check its assumptions. Since under assumption (ii), \( A \) is increasing, \( \varphi \in (0, d) \) and \( 0 < d \ll 1 \), from the inequality \( \varphi^{-1}|x-y| \geq |x-y| \) for \( \varphi \in (0, d) \) it follows that

\[A(|x'-y'|) = A(\varphi^{-1}|x-y|) \geq A(|x-y|)
\]

and by (iii) we have

\[
\sum_{i,j} \|a^{ij}(\varphi)\|_{0,A;G^{2}_{0/4}} + \varphi \sum_{i} \|a^i(\varphi)\|_{0,A;G^{2}_{0/4}} \leq \sum_{i,j} \|a^{ij}\|_{0,A;G^{2}_{0/4}} + d \sum_{i} \|a^i\|_{0,A;G^{2}_{0/4}} < \infty.
\]

Further, let \( \Phi \) be a regularity preserving extension of the boundary function \( \varphi \) into a domain \( G^{2}_{\epsilon} \) for \( \epsilon > 0 \) (such an extension exists; see e.g. [5, Lemma 6.38]).

Since \( \varphi \in C^{1,A}(\partial G) \) we have

\[\|\Phi\|_{1,A;G^{2}_{\epsilon/4}} \leq c(G)\|\varphi\|_{1,A;G^{2}_{\epsilon/4}} \leq \text{const.}
\]

By definition of the norm in \( C^{1,A} \) we obtain

\[
\sup_{x,y \in G^{2}_{\epsilon/4}} \frac{|\nabla \Phi(x) - \nabla \Phi(y)|}{A(|x-y|)} \leq \|\Phi\|_{1,A;G^{2}_{\epsilon/4}} \leq c(G)\|\varphi\|_{1,A;G^{2}_{\epsilon/4}}.
\]

Now we show that by (v) and the smoothness of \( \varphi \),

\[
|\varphi(x)| \leq c|x|A(|x|), \quad |\nabla \Phi(x)| \leq cA(|x|), \quad \forall x \in G^{2}_{\epsilon/4}.
\]

Indeed, from

\[
\varphi(x) - \varphi(0) = \frac{1}{0} \frac{d}{d\tau} \Phi(\tau x) d\tau = x_i \frac{1}{0} \frac{\partial \Phi(\tau x)}{\partial (\tau x_i)} d\tau
\]

by Hölder’s inequality we have

\[
|\varphi(x) - \varphi(0)| \leq r|\nabla \Phi|.
\]
From (iv) it follows that

\begin{equation}
\int_{G_0^\varrho} (r^{2-n}|\nabla \Phi|^2 + r^{-n}|\varphi|^2) \, dx \\
= \int_{G_0^\varrho} (r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla \Phi|^2 + r^{-n-2\lambda}\mathcal{H}^{-1}(r)|\varphi|^2)(r^{2\lambda}\mathcal{H}(r)) \, dx \\
\leq \text{const} \varrho^{2\lambda}\mathcal{H}(\varrho).
\end{equation}

Since \(|\varphi(0)| \leq |\varphi(x)| + |\varphi(x) - \varphi(0)|\), by (4.6) we obtain

\(|\varphi(0)| \leq |\varphi(x)| + r|\nabla \Phi|\).

Squaring both sides, multiplying by \(r^{-n}\) and integrating over \(G_0^\varrho\) we obtain

\begin{equation}
\varphi^2(0) \int_{G_0^\varrho} r^{-n} \, dx \leq 2 \int_{G_0^\varrho} (r^{-n-2}\varphi^2(x) + r^{2-n}|\nabla \Phi|^2) \, dx < \infty
\end{equation}

by (4.7). Since

\[ \int_{G_0^\varrho} r^{-n} \, dx = \text{mes} \, \Omega \int_0^\varrho \frac{d\varrho}{\varrho} = \infty, \]

the assumption \(\varphi(0) \neq 0\) contradicts (4.8). Thus \(\varphi(0) = 0\). Then from (4.4) we have

\[ |\nabla \Phi(x) - \nabla \Phi(y)| \leq \text{const} \, A(|x - y|)\|\varphi\|_{1,A;G_0^\varrho}, \quad \forall x, y \in G_0^{2\varrho}, \]

\[ |\nabla \Phi(y)| \leq |\nabla \Phi(x) - \nabla \Phi(y)| + |\nabla \Phi(x)| \]

\[ \leq c \, A(|x - y|)\|\varphi\|_{1,A;G_0^{2\varrho}} + |\nabla \Phi(x)|. \]

