ANNALES
POLONICI MATHEMATICI
LXIX.1 (1998)

On the method of lines for a non-linear heat equation with functional dependence

by H. Leszczyński (Gdańsk)

Abstract. We consider a heat equation with a non-linear right-hand side which depends on certain Volterra-type functionals. We study the problem of existence and convergence for the method of lines by means of semi-discrete inverse formulae.

1. Introduction. Let $a > 0$, $\tau_0, \tau_1, \ldots, \tau_n \in \mathbb{R}_+$ and $[-\tau, \tau] = [-\tau_1, \tau_1] \times \ldots \times [-\tau_n, \tau_n]$. Define $E = [0, a] \times \mathbb{R}^n$, $E_0 = [-\tau_0, 0] \times \mathbb{R}^n$, $E^+ = (0, a] \times \mathbb{R}^n$ and $B = [-\tau_0, 0] \times [-\tau, \tau]$. If $u : E_0 \cup E \to \mathbb{R}$ and $(t, x) \in E$, then we define the Hale-type functional $u_{(t,x)} : B \to \mathbb{R}$ by $u_{(t,x)}(s, y) = u(t + s, x + y)$ for $(s, y) \in B$. Because we also take into account the functional dependence on the gradient $D_x u = (D_{x_1} u, \ldots, D_{x_n} u)$, we write

$$(D_x u)_{(t,x)} = ((D_{x_1} u)_{(t,x)}, \ldots, (D_{x_n} u)_{(t,x)}).$$

Denote by Δ the Laplacian, that is, $\Delta = D_{x_1 x_1} + \ldots + D_{x_n x_n}$. Define

$$\Omega := E \times C(B, \mathbb{R}) \times C(B, \mathbb{R}^n).$$

Given $f : \Omega \to \mathbb{R}$ and $\phi : E_0 \to \mathbb{R}$, we consider the Cauchy problem

$$D_t u(t, x) = \Delta u(t, x) + f(t, x, u_{(t,x)}, (D_x u)_{(t,x)}), \tag{1}$$

$$u(t, x) = \phi(t, x) \qquad \text{for } (t, x) \in E_0. \tag{2}$$

Two specific examples of (1) are equations with Volterra integral and delayed (deviated) dependence:

$$D_t u(t, x) = \Delta u(t, x) + \widetilde{f}\Big(t, x, \int_B u(t + s, x + y)\, dy\, ds, \int_{\tau_0}^{0} D_x u(t + s, x)\, ds\Big) \qquad \text{(integral dependence)},$$

$$D_t u(t, x) = \Delta u(t, x) + \widetilde{f}\big(t, x, u\big(\tfrac{1}{2} t, x\big), D_x u(t - \tau_0, x + \tau)\big) \qquad \text{(delays)},$$

where $\widetilde{f} : E \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

1991 *Mathematics Subject Classification*: Primary 65M20; Secondary 35K05, 35R10.
Key words and phrases: method of lines, stability, consistency.

Define

$$f[u](t,x) := f(t,x,u_{(t,x)},(D_x u)_{(t,x)}).$$

The following system of integral-functional equations is equivalent to the differential-functional problem (1), (2):

$$u(t,x) = \mathcal{L}_0[u](t,x), \tag{3}$$

$$D_x u(t,x) = \mathcal{L}'[u](t,x), \tag{4}$$

for $(t,x) \in E_0 \cup E$, where $\mathcal{L}' = (\mathcal{L}_1,\dots,\mathcal{L}_n)$, and

$$\begin{aligned}
\mathcal{L}_0[u](t,x) :=& \int_{\mathbb{R}^n} H(t,x-y)\phi(0,y)\,dy \\
&+ \int_0^t \int_{\mathbb{R}^n} H(t-s,x-y) f[u](s,y)\,dy\,ds, \\
\mathcal{L}_i[u](t,x) :=& \int_{\mathbb{R}^n} H(t,x-y) D_{y_i}\phi(0,y)\,dy \\
&+ \int_0^t \int_{\mathbb{R}^n} D_{x_i} H(t-s,x-y) f[u](s,y)\,dy\,ds,
\end{aligned}$$

for $(t,x) \in E^+ := (0,a] \times \mathbb{R}^n$ $(i = 1,\dots,n)$, where

$$H(t,x) = \begin{cases} \dfrac{1}{(2\sqrt{\pi t})^n} \exp\left(-\dfrac{\|x\|^2}{4t}\right) & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ 0 & \text{for } (t,x) \in (-\infty,0] \times \mathbb{R}^n, \end{cases}$$

is the Green function $H : \mathbb{R}^{1+n} \to \mathbb{R}$, and

$$\mathcal{L}_0[u](t,x) := \phi(t,x), \quad \mathcal{L}_i[u](t,x) := D_{x_i}\phi(t,x) \quad (i = 1,\dots,n),$$

for $(t,x) \in E_0$.

We intend to formulate a semi-discrete problem which corresponds to (1), (2), and next to find a semi-discrete version of (3), (4). This seems to be a new approach to the error analysis of the method of lines. We study so-called $C^{0,1}$ solutions to (1), (2), that is, $u \in C(E_0 \cup E, \mathbb{R})$ satisfying (2) in E_0 and (3) in E with the continuous gradient $D_x u$. However, it will occur that consistency requirements lead to classical, even sufficiently regular, solutions.

