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Hardy class of functions defined by the Salagean operator

by NORIO NIWA (Niigata), TOSHIYA JIMBO (Nara)
and SHIGEYOSHI OWA (Osaka)

Abstract. We derive some properties of the Hardy class of analytic functions defined by the Salagean operator.

1. Introduction. Let A be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the open unit disk $U = \{z : |z| < 1\}$.

For $f(z) \in A$, the *Salagean operator* D^n (cf. [6]) is defined by

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}). \tag{1.4}$$

A function $f(z)$ belonging to A is said to be *starlike of order* α if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.5}$$

for some α $(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order α in U.

A function $f(z) \in A$ is said to be *convex of order* α if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U) \tag{1.6}$$

for some α $(0 \le \alpha < 1)$. Also we denote by $K(\alpha)$ the subclass of A consisting of all such functions. Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$ for $0 \le \alpha < 1$.

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Let H^p $(0 < p \le \infty)$ be the class of all analytic functions in U such that

$$\|f\|_p = \lim_{r\to 1^-} M_p(r,f) < \infty, \tag{1.7}$$

where (cf. [1])

$$M_p(r,f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right)^{1/p} & (0 < p < \infty), \\ \max\limits_{|z|\le r} |f(z)| & (p = \infty). \end{cases} \tag{1.8}$$

2. Some lemmas. To discuss our problems for the Hardy class H^p, we need the following lemmas.

Lemma 1 ([7]). *If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta)$, where*

$$\beta = \beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2(2^{1-2\alpha}-1)} & (\alpha \ne 1/2), \\ \dfrac{1}{2\log 2} & (\alpha = 1/2). \end{cases} \tag{2.1}$$

This result is sharp.

Lemma 2 ([2]). *If $f(z) \in S^*(\alpha)$ and is not of the form*

$$f(z) = \frac{z}{(1 - ze^{it})^{2(1-\alpha)}}, \tag{2.2}$$

then there exists $\delta = \delta(f) > 0$ such that $f(z)/z \in H^{\delta + 1/(2(1-\alpha))}$.

Lemma 3 ([5]). *If $p(z)$ is analytic in U with $p(0) = 1$ and*

$$\mathrm{Re}(p(z) + zp'(z)) > \frac{1 - 2\log 2}{2(1-\log 2)} \quad (z \in U), \tag{2.3}$$

then $\mathrm{Re}(p(z)) > 0$ $(z \in U)$.

Remark. We have

$$\frac{1-2\log 2}{2(1-\log 2)} \risingdotseq -0.629\ldots$$

Lemma 4 ([1]). *Every analytic function $p(z)$ with positive real part in U is in H^p for all $0 < p < 1$.*

Lemma 5 ([4]). *If $f(z) \in A$ satisfies $z^r f(z) \in H^p$ $(0 < p < \infty)$ for a real r, then $f(z) \in H^p$ $(0 < p < \infty)$.*

Lemma 6 ([1]). *If $f'(z) \in H^p$ for some p $(0 < p < 1)$, then $f(z) \in H^q$ $(q = p/(1-p))$.*

Lemma 7 ([3]). *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ $(0 \le r < 1)$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0), \qquad \text{where } k \text{ is real and } k \ge 1.$$

3. Hardy class. Our first result for the Hardy class is

THEOREM 1. *Let* $f(z) \in A$ *satisfy*

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha_0 \quad (z \in U) \tag{3.1}$$

for some α_0 $(0 \le \alpha_0 < 1)$, *and let*

$$\alpha_j = \begin{cases} \dfrac{1-2\alpha_{j-1}}{2(2^{1-2\alpha_{j-1}}-1)} & (\alpha_{j-1} \ne 1/2), \\ \dfrac{1}{2\log 2} & (\alpha_{j-1} = 1/2), \end{cases} \tag{3.2}$$

for $j = 1, \ldots, n$. *If* $D^{n-j}f(z)$ *is not of the form*

$$D^{n-j}f(z) = \frac{z}{(1-ze^{it})^{2(1-\alpha_j)}}, \tag{3.3}$$

then there exists $\delta > 0$ *such that* $D^{n-j}f(z) \in H^{\delta+1/(2(1-\alpha_j))}$.