Hence considering \(y\) to be fixed in \(G_0^{2\varrho}/4\) and \(x\) variable, we get

\[ |\nabla \Phi(y)|^2 \int_{G_0^{2\varrho}/4} r^{2-n} \, dx \leq 2c^2\|\varphi\|_{1,A;G_0^{2\varrho}/4} \int_{G_0^{2\varrho}/4} r^{2-n} \, dx \\
+ 2 \int_{G_0^{2\varrho}/4} r^{2-n} |\nabla \Phi(x)|^2 \, dx \]

or by (4.7),

\[ \varrho^2|\nabla \Phi(y)|^2 \leq c(\text{mes} \, \Omega, k_1)(\varrho^2A^2(\varrho) + \varrho^{2\lambda}\mathcal{H}(\varrho)), \quad \forall y \in G_0^{2\varrho}/4. \]

Hence the assumption (4.1) yields the second inequality of (4.5). Now the first inequality of (4.5) follows from (4.6) and \(\varphi(0) = 0\). Thus (4.5) is proved.
Now we obtain
\[
(4.9) \quad \varrho^{-1} \mathcal{A}^{-1}(\varrho) \|\varphi(\cdot\,t)\|_{1,\mathcal{A} G^2_{1/4}} \\
\leq c \varrho^{-1} \mathcal{A}^{-1}(\varrho) \|\varphi(\cdot\,t)\|_{1,\mathcal{A} G^2_{1/4}} \\
= c \varrho^{-1} \mathcal{A}^{-1}(\varrho) \left\{ \sup_{x' \in G^2_{1/4}} |\varphi(x')| + \sup_{x' \in G^2_{1/4}} |\nabla' \varphi(x')| \\
+ \sup_{x',y' \in G^2_{1/4}} \frac{|\nabla' \varphi(x') - \nabla' \varphi(y')|}{\mathcal{A}(|x' - y'|)} \right\} \\
\leq c_1 + c \mathcal{A}^{-1}(\varrho) \sup_{x,y \in G^2_{1/4}} \frac{\mathcal{A}(x - y)}{\mathcal{A}(g^{-1}|x - y|)} \\
= c_1 + c [\nabla \varphi]_{0,\mathcal{A} G^2_{1/4}} \mathcal{A}^{-1}(\varrho) \sup_{0 < t < 4} \frac{\mathcal{A}(t)}{\mathcal{A}(g^{-1}t)} \\
\leq \text{const, \ \forall \varrho \in (0, d),}
\]
by (4.5), since by (1.6),
\[
\sup_{0 < t < 4} \frac{\mathcal{A}(t)}{\mathcal{A}(g^{-1}t)} = \sup_{0 < \tau < 4} \frac{\mathcal{A}(\tau \varrho)}{\mathcal{A}(\tau)} \leq c \mathcal{A}(\varrho).
\]

In the same way we have
\[
(4.10) \quad \mathcal{A}^{-1}(\varrho) \|f_j\|_{0,\mathcal{A} G^2_{1/4}} \\
= \mathcal{A}^{-1}(\varrho) \left( |f_j|_{0,\mathcal{A} G^2_{1/4}} + \sup_{x,y \in G^2_{1/4}} \frac{|f_j(x) - f_j(y)|}{\mathcal{A}(g^{-1}|x - y|)} \right).
\]
Since \(f_j \in C^0, \mathcal{A}(\mathcal{G})\), we get
\[
(4.11) \quad |f_j(x) - f_j(y)| \leq \tilde{c}_j \mathcal{A}(|x - y|), \quad \forall x, y \in G^2_{0/4},
\]
\[
(4.12) \quad \int_{G^2_{0/4}} r^{2-n} |f_j(x)|^2 \, dx = \int_{G^2_{0/4}} (r^{2-n-2\lambda} H^{-1}(r)|f_j(x)|^2) (H(r)r^{2\lambda}) \, dx \\
\leq \text{const \ \varrho^{2\lambda} \mathcal{A}(\varrho)}
\]
by (v). Now fix \(y \in G^2_{0/4}\). Then
\[
|f_j(y)| \leq |f_j(x)| + |f_j(x) - f_j(y)| \leq |f_j(x)| + \tilde{c}_j \mathcal{A}(|x - y|).
\]
Hence
\[
|f_j(y)|^2 \int_{G^2_{0/4}} r^{2-n} \, dx \leq 2 \int_{G^2_{0/4}} r^{2-n} |f_j(x)|^2 \, dx + 2 \tilde{c}_j^2 \int_{G^2_{0/4}} r^{2-n} \mathcal{A}^2(|x - y|) \, dx.
\]
Calculations and (4.12) give
\[ \varrho^2 |f^j(y)|^2 \leq c(\bar{c}_j, k_1, \text{mes } \Omega)(\varrho^2 A^2(\varrho) + \varrho^{2\lambda} \mathcal{H}(\varrho)), \quad \forall y \in G^{2\varrho}_{\varrho/4}. \]