In [12] we obtained some existence results by means of the Banach contraction principle and discussed the question of their continuous differentiability in the set E^+, getting $C^{0,1}$ and classical solutions to the Cauchy problem. Fundamental notions, ideas and existence results in the theory of parabolic equations can be found in [8, 10].

Theoretical search for some iterative methods, esp. monotone iterative techniques (cf. [3, 9]), reveals certain advantages of Chaplygin's method (see

[2, 4]). This method can guarantee the second-order convergence under some natural assumptions. However, one would like to avoid using it because of the inevitable necessity to solve quasi-linear Cauchy problems with functional dependence at each stage of the iterative method. There are well known and frequently applied finite difference methods (FDM) (cf. [5, 6, 11, 13–16]). Explicit FDMs for parabolic equations require a very specific condition on the time and space steps, whereas there appear large non-linear systems at each stage of any implicit FDM.

Enormous progress in parallel software has pointed to semi-discrete methods such as the Rothe method and the method of lines. The latter gives a large-scale structured system of non-linear ordinary differential equations which can be solved by means of an effective Runge–Kutta method (see [17]). Since the early sixties, the method of lines has become very attractive for many mathematicians and engineers. We draw the reader's attention to some references on parabolic differential and differential-functional equations such as [7, 18, 19]. Because these papers develop a sort of maximum principle as a main tool of their convergence proof, the present paper essentially differs from them. Namely our method can be extended not only to parabolic equations with functional dependence at spatial derivatives, but also to strongly coupled systems of parabolic differential-functional equations. The assumptions in the present paper correspond to those in the existence results.

Now, we introduce a natural mesh and a family of semi-discrete schemes. First, we take the steps $h = (h_1, \dots, h_n) \in \mathbb{R}^n_+$. Define $x^\beta = h \star \beta := (h_1\beta_1, \dots, h_n\beta_n)$ for $\beta \in \mathbb{Z}^n$. Let

$$Z_h = \{x^\beta \mid \beta \in \mathbb{Z}^n\}.$$

Define some discrete sets associated with the sets E_0 and E:

$$E_h^0 = [-\tau_0, 0] \times Z_h, \quad E_h = [0, a] \times Z_h, \quad \widetilde{E}_h = E_h^0 \cup E_h, \quad E_h^+ = (0, a] \times Z_h.$$

If $u : E_0 \cup E \to \mathbb{R}$, then we write $u^\beta(t) = u(t, x^\beta)$ for $(t, x^\beta) \in \widetilde{E}_h$.

We also need some further notation. Let $e_l = (\delta_{1,l}, \dots, \delta_{n,l})$, where $\delta_{j,l}$ is the Kronecker symbol. Define the difference operators

$$\varDelta_h = (\varDelta_{1,h}, \dots, \varDelta_{n,h}), \quad \varDelta_h^2 = \varDelta_{1,h}^2 + \dots + \varDelta_{n,h}^2,$$

as follows:

$$\begin{aligned} &\varDelta_{l,h} u^\beta(t) = (2h_l)^{-1}(u^{\beta+e_l}(t) - u^{\beta-e_l}(t)), \\ &\varDelta_{l,h}^2 u^\beta(t) = h_l^{-2}(u^{\beta+e_l}(t) - 2u^\beta(t) + u^{\beta-e_l}(t)) \qquad (l = 1, \dots, n), \end{aligned}$$

for $(t, x^\beta) \in \widetilde{E}_h$.

Suppose that we are given the interpolation operators $T_h : C(\widetilde{E}_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ and $T'_h = (T_h^1, \dots, T_h^n) : C(\widetilde{E}_h, \mathbb{R}^n) \to C(E_0 \cup E, \mathbb{R}^n)$, where

$$(T_h u)(t,x) = \sum_{x^\beta \in Z_h} u^\beta(t) p_h^\beta(x),$$
$$(T'_h U)(t,x) = ((T_h u_1)(t,x), \dots, (T_h u_n)(t,x))$$

for $(t,x) \in E_0 \cup E$ and $u \in C(E_0 \cup E, \mathbb{R})$, $U = (u_1, \dots, u_n) \in C(E_0 \cup E, \mathbb{R}^n)$, where $p_h^\beta(t) \in C(\mathbb{R}^n, \mathbb{R})$ for $h \in \mathbb{R}_+^n$ and $\beta \in \mathbb{Z}^n$.

We formulate the *method of lines*:

(5) $$\frac{d}{dt} u^\beta(t) = \Delta_h^2 u^\beta(t) + f(t, x^\beta, (T_h u)_{(t,x^\beta)}, (T'_h \Delta u)_{(t,x^\beta)}) \quad \text{on } E_h^+,$$

(6) $$u^\beta(t) = \overline{\phi}^\beta(t) \quad \text{on } E_h^0,$$

where $\overline{\phi} : \widetilde{E}_h \to \mathbb{R}$ is a discrete perturbed counterpart of the function ϕ.

Finite difference schemes for parabolic problems were considered in [1, 11, 16]. The convergence theorems were proved there by means of difference inequalities or a sort of maximum principle. In [13] we prove a convergence theorem for finite difference schemes that approximate unbounded solutions to parabolic problems with differential-functional dependence by means of a comparison lemma, which was possible in absence of functionals acting on partial derivatives. Nevertheless, there were some technical problems. The present paper shows new ways to solve parabolic equations with more complex functional dependence, such as delay and Volterra type integrals, esp. acting also on partial derivatives.

