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Totally real minimal submanifolds in a quaternion projective space

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Abstract. We prove some pinching theorems with respect to the scalar curvature of 4-dimensional conformally flat (concircularly flat, quasi-conformally flat) totally real minimal submanifolds in $QP^4(c)$.

1. Introduction. A *quaternion Kähler manifold* [3] is defined as a $4n$-dimensional Riemannian manifold whose holonomy group is a subgroup of $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$. A *quaternion projective space* $QP^n(c)$ is a quaternion manifold with constant quaternion sectional curvature $c \geq 0$.

Let M be an n-dimensional Riemannian manifold and $J : M \to QP^n(c)$ an isometric immersion. If each tangent 2-subspace of M is mapped by J into a totally real plane of $QP^n(c)$, then M is called a *totally real submanifold* of $QP^n(c)$. Funabashi [2], Chen and Houh [1] and Shen [4] studied this class of submanifolds and got many interesting curvature pinching theorems. The purpose of this paper is to give some pinching theorems with respect to the scalar curvature of 4-dimensional conformally flat (concircularly flat, quasi-conformally flat) totally real minimal submanifolds in $QP^4(c)$.

2. Preliminaries. We give here a quick review of basic formulas for totally real submanifolds in a quaternion Kähler manifold; for details see [1].

Let (N, g) be a $4m$-dimensional quaternion Kähler manifold with quaternion structure I, J and K satisfying

$$IJ = K, \quad JK = I, \quad KI = J, \quad I^2 = J^2 = K^2 = -1.$$

For a unit vector X on N, let $Q(X)$ denote the 4-plane spanned by X, IX, JX and KX, which is called the *quaternion-section* determined by X. Any 2-plane in a quaternion-section is called the *quaternion-section-plane*,

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whose sectional curvature is called the *quaternion sectional curvature*. For any two vectors X and Y on N, if $Q(X)$ and $Q(Y)$ are mutually orthogonal, the 2-plane spanned by X and Y is called a *totally real plane* of N. It is well known that (N,g) has constant quaternion curvature c if and only if the curvature $\overline{R}$ of N is of the following form:

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}(&g(Y,Z)X - g(X,Z)Y + g(IY,Z)IX - g(IX,Z)IY\\ &+ 2g(X,IY)IZ + g(JY,Z)JX - g(JX,Z)JY\\ &+ 2g(X,JY)JZ + g(KY,Z)KX\\ &- g(KX,Z)KY + 2g(X,KY)KZ).\end{aligned}$$

Let M be an n-dimensional Riemannian manifold and $J : M \to N$ an isometric immersion. If each tangent 2-plane of M is mapped by J into a totally real plane in N, then M is called a *totally real submanifold* of N.

In the following, let $QP^n(c)$ denote a $4n$-dimensional quaternion projective space with constant quaternion sectional curvature $c \geq 0$. Let M be an n-dimensional totally real submanifold in $QP^n(c)$ with $n \geq 2$. We choose a local field of orthonormal frames in $QP^n(c)$:

$$e_1,\dots,e_n;\ e_{I(1)} = Ie_1,\dots,e_{I(n)} = Ie_n;\ e_{J(1)} = Je_1,\dots,e_{K(n)} = Ke_n$$

in such a way that, restricted to M, $e_1,\dots,e_n$ are tangent to M. We will use the following convention on the range of indices unless otherwise stated:

$$\begin{aligned}A,B,C,\dots &= 1,\dots,n,I(1),\dots,I(n),J(1),\dots,K(n);\\ i,j,k,\dots &= 1,\dots,n;\\ u,v,\dots &= I(1),\dots,K(n);\\ \phi &= I, J \text{ or } K.\end{aligned}$$

Let ω^A and ω^A_B be the dual frame field and the connection forms with respect to the frame field chosen above. Then the structure equations of $QP^n(c)$ are

$$\begin{aligned}d\omega^A &= -\sum \omega^A_B \wedge \omega^B, \quad \omega^A_B + \omega^B_A = 0,\\ d\omega^A_B &= -\sum \omega^A_C \wedge \omega^C_B + \frac{1}{2}\sum \overline{R}_{ABCD}\omega^C \wedge \omega^D.\end{aligned}$$

Restricting these forms to M, we have

$$\omega^u = 0, \quad \omega^u_i = \sum h^u_{ij}\omega^j, \quad h^u_{ij} = h^u_{ji}, \quad h^{\phi(i)}_{jk} = h^{\phi(j)}_{ki} = h^{\phi(k)}_{ij}. \tag{1}$$

The second fundamental form σ of M in $QP^n(c)$ is defined as

$$\sigma = \sum h^u_{ij}\omega^i \otimes \omega^j \otimes e_u. \tag{2}$$

Its length square is $\|\sigma\|^2 = \sum (h^u_{ij})^2$.