Proof. Note that

$$\begin{aligned} D^{n+1}f(z) &= D(D^n f(z)) = z(D^n f(z))' \\ &= z(D^{n-1}f(z))' + z^2(D^{n-1}f(z))'' \end{aligned} \tag{3.4}$$

and

$$D^n f(z) = z(D^{n-1}f(z))'. \tag{3.5}$$

This implies that

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \operatorname{Re}\left\{1 + \frac{z(D^{n-1}f(z))''}{(D^{n-1}f(z))'}\right\} > \alpha_0, \tag{3.6}$$

so that $D^{n-1}f(z) \in K(\alpha_0)$. Therefore, an application of Lemma 1 leads to

$$\begin{aligned} D^{n-1}f(z) \in K(\alpha_0) &\Rightarrow D^{n-1}f(z) \in S^*(\alpha_1) \\ &\Leftrightarrow D^{n-2}f(z) \in K(\alpha_1) \\ &\Rightarrow D^{n-2}f(z) \in S^*(\alpha_2) \\ &\ldots \\ &\Leftrightarrow D^{n-j}f(z) \in K(\alpha_{j-1}) \\ &\Rightarrow D^{n-j}f(z) \in S^*(\alpha_j). \end{aligned}$$

Further, by using Lemmas 2 and 5, we know that there exists $\delta > 0$ such that $D^{n-j}f(z) \in H^{\delta+1/(2(1-\alpha_j))}$. ∎

Taking $j = n$ in Theorem 1, we have

COROLLARY 1. *Let* $f(z) \in A$ *satisfy* (3.1) *for some* α_0 $(0 \le \alpha_0 < 1)$, *and let* α_n *be as in* (3.2). *If* $f(z)$ *is not of the form* (3.3), *then there exists* $\delta > 0$ *such that* $f(z) \in H^{\delta+1/(2(1-\alpha_n))}$.

Next, we derive

THEOREM 2. *Let $f(z) \in A$ satisfy*

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{z}\right\} > \frac{1-2\log 2}{2(1-\log 2)} \quad (z \in U). \tag{3.7}$$

Then there exists p_j $(j = 1, \ldots, n+1)$ such that $D^{n-j+1}f(z) \in H^{p_j}$, where

$$p_k < \frac{1}{j-k+1} \quad (k = 1, \ldots, j). \tag{3.8}$$

Proof. Define

$$p(z) = D^n f(z)/z. \tag{3.9}$$

Then $p(z)$ is analytic in U and $p(0) = 1$. Since

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{z}\right\} = \operatorname{Re}(p(z) + zp'(z)) > \frac{1-2\log 2}{2(1-\log 2)}, \tag{3.10}$$

Lemma 3 gives

$$\operatorname{Re}(p(z)) = \operatorname{Re}\{D^n f(z)/z\} > 0 \quad (z \in U). \tag{3.11}$$

Since $D^n f(z)/z = (D^{n-1}f(z))'$, an application of Lemma 4 implies that $(D^{n-1}f(z))' \in H^{p_1}$, so by Lemma 6,

$$D^{n-1}f(z) \in H^{p_2} \quad (p_2 = p_1/(1-p_1)).$$

Further, since $D^{n-1}f(z) = z(D^{n-2}f(z))'$, using Lemma 5, we obtain $(D^{n-2}f(z))' \in H^{p_2}$. Continuing this process, we conclude that $D^{n-j+2}f(z) \in H^{p_{j-1}}$ and $0 < p_{j-1} < 1/2$. Thus, finally we have $D^{n-j+1}f(z) \in H^{p_j}$ $(0 < p_j < 1)$. This completes the proof. ■

Letting $j = n+1$ in Theorem 2, we have

COROLLARY 2. *Let $f(z) \in A$ satisfy* (3.7). *Then there exists p_{n+1} such that $f(z) \in H^{p_{n+1}}$, where*

$$p_k < \frac{1}{n-k+2} \quad (k = 1, \ldots, n+1).$$

4. Hardy class of bounded functions. Our next theorem for the Hardy class of bounded functions is

THEOREM 3. *Let $f(z) \in A$ satisfy*

$$\left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right| < \frac{5\alpha_0 - 2\alpha_0^2 - 1}{2\alpha_0} \quad (z \in U) \tag{4.1}$$

for some α_0 $(1/3 \le \alpha_0 \le 1/2)$, or

$$\left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right| < \frac{\alpha_0 - 2\alpha_0^2 + 1}{2\alpha_0} \quad (z \in U) \tag{4.2}$$

for some α_0 $(1/2 \le \alpha_0 < 1)$. *If* $D^{n-j}f(z)$ *is not of the form* (3.3), *then there exists* $\delta > 0$ *such that* $D^{n-j}f(z) \in H^{\delta+1/(2(1-\alpha_j))}$ $(j = 1, \ldots, n)$, *where* α_j *is given by* (3.2).