Define the set $\mathcal{F}_p^\beta$ for $p = 1, 2, \dots$ and $\beta \in \mathbb{Z}^n$ as follows:

$$\sigma \in \mathcal{F}_p^\beta \quad \text{if } \sigma = (\sigma_1, \dots, \sigma_p) \text{ and } \sigma_1, \dots, \sigma_p \in \{\pm e_k \mid k = 1, \dots, n\}.$$

Set

$$C_{0,h}^\beta = \begin{cases} 1 & \text{for } \beta = 0, \\ 0 & \text{for } \beta \neq 0, \end{cases} \qquad C_{p,h}^\beta = \sum_{\sigma \in \mathcal{F}_p^\beta} h_{\sigma_1}^{-2} \dots h_{\sigma_p}^{-2} \qquad (p = 1, 2, \dots),$$

and

$$[h]_2 = \sum_{j=1}^n h_j^{-2} \qquad \text{for } h \in \mathbb{R}_+^n.$$

Define

(7), (8) $$H_p^\beta(t) = \begin{cases} C_{p,h}^\beta \exp(-2tn[h]_2) t^p / p! & \text{for } t > 0,\ p = 0, 1, \dots, \\ 0 & \text{for } t \le 0,\ p = 0, 1, \dots, \end{cases}$$

If $u : \widetilde{E}_h \to \mathbb{R}$ satisfies the equation

(9) $$\frac{d}{dt} u^\beta(t) = \Delta_h^2 u^\beta(t) + g^\beta(t) \quad \text{on } E_h^+,$$

where $g : E_h^+ \to \mathbb{R}$, then we can rewrite (9) as follows:

$$\frac{d}{dt}(u^\beta(t)\exp(2tn[h]_2)) = \Big\{\sum_{l=1}^{n} h_l^{-2}(u^{\beta+e_l}(t) + u^{\beta-e_l}(t)) + g^\beta(t)\Big\}\exp(2tn[h]_2)$$

and next integrating from 0 to t we obtain

$$u^\beta(t) = u^\beta(0)\exp(-2tn[h]_2) + \int_0^t \Big\{\sum_{l=1}^{n} h_l^{-2}(u^{\beta+e_l}(s) + u^{\beta-e_l}(s)) + g^\beta(s)\Big\}\exp(-2(t-s)n[h]_2)\,ds,$$

or, in explicit form,

$$u^\beta(t) = \sum_{p=0}^{\infty}\sum_{\eta\in\mathbb{Z}^n} H_p^{\beta-\eta}(t)u^\eta(0) + \sum_{p=0}^{\infty}\sum_{\eta\in\mathbb{Z}^n}\int_0^t H_p^{\beta-\eta}(t-s)g^\eta(s)\,ds \quad \text{on } E_h,$$

where $H_p^\beta(t)$ are defined by (7)–(8).

If we take $g^\beta(t) := f(t, x^\beta, \ldots)$ in (9), then we get the following discrete inverse formula for the scheme (5), (6):

$$u^\beta(t) = \sum_{p=0}^{\infty}\sum_{\eta\in\mathbb{Z}^n} H_p^{\beta-\eta}(t)\overline{\phi}^\eta(0) + \sum_{p=0}^{\infty}\sum_{\eta\in\mathbb{Z}^n}\int_0^t H_p^{\beta-\eta}(t-s)f^\eta[u](s)ds \quad \text{on } E_h,$$

where

$$f^\beta[u](t) = f(t, x^\beta, (T_h u)_{(t,x^\beta)}, (T'_h \Delta u)_{(t,x^\beta)}).$$

Define the residual expression

$$\Theta^\beta[u;h](t) = \frac{d}{dt}u^\beta(t) - \Delta_h^2 u^\beta(t) - f^\beta[u](t) \tag{10}$$

for $(t, x^\beta) \in E_h^+$ and $u \in C(\widetilde{E}_h, \mathbb{R})$ differentiable with respect to t. Observe that $f^\beta[u](t)$ is a semi-discrete version of the Nemytskiĭ's operator.

2. Existence and convergence results. We use the symbol C_B to indicate classes of bounded continuous functions. Write

$$\mathcal{X}[\phi] := \{(u,U) \in C_B(E_0 \cup E, \mathbb{R}^1) \times C_B(E_0 \cup E, \mathbb{R}^n) \mid u(t,x) = \phi(t,x),\ U(t,x) = D_x\phi(t,x) \text{ for } (t,x) \in E_0\}.$$

Denote by $\|\cdot\|_0$ the supremum norm.

Assumption 1. Suppose that:

1) $\phi, \overline{\phi} \in C_B(E_0, \mathbb{R})$, $D_x\phi, D_x\overline{\phi} \in C_B(E_0, \mathbb{R})$, $f(\cdot,\cdot,0,0) \in C_B(E, \mathbb{R})$.

2) There are $L_1, L_2 \in \mathbb{R}_+$ such that
$$|f(t,x,w,W) - f(t,x,\overline{w},\overline{W})| \le L_1\|w-\overline{w}\|_0 + L_2\|W-\overline{W}\|_0$$
for $(t,x,w,W), (t,x,\overline{w},\overline{W}) \in \Omega$ (recall that $\Omega = E \times C(B,\mathbb{R}) \times C(B,\mathbb{R}^n)$).