Hence by the assumption (4.1) it follows that
\[ |f^j(x)| \leq c_j A(\varrho), \quad \forall x \in G^{2\varrho}_{\varrho/4}, \ j = 1, \ldots, n. \]

Further, in the same way as in the proof of (4.9),
\[ \sup_{x,y \in G^{2\varrho}_{\varrho/4} \atop x \neq y} \frac{|f^j(x) - f^j(y)|}{A(\varrho^{-1}|x - y|)} \leq [f^j]_{0, A; G^{2\varrho}_{\varrho/4}} \sup_{0 < t < 4 \varrho} \frac{A(t)}{A(\varrho^{-1} t)} \]
\[ \leq c A(\varrho) [f^j]_{0, A; G^{2\varrho}_{\varrho/4}}. \]

Now from (4.10), (4.13) and (4.14) we obtain
\[ A^{-1}(\varrho) \sum_{j=1}^{n} |f^j|_{0, A; G^{2\varrho}_{\varrho/4}} \leq \text{const}. \]

It remains to verify the finiteness of \( |g A(\varrho)^{-1} g(\varrho \cdot)|_{n/(1-\alpha); G^{2\varrho}_{\varrho/4}} \). We have
\[ \varrho A^{-1}(\varrho) \left( \int_{G^{2\varrho}_{\varrho/4}} |g(\varrho \cdot)|^{n/(1-\alpha)} \, dx \right)^{(1-\alpha)/n} \]
\[ = \varrho^\alpha A^{-1}(\varrho) \left( \int_{G^{2\varrho}_{\varrho/4}} |g(\varrho \cdot)|^{n/(1-\alpha)} \, dx \right)^{(1-\alpha)/n} \]
\[ \leq d^\alpha A^{-1}(d) \left( \int_{G^{2\varrho}_{\varrho/4}} |g(\varrho \cdot)|^{n/(1-\alpha)} \, dx \right)^{(1-\alpha)/n} \]
\[ \leq \text{const}, \quad \forall \varrho \in (0, d), \]
by the condition (1.1). Thus the conditions of Theorem 2 are satisfied.

By this theorem we have
\[ \|v\|_{1, B; G^{1/2}_{1/4}} \]
\[ \leq c\{n, \nu, G, \max_{i,j=1,\ldots,n} (\|a^{ij}(\varrho \cdot)\|_{0, A; G^{2\varrho}_{\varrho/4}}, \|a^i(\varrho \cdot)\|_{0, A; G^{2\varrho}_{\varrho/4}}, A(2\varrho)) \}
\times \left( |v|_{0; G^{2\varrho}_{\varrho/4}} + \varrho^{-1} A^{-1}(\varrho) \|\varphi(\varrho \cdot)\|_{1, A; G^{2\varrho}_{\varrho/4}} + \varrho A^{-1}(\varrho) \|g(\varrho \cdot)\|_{n/(1-\alpha); G^{2\varrho}_{\varrho/4}} \right.
\[ + A^{-1}(\varrho) \sum_{j=1}^{n} |f^j(\varrho \cdot)|_{0, A; G^{2\varrho}_{\varrho/4}} \right), \quad \forall \varrho \in (0, d). \]
2°. To estimate $|v|_{0;G_2^2}$ we use the local estimate at the boundary of the maximum of the modulus of a solution [5, Theorem 8.25]. We check the assumptions of that theorem. To this end, set