If M is minimal in $QP^n(c)$, i.e., trace $\sigma = 0$, we have

$$\varrho = \frac{c}{4}n(n-1) - \|\sigma\|^2 \tag{3}$$

where ϱ is the scalar curvature of M.

Let A_u and Δ denote the $(n \times n)$-matrix (h^u_{ij}) and the Laplacian on M, respectively. We have the following formula [1]:

$$\begin{aligned}\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum \mathrm{tr}(A_uA_v - A_vA_u)^2 \\ - \sum(\mathrm{tr}\, A_uA_v)^2 + \frac{c}{4}(n+1)\|\sigma\|^2.\end{aligned} \tag{4}$$

Since $\sum \mathrm{tr}(A_uA_v - A_vA_u)^2 = -\sum(\sum_m(h^u_{km}h^v_{lm} - h^v_{km}h^u_{lm}))^2$, this together with the equation of Gauss implies

$$\sum \mathrm{tr}(A_uA_v - A_vA_u)^2 = -\|R\|^2 + c\varrho - \frac{n-1}{8}nc^2. \tag{5}$$

Similarly, we have

$$\sum(\mathrm{tr}\, A_uA_v)^2 = \|S\|^2 - \frac{n-1}{2}c\varrho + n\left(\frac{n-1}{4}c\right)^2 \tag{6}$$

where S is the Ricci tensor of M.

Combining (3)–(6), we obtain

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|S\|^2 + \frac{n+1}{4}c\varrho. \tag{7}$$

3. Conformally flat totally real minimal submanifolds. Suppose M is an n-dimensional compact oriented totally real minimal submanifold in $QP^n(c)$. If M is conformally flat, then its conformal curvature tensor C [6] satisfies

$$\begin{aligned}C(X,Y,Z.W) = R(X,Y,Z,W) - \varrho(g(X,W)g(Y,Z) \\ - g(Y,W)g(X,Z))/(n(n-1)) \\ - (g(X,W)g(Y,Z) - g(Y,W)G(X,Z) \\ + g(Y,Z)G(X,W) - g(X,Z)G(Y,W))/(n-2) = 0\end{aligned} \tag{8}$$

where $G(X,Y) = S(X,Y) - \varrho g(X,Y)/n$. From (8) we have

$$\|R\|^2 = \frac{4}{n-2}\|S\|^2 - \frac{2\varrho^2}{(n-1)(n-2)}. \tag{9}$$

Taking the integrals of the both sides of (7) and using (9), by the Green theorem, we have

$$\int_M \|\nabla'\sigma\|^2\, dV = \int_M \left(\frac{n+2}{n-2}\|S\|^2 - \frac{n+1}{4}c\varrho - \frac{2\varrho^2}{(n-1)(n-2)}\right) dV. \tag{10}$$

On the other hand, by the Gauss–Bonnet theorem, when $n = 4$, the Euler number $\chi(M)$ of M is given by

$$\chi(M) = \frac{1}{32\pi^2} \int_M (\|R\|^2 - 4\|S\|^2 + \varrho^2)\, dV. \tag{11}$$

From (9)–(11) we get, when $n = 4$,

$$48\pi^2\chi(M) + \int_M \|\nabla'\sigma\|^2\, dV = \int_M \left(\frac{2}{3}\varrho^2 - \frac{5}{4}c\varrho\right) dV. \tag{12}$$

This yields the following theorem.

Theorem A. *Let M be a* 4*-dimensional compact oriented conformally flat totally real minimal submanifold in $QP^4(c)$. If M has nonnegative Euler number and the scalar curvature ϱ of M is between* 0 *and* $15c/8$, *then ϱ is* 0 *or* $15c/8$.