Assumption 2. For every step h and for all $x^\beta \in Z_h$ we have $p_h^\beta \in C(\mathbb{R}^n, \mathbb{R})$, and there are $\lambda \ge 1$ and $M \in \mathbb{R}_+$ such that
$$p_h^\beta(x) = 0 \quad \text{for } \|x - x^\beta\|_0 > h\lambda,$$
$$\sum_{x^\beta \in Z_h} p_h^\beta(x) = 1 \quad \text{and} \quad \|p_h^\beta(x)\|_0 \le M \quad \text{on } \mathbb{R}^n.$$

Assumption 3. Suppose that $v \in C_B(E_0 \cup E, \mathbb{R})$ is a classical solution to the problem (1), (2) such that $(v, D_x v) \in \mathcal{X}[\phi]$ and $D_{x_i x_i} v \in C_B(E_0 \cup E, \mathbb{R})$ for $i = 1, \ldots, n$. Assume that the function $D_x v$ is uniformly continuous with respect to x in $E_0 \cup E$, and $D_{x_i x_i} v$ is uniformly continuous with respect to x_i for $i = 1, \ldots, n$. Denote their moduli of continuity by σ_x and σ_{xx}, respectively.

First, we formulate a lemma on global estimates for a system of integral equations.

Lemma 1. *Suppose that* ε_0, ε_1, P, Q, L, L', Q_j, S_j, $S_j' \in \mathbb{R}_+$ *for* $j = 1, \ldots, n$. *If*
$$L'S < 1 \quad \textit{for} \quad S := \max_{j=1,\ldots,n} \left(\frac{S_j}{Q} + 2\frac{S_j'}{Q_j^3}\right),$$
and $W_0, W_j \in C([0,a], \mathbb{R}_+)$, *where*
$$W_0(t) = \varepsilon_0 + \int_0^t \{P + LW_0(s) + L'W_1(s)\}\,ds, \tag{11}$$
$$W_j(t) = \varepsilon_1 + \int_0^t \{S_j e^{-Q(t-s)} + S_j'(t-s)^2 e^{-Q_j(t-s)}\} \times \{P + LW_0(s) + L'W_1(s)\}\,ds \quad (j = 1, \ldots, n), \tag{12}$$
then
$$W_0(t) \le \varepsilon_0 + \frac{\widetilde{\varepsilon}}{L}\left\{\exp\left(\frac{tL}{1-L'S}\right) - 1\right\},$$
$$W_j(t) \le \varepsilon_1 + \widetilde{\varepsilon}\left\{S_j \frac{\exp\left(\frac{tL}{1-L'S}\right) - \exp(-Qt)}{(1-L'S)Q + L} + \frac{2S_j' \exp\left(\frac{tL}{1-L'S}\right)}{\left(Q_j + \frac{L}{1-L'S}\right)^3}\right\} \quad (j = 1, \ldots, n),$$
where $\widetilde{\varepsilon} = P + L\varepsilon_0 + L'\varepsilon_1$.

P r o o f. Define

$$\widetilde{W}(t) = P + LW_0(t) + L' \max_{j=1,\ldots,n} W_j(t) \qquad (t \in [0,a]).$$

Then $\widetilde{W}$ satisfies the integral inequality

$$\begin{aligned}\widetilde{W}(t) &\le \widetilde{\varepsilon} + \max_{j=1,\ldots,n} \Big\{ \int_0^t \widetilde{W}(s)\{L + L'(S_j e^{-Q(t-s)} + S'_j (t-s)^2 e^{-Q_j(t-s)})\}\, ds \Big\} \\ &\le \widetilde{\varepsilon} + \int_0^t \widetilde{W}(s) L\, ds \\ &\quad + \max_{j=1,\ldots,n} \Big\{ \int_0^t \widetilde{W}(t) L'(S_j e^{-Q(t-s)} + S'_j (t-s)^2 e^{-Q_j(t-s)})\, ds \Big\} \\ &\le \widetilde{\varepsilon} + \int_0^t \widetilde{W}(s) L\, ds + \widetilde{W}(t) L' \max_{j=1,\ldots,n} \Big\{ \int_0^\infty \frac{S_j}{Q} e^{-\xi}\, d\xi + \int_0^\infty \frac{S'_j}{Q_j^3} \xi^2 e^{-\xi}\, d\xi \Big\}.\end{aligned}$$

Hence (by the Gronwall lemma) we get

$$\widetilde{W}(t) \le \frac{\widetilde{\varepsilon}}{1 - L'S} \exp\Big(\frac{tL}{1 - L'S}\Big). \tag{13}$$

From (11)–(13), we get the assertions of our lemma. ∎

Lemma 2 (Existence). *If Assumptions* 1–2 *are satisfied*, *then there exists a unique bounded and continuous solution to* (5), (6).

P r o o f. The right-hand side of the system satisfies the Lipschitz condition in the Banach space of all bounded continuous functions. Apply the Banach contraction principle. ∎

We say that a particular method of lines (e.g. (5)) is *stable* if small perturbations of its right-hand side and initial data result in a correspondingly small variation of its solutions. The method (5) will be called *consistent* with the differential equation if, given a regular solution v to (1), we get

$$|f^\beta[v](t) - f[v](t, x^\beta)| \le C'_h$$

for all $(t, x^\beta) \in E_h^+$, where $\mathbb{R}_+ \ni C'_h \to 0$ as $\|h\|_0 \to 0$.