$$z(x') = v(x') - q^{-1}A^{-1}(q)\Phi(qx')$$

and write the problem for the function $z$:

$$\begin{cases}
\frac{\partial}{\partial x_i'}(a^{ij}(qx')z_{x_j'}) + qa^i(qx')z + qb^i(qx')z_{x_i'} + q^2c(qx')z = G(x'), & x' \in G_2^1, \\
z(x') = 0, & x' \in G_1^2.
\end{cases}$$

where

\begin{align}
G(x') &\equiv qA^{-1}(q)g(qx') - A^{-1}(q)b'(qx')\Phi'(qx') \\
&- qA^{-1}(q)c(qx')\Phi(qx'), \\
F'(x') &\equiv A^{-1}(q)f'(qx') - q^{-1}A^{-1}(q)a^{ij}(qx')\Phi'(qx') \\
&- A^{-1}(q)a^i(qx')\Phi(qx') \\
&\quad (i = 1, \ldots, n).
\end{align}

First we verify the smoothness of the coefficients (see the remark at the end of [5, 8.10]). Let $q > n$. We have

\begin{align}
(4.17) \quad &\int_{G_1^2} |qa^i(qx')|^q dx' = \frac{q^q}{a^i(x)} \int_{G_2^1} |a^i(x)|^q dx \\
&\leq c_2(G)d^q\|a^i\|_{L^q_0, A; G}, \quad \forall q \in (0, d).
\end{align}

By (iii) we also obtain

\begin{align}
(4.20) \quad &\int_{G_1^2} |qb^i(qx')|^q dx' = \frac{q^q}{b^i(x)} \int_{G_2^1} |b^i(x)|^q dx \leq 4^q q^{-n} \int_{G_2^1} |rb^i(x)|^q dx \\
&\leq 4^q q^{-n} \int_{G_2^1} A^q(r) dx \leq 2^{n+2q} \int_{G_2^1} r^{-n} A^q(r) dx \\
&= 2^{n+2q} \text{mes} \Omega \int_{G_2^1} \frac{A^q(r)}{r^d} dr \\
&\leq 2^{n+2q} \text{mes} \Omega \cdot A^{-1}(2d) \int_0^{2d} \frac{A(r)}{r^d} dr,
\end{align}

\begin{align}
(4.21) \quad &\int_{G_1^2} |q^2c(qx')|^{q/2} dx' = \frac{q^{q-n}}{c(x)} \int_{G_2^1} |c(x)|^{q/2} dx
\end{align}
for \( q > n \) and all \( \theta \in (0, d) \).

In the same way from (4.17) we get

\[
\varrho A^{-1}(\theta) \left| G(x') \right| \frac{1}{\theta^{1/2} G_{\theta/4}^2} \\
= \varrho A^{-1}(\theta) \left( \int_{G_{\theta/4}^2} \varrho^{-n} \left( |g(x)|^{q/2} \\
+ \left( \sum_{i=1}^{n} |\beta_i(x)| \right)^{q/2} \left| \nabla \Phi \right|^{q/2} + |c(x)|^{q/2} |\Phi(x)|^{q/2} \right) dx \right)^{2/q}.
\]

By (iv) setting \( q = n/(1 - \alpha) > n \) and applying Hölder’s inequality we obtain

\[
\varrho A^{-1}(\theta) \left( \int_{G_{\theta/4}^2} \varrho^{-n} |g(x)|^{q/2} dx \right)^{2/q} \\
\leq c \varrho^\alpha A^{-1}(\theta) \left( \int_{G_{\theta/4}^2} \varrho^{-n/2} |g(x)|^{q/2} dx \right)^{2/q} \\
\leq c \varrho^\alpha A^{-1}(\theta) \left| G_{\theta/4}^2 \right| (\text{mes } \Omega \ln 8)^{1/q} \\
\leq c(d, \alpha, q, \text{mes } \Omega, A(d)) \left| G_{\theta/4}^2 \right|,
\]

since by (1.1), \( \varrho^\alpha A^{-1}(\theta) \leq d^\alpha A^{-1}(d) \) for all \( \varrho \in (0, d) \). Similarly,