4. Concircularly flat totally real minimal submanifolds. Suppose M is an n-dimensional compact oriented totally real minimal submanifold in $QP^n(c)$. If M is concircularly flat, then its concircular curvature tensor B [5] satisfies

$$\begin{aligned} B(X,Y,Z.W) = R(X,Y,Z,W) - \varrho(g(X,W)g(Y,Z) \\ - g(Y,W)g(X,Z))/(n(n-1)) = 0. \end{aligned} \tag{13}$$

From (13) we have

$$\|R\|^2 = 2\varrho^2/(n(n-1)). \tag{14}$$

Since M is compact and oriented, from (7), (14) and the Green theorem we can obtain

$$\int_M \|\nabla'\sigma\|^2\, dV = \int_M \left(\|S\|^2 + \frac{2\varrho^2}{n(n-1)} - \frac{n+1}{4}c\varrho\right) dV. \tag{15}$$

When $n = 4$, from (11), (14) and (15) we have

$$32\pi^2\chi(M) + 4\int_M \|\nabla'\sigma\|^2\, dV = \int_M \frac{11}{6}\varrho\left(\varrho - \frac{30}{11}c\right) dV. \tag{16}$$

When $\chi(M)$ is nonnegative, we obtain the following theorem.

Theorem B. *Let M be a* 4*-dimensional compact oriented concircularly flat totally real minimal submanifold in $QP^4(c)$. If M has nonnegative Euler number and the scalar curvature $\varrho \geq 0$, then M is totally geodesic.*

5. Quasi-conformally flat totally real minimal submanifolds. Let M be an n-dimensional compact oriented totally real minimal submanifold in $QP^n(c)$. If M is quasi-conformally flat, then its quasi-conformal curvature tensor H [6] satisfies

$$H(X,Y,Z.W) = B(X,Y,Z,W) - \mu(g(X,W)G(Y,Z) - g(Y,W)G(X,Z) + g(Y,Z)G(X,W) - g(X,Z)G(Y,W)) = 0 \tag{17}$$

where $G(X,Y) = S(X,Y) - \varrho g(X,Y)/n = 0$ and μ is a constant. From (17) we have

$$\|R\|^2 = 4\mu^2(n-2)\|S\|^2 + \left(\frac{2}{n-1} - 4\mu^2(n-2)\right)\varrho^2/n. \tag{18}$$

From (7), (18) and the Green theorem we get

$$\int_M \|\nabla'\sigma\|^2\,dV = \int_M \left((4\mu^2(n-2)+1)\|S\|^2 - 4\mu^2(n-2) - \frac{2}{n-1}\frac{\varrho^2}{n} - (n+1)\frac{c}{4}\varrho\right) dV. \tag{19}$$

When $n=4$, from (11), (18) and (19) we have

$$32(1+8\mu^2)\pi^2\chi(M) + 4(1-2\mu^2)\int_M \|\nabla'\sigma\|^2\,dV = \int_M \left(\varrho - \frac{1-2\mu^2}{11-12\mu^2}30c\right)\frac{\varrho}{6}(11-12\mu^2)\,dV. \tag{20}$$

Case (I): $\mu^2 < \frac{8}{61}$, $\frac{1-2\mu^2}{11-12\mu^2}30c > \frac{n(2n^2-5n-1)}{2(6n-1)}c$ $(n=4)$. Then from Theorem 4 of [1], we have

THEOREM C. *Let M be a 4-dimensional compact oriented quasi-conformally flat totally real minimal submanifold in $QP^4(c)$. If M has nonnegative Euler number and scalar curvature $\varrho \geq 0$, then M is totally geodesic.*

Case (II): $\frac{8}{61} \leq \mu^2 \leq \frac{1}{2}$, $1-2\mu^2 \geq 0$. Then we have

THEOREM C′. *Let M be a 4-dimensional compact oriented quasi-conformally flat totally real minimal submanifold in $QP^4(c)$ $(\frac{8}{61} \leq \mu^2 \leq \frac{1}{2})$. If M has nonnegative Euler number and the scalar curvature ϱ of M is between 0 and $\frac{1-2\mu^2}{11-12\mu^2}30c$, then ϱ is 0 or $\frac{1-2\mu^2}{11-12\mu^2}30c$.*

REMARK. If $\mu = 0$, M is concircularly flat, then Theorem C becomes Theorem B.

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