Given $K \in \mathbb{R}_+$ and $\overline{h} \in \mathbb{R}_+^n$, define

$$I_K(\overline{h}) = \{h \in \mathbb{R}_+^n \mid h \le \overline{h},\ h_j/h_l \le K\ (j,l = 1,\ldots,n)\}.$$

Lemma 3 (Stability). *Suppose that* $u, v \in C_B(\widetilde{E}_h, \mathbb{R})$ *and there are* $C_h, \overline{C}_h, P_h \in \mathbb{R}_+$ *such that*

$$|v^\beta(t) - u^\beta(t)| \le C_h \to 0 \quad \text{on } E_h^0,$$

$$\|\Delta_h v^\beta(t) - \Delta_h u^\beta(t)\|_0 \le \overline{C}_h \to 0 \quad \text{on } E_h^0,$$

$$\Theta^\beta[u;h](t)=0,\qquad |\Theta^\beta[v;h](t)|\le P_h\to 0\qquad \textit{on } E_h^+.$$

If Assumptions 1–2 *are satisfied and* $K\ge 1$, $\overline{h}\in\mathbb{R}^n_+$, *then*

$$\sup_{\beta\in\mathbb{Z}^n}\|(v^\beta-u^\beta,\Delta_h(v-u)^\beta)\|_0\to 0\quad \textit{as } \|h\|_0\to 0,\ h\in I_K(\overline{h}).$$

P r o o f. Set $\omega^\beta(t):=v^\beta(t)-u^\beta(t)$. Then

$$|\Theta^\beta[v;h](t)|\le P_h\quad\text{on } E_h^+,$$
$$|\omega^\beta(t)|\le C_h,\quad \|\Delta_h\omega^\beta(t)\|_0\le \overline{C}_h\quad\text{on } E_h^0,$$

and

$$\begin{aligned}\omega^\beta(t)&=\sum_{p=0}^\infty\sum_{\eta\in\mathbb{Z}^n}H_p^{\beta-\eta}(t)\omega^\eta(0)\\&\quad+\sum_{p=0}^\infty\sum_{\eta\in\mathbb{Z}^n}\int_0^t H_p^{\beta-\eta}(t-s)\\&\quad\times\{(f^\eta[v](s)-f^\eta[u](s))+\Theta^\eta[v;h](s)\}\,ds,\end{aligned}\tag{14}$$

$$\begin{aligned}\Delta_{j,h}\omega^\beta(t)&=\sum_{p=0}^\infty\sum_{\eta\in\mathbb{Z}^n}\Delta_{j,h}H_p^{\beta-\eta}(t)\omega^\eta(0)\\&\quad+\sum_{p=0}^\infty\sum_{\eta\in\mathbb{Z}^n}\int_0^t \Delta_{j,h}H_p^{\beta-\eta}(t-s)\\&\quad\times\{(f^\eta[v](s)-f^\eta[u](s))+\Theta^\eta[v;h](s)\}\,ds\end{aligned}\tag{15}$$

for $(t,x^\beta)\in E_h$ and $j=1,\ldots,n$. Observe that

$$\sum_{p=0}^\infty\sum_{\eta\in\mathbb{Z}^n}H_p^{\beta-\eta}(t)=1\quad\text{on}\quad E_h^+,\tag{16}$$

$$\begin{aligned}&\sum_{p=0}^\infty\sum_{\eta\in\mathbb{Z}^n}|\Delta_{j,h}H_p^{\beta-\eta}(t)|\\&\quad\le h_j^{-1}\sum_{p=0}^\infty\exp(-2t[h]_2)\frac{t^p}{p!}\\&\qquad\times\Big\{2^{n-1}([h]_2)^p+p(p-1)(2[h]_2-h_j^{-2})^{p-2}\Big|h_j^{-4}-\sum_{l=1,\,l\ne j}^n h_l^{-4}\Big|\Big\}\\&\quad=h_j^{-1}\Big\{2^{n-1}\exp(-t[h]_2)+t^2\exp(-th_j^{-2})\Big|h_j^{-4}-\sum_{l=1,\,l\ne j}^n h_l^{-4}\Big|\Big\}\\&\qquad\qquad\text{on } E_h^+\ (j=1,\ldots,n).\end{aligned}\tag{17}$$

Define $M_\lambda = M(2\lambda+1)^n$ and

$$\|\gamma\|_0(t) = \sup_{s\le t,\, x^\eta\in Z_h} \|\gamma^\eta(s)\|_0 \quad \text{for } \gamma\in C_B(\widetilde{E}_h,\mathbb{R}^k).$$

It follows from Assumption 2 that

$$\begin{aligned}\|(T_h\omega)_{(t,x^\beta)}\|_0 &= \sup_{(s,y)\in B} |(T_h\omega)(s+t,y+x^\beta)| \\ &\le \sum_{x^\beta\in Z_h} |\omega^\beta(s)|\,|p_h^\eta(s+t,y+x^\beta)| \\ &\le M(2\lambda+1)^n \sup_{s,\,x^\eta\in Z_h} |\omega^\eta(s)| = M_\lambda\|\omega\|_0(t).\end{aligned}$$