\[
\varrho A^{-1}(\theta) \left( \int_{G_{\theta/4}^2} r^{-n} \\
\times \left( \left( \sum_{i=1}^{n} |\beta_i(x)| \right)^{q/2} \left| \nabla \Phi \right|^{q/2} + |c(x)|^{q/2} |\Phi(x)|^{q/2} \right) dx \right)^{2/q} \\
\leq c(\text{mes } \Omega)^{2/q} \left\| \varphi \right\|_{1, A; G_{\theta/4}^2} A^{(q-2)/q}(\theta) \int_{\theta/4}^{2\theta} \frac{A(r)}{r} dr.
\]
From (4.22)–(4.24) we obtain
\[ \|G(q)\|_{q/2;G^{2}\varepsilon/4} \]
\[ \leq \text{const} \left( q, \alpha, d, \mu \Omega, \mathcal{A}(d), \int_{\varepsilon/4}^{2\varepsilon} \frac{\mathcal{A}(r)}{r} dr \right) \]
\[ \times (\Phi_{q,G^{2}\varepsilon/4} + \|\varphi\|_{1,A;I^{2}\varepsilon/4}, \quad q = n/(1 - \alpha) > n. \]

Finally, in the same way from (4.18) we have
\[ \sum_{i=1}^{n} \int_{G^{2}\varepsilon/4} |F^{i}(x')|^q dx' \]
\[ \leq c\left( q, G, \max_{j=1,\ldots,n} \left\{ \sum_{i=1}^{n} \|a_{ij}\|_{0,A,G}^{q}, \sum_{i=1}^{n} \|a_{i}\|_{0,A,G}^{q} \right\} \right) \]
\[ \times \int_{G^{2}\varepsilon/4} r^{-n} A^{-q}(r) \left( \sum_{i=1}^{n} |f^{i}(x)|^q + |\nabla \Phi|^q + |\Phi|^q \right) dx. \]

It follows from (4.5) as \( \varepsilon \to 0 \) that \( |\nabla \Phi(0)| = 0 \). Therefore
\[ |\nabla \Phi(x)| = |\nabla \Phi(x) - \nabla \Phi(0)| \leq A(|x|)|\varphi|_{1,A;I^{2}\varepsilon/4}, \quad \forall x \in G^{2}\varepsilon/4, \]
and hence
\[ |\Phi(x)| \leq r|\nabla \Phi| \leq |x|A(|x|)|\varphi|_{1,A;I^{2}\varepsilon/4}, \quad \forall x \in G^{2}\varepsilon/4. \]

Similarly it follows from (4.13) as \( \varepsilon \to 0 \) that \( f^{j}(0) = 0 \) for \( j = 1,\ldots,n \). Therefore we have for all \( x \in G^{2}\varepsilon/4, \)
\[ |f^{j}(x)| = |f^{j}(x) - f^{j}(0)| \leq A(r)|f^{j}|_{0,A;G^{2}\varepsilon/4}. \]

Consequently, estimating the right side of (4.26) and taking into account the inequalities obtained, we have
\[ \sum_{i=1}^{n} \int_{G^{2}\varepsilon/4} |F^{i}(x')|^{q} dx' \leq c\left( q, G, \max_{j=1,\ldots,n} \left\{ \sum_{i=1}^{n} \|a_{ij}\|_{0,A,G}^{q}, \sum_{i=1}^{n} \|a_{i}\|_{0,A,G}^{q} \right\} \right) \]
\[ \times \text{mes } \Omega \cdot \left( \sum_{i=1}^{n} |f^{i}|_{0,A;G^{2}\varepsilon/4} + \|\varphi\|_{1,A;I^{2}\varepsilon/4} \right). \]

So all conditions of [5, Theorem 8.25] are satisfied. By this theorem we get
Our assumptions guarantee that the integral on the right side is finite. Since

\begin{equation}
\sup_{x' \in G_{1/2}^1} |z(x')|
\end{equation}

\begin{equation}
\leq c \left( \|z\|_{L^2;G_{1/4}^2} + \|G\|_{n/(2(1-\alpha));G_{1/4}^2} + \sum_{i=1}^{n} \|F^i\|_{n/(1-\alpha);G_{1/4}^2} \right)
\end{equation}

\begin{equation}
\leq c \left( \|z\|_{L^2;G_{1/4}^2} + \|G\|_{n/(1-\alpha);G_{1/4}^2} + \sum_{i=1}^{n} \|F^i\|_{0,A;G_{e/4}^2} + \|\varphi\|_{1,A;G_{e/4}^2} \right).
\end{equation}