Proof. Define the function $w(z)$ by

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+(1-2\alpha_0)w(z)}{1-w(z)} \quad (w(z) \neq 1). \tag{4.3}$$

Then $w(z)$ is analytic in U and $w(0) = 0$. It follows from (4.3) that

$$\begin{aligned}\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 = \left(\frac{w(z)}{1-w(z)}\right)\Bigg(& 2(1-\alpha_0) + \frac{zw'(z)}{w(z)} \\ &+ \frac{(1-2\alpha_0)(1-w(z))}{1+(1-2\alpha_0)w(z)}\left(\frac{zw'(z)}{w(z)}\right)\Bigg).\end{aligned} \tag{4.4}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).$$

Then Lemma 7 yields $w(z_0) = e^{i\theta}$ and

$$z_0 w'(z_0) = k w(z_0) \quad (k \ge 1).$$

Therefore, we have

$$\begin{aligned}\left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1\right| &= \left|\frac{w(z_0)}{1-w(z_0)}\right| \Bigg| 2(1-\alpha_0) + \frac{z_0 w'(z_0)}{w(z_0)} \\ &\qquad + \frac{(1-2\alpha_0)(1-w(z_0))}{1+(1-2\alpha_0)w(z_0)}\left(\frac{z_0 w'(z_0)}{w(z_0)}\right)\Bigg| \\ &= \left|\frac{e^{i\theta}}{1-e^{i\theta}}\right| \left| 2(1-\alpha_0) + k + k\frac{(1-2\alpha_0)(1-e^{i\theta})}{1+(1-2\alpha_0)e^{i\theta}}\right| \\ &\ge \frac{2(1-\alpha_0)+k}{|1-e^{i\theta}|} - \frac{k|1-2\alpha_0|}{|1+(1-2\alpha_0)e^{i\theta}|} \\ &\ge \frac{2(1-\alpha_0)+k}{2} - \frac{k|1-2\alpha_0|}{2\alpha_0}.\end{aligned} \tag{4.5}$$

For $1/3 \le \alpha_0 \le 1/2$, we have

$$\left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1\right| \ge \frac{5\alpha_0 - 2\alpha_0^2 - 1}{2\alpha_0}, \tag{4.6}$$

and for $1/2 \le \alpha_0 < 1$, we have

$$\left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1\right| \ge \frac{\alpha_0 - 2\alpha_0^2 + 1}{2\alpha_0}. \tag{4.7}$$

Since the above contradicts our assumptions (4.1) and (4.2), we conclude that $|w(z)| < 1$ for all $z \in U$. This implies that

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha_0 \quad (z \in U). \tag{4.8}$$

Note that (4.8) is equivalent to $D^n f(z) \in S^*(\alpha_0)$. In the same manner, in the proof of Theorem 1, we conclude that $D^{n-j} f(z) \in S^*(\alpha_j)$. Thus, applying Lemmas 2 and 5, we can complete the proof. ■

If we put $j = n$ in Theorem 3, then we have

COROLLARY 3. *Let $f(z) \in A$ satisfy the condition* (4.1) *for some α_0 $(1/3 \le \alpha_0 \le 1/2)$ or* (4.2) *for some α_0 $(1/2 \le \alpha_0 < 1)$. If $f(z)$ is not of the form* (3.3), *then there exists $\delta > 0$ such that $f(z) \in H^{\delta+1/(2(1-\alpha_n))}$, where α_n is given by* (3.2).

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Department of Mathematical Science
Niigata University
Niigata 950-2181, Japan
E-mail: niwa@scux.sc.niigata-u.ac.jp

Department of Mathematics
Kinki University
Higashi-Osaka
Osaka 577-8502, Japan
E-mail: owa@math.kindai.ac.jp

Department of Mathematics
Nara University of Education
Takabatake
Nara 630-8301, Japan
E-mail: jinbo@nara-edu.ac.jp

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