Hence

$$\|(T_h\Delta_{j,h}\omega)_{(t,x^\beta)}\|_0 \le M_\lambda\|\Delta_h\omega\|_0(t) \quad (j=1,\ldots,n),$$

for $(t,x^\beta)\in\widetilde{E}_h$. Taking supremum on both sides of (14), (15) and applying (16), (18), we obtain the integral estimates

$$\|\omega\|_0(t) \le C_h + \int_0^t \{P_h + M_\lambda L_1\|\omega\|_0(s) + M_\lambda L_2\|\Delta_h\omega\|_0(s)\}\,ds,$$

$$\begin{aligned}\|\Delta_{j,h}\omega\|_0(t) \le \overline{C}_h + \int_0^t & h_j^{-1}\{P_h + M_\lambda L_1\|\omega\|_0(s) + M_\lambda L_2\|\Delta_h\omega\|_0(s)\} \\ &\times \Big\{2^{n-1}\exp(-(t-s)[h]_2) \\ &+ t^2\exp(-(t-s)h_j^{-2})\Big|h_j^{-4} - \sum_{l=1,\,l\ne j}^n h_l^{-4}\Big|\Big\}\,ds \\ &\qquad\qquad (j=1,\ldots,n),\end{aligned}$$

for $0\le t\le a$. Take W_0, W_j $(j=1,\ldots,n)$ given by (11), (12) with

$$\begin{aligned}&L = M_\lambda L_1,\quad L' = M_\lambda L_2,\quad Q=[h]_2,\\ &P=P_h,\quad \varepsilon_0 = C_h,\quad \varepsilon_1 = \overline{C}_h,\\ &Q_j = h_j^{-2},\quad S_j = h_j^{-1}2^{n-1},\\ &S_j' = h_j^{-1}\Big|h_j^{-4} - \sum_{l=1,\,l\ne j}^n h_l^{-4}\Big| \quad (j=1,\ldots,n).\end{aligned}$$

There exists $\overline{h}'\in I_K(\overline{h})$ such that

$$\theta_K(h) := M_\lambda L_2 \max_{j=1,\ldots,n}\Big\{\frac{h_j^{-1}2^{n-1}}{[h]_2} + 2h_j^5\Big|h_j^{-4} - \sum_{l=1,\,l\ne j}^n h_l^{-4}\Big|\Big\} < 1$$

for each $h \in I_K(\overline{h})$, $h \le \overline{h}'$. In view of Lemma 1, there is a global estimate for the comparison problem, thus we have the estimates

$$(18)\quad \|\omega\|_0(t) \le W_0(t) \le C_h + \frac{\widetilde{C}_h}{M_\lambda L_1}\Big\{\exp\Big(\frac{tM_\lambda L_1}{1-M_\lambda L_2\theta_K(h)}\Big) - 1\Big\},$$

$$(19)\quad \|\Delta_{j,h}\omega\|_0(t)$$
$$\le W_j(t) \le \overline{C}_h + \widetilde{C}_h\Big\{h_j^{-1}2^{n-1}\frac{\exp\big(\frac{tM_\lambda L_1}{1-M_\lambda L_2\theta_K(h)}\big) - \exp(-t[h]_2)}{(1-M_\lambda L_2\theta_K(h))[h]_2 + M_\lambda L_1}$$
$$+ \frac{2h_j^{-1}\exp\big(\frac{tM_\lambda L_1}{1-M_\lambda L_2\theta_K(h)}\big)}{\big(h_j^{-2} + \frac{M_\lambda L_1}{1-M_\lambda L_2\theta_K(h)}\big)^3}\Big|h_j^{-4} - \sum_{l=1,\, l\ne j}^{n} h_l^{-4}\Big|\Big\} \quad (j = 1,\ldots,n),$$

for $h \in I_K(\overline{h}')$, where

$$(20)\qquad \widetilde{C}_h = P_h + L_1C_h + L_2\overline{C}_h.$$

It is clear that $\|W_j\|_0 \to 0$ as $\|h\|_0 \to 0$, $h \in I_K(\overline{h}')$ $(j = 0,\ldots,n)$, which completes the proof. ∎

LEMMA 4 (Consistency). *If Assumptions* 1–3 *are satisfied*, *then the scheme* (5) *is consistent with the differential-functional problem.*

P r o o f. Take v as in Assumption 3. Suppose that σ_x and σ_{xx} are the moduli of continuity for D_xv and $D_{x_ix_i}v$ respectively. Let $(t,x^\beta) \in E_h^+$ and $(s,y) \in B$. Then we can use the Taylor expansion at $x = x^\beta + y$ to derive

$$(T_hv)_{(t,x^\beta)}(s,y) - v(t+s,x)$$
$$= \sum_{x^\eta\in Z_h} p_h^\eta(x)(v^\eta(t+s) - v(t+s,x))$$
$$= \sum_{x^\eta\in Z_h} p_h^\eta(x)\int_0^1 D_xv(t+s,\zeta x^\eta + (1-\zeta)x)\circ(x^\eta - x)\,d\zeta,$$

where $z\circ\overline{z}$ denotes the scalar product in $\mathbb{R}^n$, and

$$(T_h\Delta_{j,h}v)_{(t,x^\beta)}(s,y) - D_{x_j}v(t+s,x)$$
$$= \sum_{x^\eta\in Z_h} p_h^\eta(x)\{[(\Delta_{j,h}v)^\eta(t+s) - D_{x_j}v(t+s,x^\eta)]$$
$$+ [D_{x_j}v(t+s,x^\eta) - D_{x_j}v(t+s,x)]\}$$
$$= \sum_{x^\eta\in Z_h} p_h^\eta(x)\Big\{[D_{x_j}v(t+s,x^\eta) - D_{x_j}v(t+s,x)]$$