Setting \( w(x) = u(x) - \varphi(x) \) we have for \( w(x) \) the problem

\begin{equation}
(\text{DL})_{0,2d}
\begin{cases}
\frac{\partial}{\partial x_i} (a^{ij}(x)w_{x_j} + a^i(x)w) + b^i(x)w_{x_i} + c(x)w = G(x) + \frac{\partial F^j}{\partial x_j}, \quad x \in G_{0}^{2d}, \\
w(x) = 0, \quad x \in I_{0}^{2d} \subset \partial G_{0}^{2d},
\end{cases}
\end{equation}

where

\begin{align*}
G(x) &= g(x) - b^i(x)\Phi_{x_i} - c(x)\Phi(x), \\
F^i(x) &= f^i(x) - a^{ij}(x)\Phi_{x_j} - a^i(x)\Phi(x).
\end{align*}

Moreover, by assumptions (i), (ii),

\begin{equation}
|a^{ij}(x) - \delta^i_j| \leq \|a^{ij}\|_{0,A;G,A(|x|)}, \quad x \in G.
\end{equation}

By [6, Theorem 1] there is a constant \( c > 0 \) independent of \( w, G, F^i \) such that

\begin{equation}
\int_{G_0^\theta} r^{2-n-2\lambda} |\nabla w|^2 \, dx \leq c_0^{2\lambda} \int_{G_0^{2d}} \left\{ |w(x)|^2 + |\nabla w|^2 + G^2(x) + \sum_{i=1}^{n} |F^i(x)|^2 \right. \right. \\
+ r^{4-n-2\lambda} \mathcal{H}^{-1}(r)G^2(x) + r^{2-n-2\lambda} \\
\left. \left. \times \mathcal{H}^{-1}(r) \sum_{i=1}^{n} |F^i(x)|^2 \right\} \, dx, \quad \forall \theta \in (0,d).
\end{equation}

Our assumptions guarantee that the integral on the right side is finite. Since \( z(x') = g^{-1}A^{-1}(\varphi)w(\varphi x') \) we obtain from (4.29),

\begin{equation}
\int \left|\nabla' z\right|^2 \, dx' \leq 2^{n-2} \varrho^{-2}A^{-2}(\varphi) \int \left|\nabla' w\right|^2 \, dx \\
\leq c_\varphi^{2\lambda-2}A^{-2}(\varphi) \int |w|^2 + |\nabla w|^2 + G^2(x)
\end{equation}
\begin{align*}
+ \sum_{i=1}^{n} |F_i(x)|^2 + r^{4-n-2\lambda}H^{-1}(r)G^2(x) \\
+ r^{2-n-2\lambda}H^{-1}(r) \sum_{i=1}^{n} |F_i(x)|^2 \right\} dx.
\end{align*}

By assumptions (i)–(iv) we have
\begin{equation} \label{eq:4.31}
|G(x)|^2 \leq c\{ |g|^2 + A^2(r)(r^{-2}|\nabla \Phi|^2 + r^{-4}\phi^2) \},
\end{equation}
\begin{equation} \label{eq:4.32}
\sum_{i=1}^{n} |F_i(x)|^2 \leq c\left\{ \sum_{i=1}^{n} |f_i(x)|^2 \\
+ \max_{i,j=1,\ldots,n}(\|a^i_{ij}\|_{0,A;G},\|a^j_{ij}\|_{0,A;G})(|\nabla \Phi|^2 + \phi^2) \right\}.
\end{equation}