$$+\frac{1}{2h_j}\int_0^1 (1-\zeta)[D_{x_jx_j}v(t+s,x^\eta+\zeta h_je_j)h_j^2$$
$$-D_{x_jx_j}v(t+s,x^\eta-\zeta h_je_j)h_j^2]\,d\zeta\Big\}$$

for $j=1,\dots,n$. Thus, we get

$$\|(T_hv)_{(t,x^\beta)}-v_{(t,x^\beta)}\|_0\le M_\lambda\|D_xv\|_0n\lambda\|h\|_0,$$
$$\|(T_h'\Delta_hv)_{(t,x^\beta)}-(D_xv)_{(t,x^\beta)}\|_0$$
$$\le M_\lambda\Big\{\sigma_x(\lambda\|h\|_0)+\frac{1}{2}\int_0^1(1-\zeta)\sigma_{xx}(2\|h\|_0)\|h\|_0\,d\zeta\Big\}.$$

Finally, we obtain

$$|f^\beta[v](t)-f[v](t,x^\beta)|$$
$$\le M_\lambda\Big\{L_1\|D_xv\|_0n\lambda\|h\|_0+L_2\Big[\sigma_x(\lambda\|h\|_0)+\sigma_{xx}(2\|h\|_0)\frac{\|h\|_0}{4}\Big]\Big\}$$

and

$$|\Delta_{j,h}^2v^\beta(t)-D_{x_jx_j}v(t,x^\beta)|\le\sigma_{xx}(h_j)\qquad(j=1,\dots,n).$$

These estimates complete the proof. ∎

THEOREM 1 (Convergence result). *Suppose that Assumptions* 1–3 *are satisfied*, *and there are* $C_h,\overline{C}_h\in\mathbb{R}_+$ *such that*

$$|\phi^\beta(t)-\overline{\phi}^\beta(t)|\le C_h\to0\qquad\text{on } E_h^0,$$
$$\|\Delta_h\phi^\beta(t)-\Delta_h\overline{\phi}^\beta(t)\|_0\le\overline{C}_h\to0\qquad\text{on } E_h^0.$$

Let $K\ge1$, $\overline{h}\in I_K(\overline{h})$ *and* $\theta_K(\overline{h})<1$. *If* $u\in C_B(\widetilde{E}_h,\mathbb{R})$ *is a solution to* (5), (6), *then*

$$\sup_{\beta\in\mathbb{Z}^n}\|(v^\beta-u^\beta,\Delta_h(v-u)^\beta)\|_0\to0\qquad\text{as } \|h\|_0\to0,\ h\in I_K(\overline{h}). \tag{21}$$

Proof. The existence of bounded solutions to (5), (6) is a consequence of Lemma 2. It is obvious that $C_h\to0$ and $\overline{C}_h\to0$ as $\|h\|_0\to0$. In view of Lemma 4, we can define

$$P_h=\sigma_{xx}(\|h\|_0)+M_\lambda\Big\{L_1\|D_xv\|_0n\lambda\|h\|_0+L_2\Big[\sigma_x(\lambda\|h\|_0)+\sigma_{xx}(2\|h\|_0)\frac{\|h\|_0}{4}\Big]\Big\}. \tag{22}$$

Then P_h and $\widetilde{C}_h$, defined by (20), tend to 0 as $\|h\|_0\to0$. Assertion (21) follows immediately from estimates (18), (19) in the proof of Lemma 3 and from the evident fact that $\|D_xv-\Delta_hv\|_0\to0$ as $\|h\|_0\to0$. ∎

The right-hand sides of estimates (18), (19), show that the convergence rate depends on C_h, $\overline{C}_h$, P_h. We formulate a higher-order convergence statement.

COROLLARY 1. *Suppose that the assumptions of Theorem* 1 *are satisfied, and*

1) $C_h/\|h\|_0^2$ *and* $\overline{C}_h/\|h\|_0^2$ *are uniformly bounded.*
2) *For every* $x \in \mathbb{R}^n$ *and for every* $h \in \mathbb{R}_+^n$, *we have*

$$\sum_{x^\beta \in Z_h} p_h^\beta(x)(x^\beta - x) = 0$$

3) *There are bounded and continuous derivatives* $D_{xxx}v$ *in* $E_0 \cup E$ *and* $D_{x_j x_j x_j x_j}v$ *in* $\overline{E}$ $(j = 1, \ldots, n)$.

Then the second-order convergence of the method of lines holds true.

Proof. We verify the estimates of the two terms which appear in the proof of Lemma 4, that is,

$$\begin{aligned}
&\Big| \sum_{x^\eta \in Z_h} p_h^\eta(x)\{D_{x_j}v(t+s, x^\eta) - D_{x_j}v(t+s, x)\}\Big| \\
&\quad = \Big| D_{x_j x}v(t+s, x) \circ (x^\eta - x) \\
&\qquad + \int_0^1 (D_{x_j x}v(t+s, x + \zeta(x^\eta - x)) - D_{x_j x}v(t+s, x)) \circ (x^\eta - x)\, d\zeta \Big| \\
&\quad \le M_\lambda \|D_{xxx}v\|_0 (\lambda \|h\|_0)^2,
\end{aligned}$$

where $z \circ x$ is the scalar product of the vectors z and x, and

$$|D_{x_j}v(t, x^\beta) - \varDelta_{j,h}v^\beta(t)| \le \frac{h_j^2}{6}\|D_{x_j x_j x_j}v\|_0 \qquad (j = 1, \ldots, n).$$

The second-order approximation of the Laplacian by its difference counterpart is standard. These estimates are crucial for getting the second-order consistency statement, hence the same convergence rate. ∎

REMARK 1. The stability and convergence results of the present paper extend, in a non-trivial way, to strongly coupled systems of differential-functional equations

$$\begin{aligned}
D_t u_k(t, x) &= \sum_{j,l=1}^n a_{jl}^{(k)} D_{x_j x_l} u_k(t, x) \\
&\qquad + f^{(k)}(t, x, u(t, x), u_{(t,x)}, D_x u(t, x), (D_x)_{(t,x)}) \qquad (k = 1, \ldots, m), \\
u_k(t, x) &= \phi_k(t, x) \qquad \text{on } E_0 \ (k = 1, \ldots, m),
\end{aligned}$$

and their further generalization:

$$D_t u_k(t,x) = \sum_{j,l=1}^{n} a_{jl}^{(k)} D_{x_j x_l} u_k(t,x)$$
$$+ f^{(k)}(t,x,u(t,x),V_{(t,x)}u, D_x u(t,x), V_{(t,x)}(D_x u)) \quad (k=1,\ldots,m),$$
$$u_k(t,x) = \phi_k(t,x) \quad \text{on } E_0 \ (k=1,\ldots,m),$$

where $u=(u_1,\ldots,u_m) : E_0 \cup E \to \mathbb{R}^m$, $\phi_k : E_0 \to \mathbb{R}$ and the real coefficients $a_{jl}^{(k)}$ are such that the matrices $A^{(k)} = [a_{jl}^{(k)}]_{j,l=1,\ldots,n}$ are positive and symmetric. The functionals $V_{(t,x)}u$ are some generalizations of the Hale-type functionals $u_{(t,x)}$.

Acknowledgements. We express our gratitude to the referee for his invaluable remarks. This work was supported by the KBN grant 2 P03A 022 13.

References

[1] P. Besala, *Finite difference approximation to the Cauchy problem for non-linear parabolic differential equations*, Ann. Polon. Math. 46 (1985), 19–26.

[2] S. Brzychczy, *Chaplygin's method for a system of nonlinear parabolic differential-functional equations*, Differentsial'nye Uravneniya 22 (1986), 705–708 (in Russian).

[3] L. Byszewski, *Monotone iterative method for a system of nonlocal initial-boundary parabolic problems*, J. Math. Anal. Appl. 177 (1993), 445–458.

[4] Z. Kamont, *On the Chaplygin method for partial differential-functional equations of the first order*, Ann. Polon. Math. 38 (1980), 27–46.

[5] Z. Kamont and H. Leszczyński, *Stability of difference equations generated by parabolic differential-functional problems*, Rend. Mat. 16 (1996), 265–287.

[6] —, —, *Numerical solutions to the Darboux problem with the functional dependence*, Georgian Math. J. (1997).

[7] Z. Kamont and S. Zacharek, *The line method for parabolic differential-functional equations with initial boundary conditions of the Dirichlet type*, Atti Sem. Mat. Fis. Univ. Modena 35 (1987), 249–262.

[8] M. Krzyżański, *Partial Differential Equations of Second Order*, PWN, Warszawa, 1971.

[9] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman Adv. Publ. Program, Pitman, Boston, 1985.

[10] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967 (in Russian); English transl.: Transl. Math. Monographs 23, Amer. Math. Soc., Providence, R.I., 1968.

[11] H. Leszczyński, *Convergence of one-step difference methods for nonlinear parabolic differential-functional systems with initial boundary conditions of Dirichlet type*, Comment. Math. Prace Mat. 30 (1991), 357–375.

[12] H. Leszczyński, *A new existence result for a non-linear heat equation with functional dependence*, ibid. 37 (1997), 155–181.
[13] —, *General finite difference approximation to the Cauchy problem for non-linear parabolic differential-functional equations*, Ann. Polon. Math. 53 (1991), 15–28.
[14] —, *Convergence results for unbounded solutions of first order non-linear differential-functional equations*, ibid. 64 (1996), 1–16.
[15] —, *Discrete approximations to the Cauchy problem for hyperbolic differential-functional systems in the Schauder canonic form*, Zh. Vychisl. Mat. Mat. Fiz. 34 (1994), 185–200 (in Russian); English transl.: Comput. Math. Math. Phys. 34 (1994), 151–164.
[16] M. Malec et A. Schiaffino, *Méthode aux différences finies pour une équation non linéaire différentielle fonctionnelle du type parabolique avec une condition initiale de Cauchy*, Boll. Un. Mat. Ital. B (7) 1 (1987), 99–109.
[17] L. F. Shampine, *ODE solvers and the method of lines*, Numer. Methods Partial Differential Equations 10 (1994), 739–755.
[18] A. Voigt, *Line method approximation to the Cauchy problem for nonlinear differential equations*, Numer. Math. 23 (1974), 23–36.
[19] —, *The method of lines for nonlinear parabolic differential equations with mixed derivatives*, ibid. 32 (1979), 197–207.

Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: hleszcz@ksinet.univ.gda.pl.

Reçu par la Rédaction le 14.4.1997
Révisé le 17.11.1997