Applying now the Friedrichs inequality and taking into account (4.1), we obtain from (4.30), (4.31),
\begin{equation} \label{eq:4.33}
|v|_{0;G^{1/4}} \leq c_0 \left\{ \int_G |w|^2 + |\nabla w|^2 + g^2(x) \\
+ \sum_{i=1}^{n} |f_i(x)|^2 + |\nabla \phi|^2 + \phi^2 + r^{4-n-2\lambda}H^{-1}(r)g^2(x) \\
+ r^{2-n-2\lambda}H^{-1}(r) \sum_{i=1}^{n} |f_i(x)|^2 \\
+ r^{2-n-2\lambda}H^{-1}(r)|\nabla \phi|^2 + r^{-2}A^2(r)|\nabla \phi|^2 \right\} dx
\end{equation}
by assumptions (iii)–(v). By the definition of \( z(x') \), inequalities (4.28), (4.32) and assumptions (i)–(v) we finally obtain
\begin{equation} \label{eq:4.34}
|v|_{0;G^{1/4}} \leq |z|_{0;G^{1/4}} + \theta^{-1}A^{-1}(\theta)|\phi(x')|_{0;G^{1/4}}
\end{equation}
\[
\leq c \left( |g|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,A;G} + \|\varphi\|_{1,A;\partial G}
\right.
\]
\[
+ \left\{ \int_{G} \left( |w|^2 + |\nabla w|^2 + r^{4-n-2\lambda} \mathcal{H}^{-1}(r) g^2(x) \right.
\right.
\]
\[
+ r^{2-n-2\lambda} \mathcal{H}^{-1}(r) \sum_{i=1}^{n} |f^i(x)|^2
\]
\[
\left. \left. + r^{2-n-2\lambda} \mathcal{H}^{-1}(r) |\nabla \varphi|^2 \right) dx \right\}^{1/2}.
\]

3°. Returning to the variables \(x, u(x)\), we now obtain from inequalities (4.16), (4.33),
\[
(4.34) \quad \varrho^{-1} A^{-1}(\varrho) \sup_{x \in G_\varrho/2} |u(x)| + A^{-1}(\varrho) \sup_{x \in G_\varrho/2} |\nabla u(x)|
\]
\[
+ \sup_{x,y \in G_\varrho/2, x \neq y} \frac{|\nabla u(x) - \nabla u(y)|}{A(\varrho) B(|x - y|)}
\]
\[
\leq c \left( |g|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,A;G} + \|\varphi\|_{1,A;\partial G}
\right.
\]
\[
+ \left\{ \int_{G} \left( |w|^2 + |\nabla w|^2 + r^{4-n-2\lambda} \mathcal{H}^{-1}(r) g^2(x) \right.
\right.
\]
\[
+ r^{2-n-2\lambda} \mathcal{H}^{-1}(r) \sum_{i=1}^{n} |f^i(x)|^2
\]
\[
\left. \left. + r^{2-n-2\lambda} \mathcal{H}^{-1}(r) |\nabla \varphi|^2 \right) dx \right\}^{1/2}.
\]

Setting \(|x| = 2\varrho/3\) we deduce from (4.34) the validity of (4.2), (4.3). This completes the proof of Theorem 3.

Remark. As an example of \(A\) that satisfies all the conditions of Theorem 3, besides the function \(r^\alpha\), one may take \(A(r) = r^\alpha \ln(1/r)\), provided \(\lambda \geq 1 + \alpha\). In the case of \(A(r) = r^\alpha\) the result of [1] follows from Theorem 3 for a single equation and the estimate (4.2) coincides with [6, (10)].

5. Global regularity and solvability

Theorem 4. Let \(A\) be an \(\alpha\)-Dini function \((0 < \alpha < 1)\) that satisfies the conditions (1.5), (1.6), (4.1). Let \(\Omega \setminus \{O\}\) be a domain of class \(C^{1,A}\), and
\( O \in \partial G \) be a conical point of \( G \). Suppose that assumptions (i)--(iv) are valid and

\[
\int_G (c(x)\eta - a^i(x)D_i\eta) \, dx \leq 0, \quad \forall \eta \geq 0, \ \eta \in C_0^1(G).
\]

Then the generalized problem (DL) has a unique solution \( u \in C^{1,A}(G) \) and we have the estimate

\[
\|u\|_{1,A;G} \leq c \left( |g|^{(1-\alpha)}G + \sum_{i=1}^n \|f^i\|_{0,A;G} + \|\varphi\|_{1,A;\partial G} \right.
\]

\[
+ \left\{ \int_G \left( r^{4-n-2\lambda}H^{-1}(r)g^2(x) + r^{2-n-2\lambda}H^{-1}(r)\sum_{i=1}^n |f^i(x)|^2 + \frac{r^{2-n-2\lambda}H^{-1}(r)|\nabla \varphi|^2}{2} \right) \right\}^{1/2} \right).
\]

**Proof.** The inequality (4.34) implies that

\[
|\nabla u(x) - \nabla u(y)| \leq cB(|x-y|) \left( |g|^{(1-\alpha)}G + \sum_{i=1}^n \|f^i\|_{0,A;G} + \|\varphi\|_{1,A;\partial G} \right.
\]

\[
+ \left\{ \int_G \left( |u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}H^{-1}(r)g^2(x) + r^{2-n-2\lambda}H^{-1}(r)|\nabla \varphi|^2 \right) \right\}^{1/2} \right).
\]

for all \( x, y \in G_{\nu/2}^d \) and all \( \nu \in (0, d) \).

From (4.34), (5.2) we now infer that \( u \in C^{1,B}(\overline{G_0^d}) \). Indeed, let \( x, y \in G_0^d \) and \( \nu \in (0, d) \). If \( x, y \in G_{\nu/2}^d \) then (5.2) holds. If \( |x-y| > |\nu| = |x| \) then by (4.34) we obtain

\[
\frac{|\nabla u(x) - \nabla u(y)|}{B(|x-y|)B(|x|)} \leq 2cA(|x|)B^{-1}(|x|) \left( |g|^{(1-\alpha)}G + \|\varphi\|_{1,A;\partial G} \right.
\]

\[
+ \sum_{i=1}^n \|f^i\|_{0,A;G} + \left\{ \int_G \left( |u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}H^{-1}(r)g^2(x) + r^{2-n-2\lambda}H^{-1}(r)|\nabla \varphi|^2 \right) \right\}^{1/2} \right).
\]
in view of (1.3). Because of the condition (1.5) for the equivalence of $A$ and $B$, we derive $u \in C^{1,A}(\overline{G_d})$ and the estimate

$$\|u\|_{1,A;\overline{G}} \leq c \left( |u|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,A;G} + \|\varphi\|_{1,A;\partial G} + \left\{ \int_G \left( |u|^2 + |
abla u|^2 + r^{4-n-2\lambda}H^{-1}(r)g^2(x) \right. \\
+ r^{2-n-2\lambda}H^{-1}(r)\sum_{i=1}^{n} |f^i(x)|^2 \right. \\
+ r^{2-n-2\lambda}H^{-1}(r)|\nabla \Phi|^2 dx \right\}^{1/2} \right),$$

following from the above arguments.

By means of a partition of unity, from the bounds (3.1) of Theorem 2 and (5.3) we derive

$$\|u\|_{1,A;G} \leq c \left( |u|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,A;G} + \|\varphi\|_{1,A;\partial G} + |u|_{0,G} + \left\{ \int_G \left( |u|^2 + |
abla u|^2 + r^{4-n-2\lambda}H^{-1}(r)g^2(x) \right. \\
+ r^{2-n-2\lambda}H^{-1}(r)\sum_{i=1}^{n} |f^i(x)|^2 \right. \\
+ r^{2-n-2\lambda}H^{-1}(r)|\nabla \Phi|^2 dx \right\}^{1/2} \right).$$

By the assumption (vi) that guarantees the uniqueness of the solution for the problem (DL), we have the bound [5, Corollary 8.7]

$$\int_G \left( |u|^2 + |
abla u|^2 \right) dx \leq C \int_G \left( g^2 + \sum_{i=1}^{n} |f^i|^2 + |\nabla \Phi|^2 + \phi^2 \right) dx,$$
which together with the global boundedness of weak solutions [5, Theorem 8.16], and the bound (5.4), leads to the desired estimate (5.1).

Finally, the global estimate (5.1) leads to the assertion on the unique solvability in $C^{1,A} (\overline{G})$. This is proved by an approximation argument using the relevant propositions from [8] in the same way as in [5, Theorem 8.34].

**Remark.** The conclusion of Theorem 4 is best possible. This is shown for $\mathcal{A}(r) = r^\alpha$, $\lambda \geq 1 + \alpha$, $\alpha \in (0, 1)$, in [6] (see also examples in [2]).

**References**


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