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On the disc-convexity of complex Banach manifolds

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Abstract. The Banach hyperbolicity and disc-convexity of complex Banach manifolds and their relations are investigated.

Introduction. The disc-convexity of complex Banach manifolds is one of the forms of complex convexity. It has been the object of interest for some time. Especially it was a useful tool to study the extension of holomorphic maps (see [6], [11], [13], ...).

Our aim in this article is to investigate the disc-convexity of complex Banach manifolds and the relations between the Banach hyperbolicity and disc-convexity of complex Banach manifolds.

We now describe more precisely the content of the paper.

In Section 1 we prove the existence of hyperbolic neighbourhoods of compact subsets in Banach manifolds which contain no complex lines (Proposition 1.2). As an easy corollary we prove the openness of Banach hyperbolicity for proper holomorphic maps between Banach analytic spaces (Proposition 1.6). These results are generalizations of the finite-dimensional case considered by Brody [1], Urata [16] and Zaĭdenberg [18].

The above-mentioned hyperbolic neighbourhoods play a central role in our approach to disc-convexity of complex Banach manifolds.

In Section 2 we study in detail the disc-convexity of complex Banach manifolds. More precisely: we give an example of a hyperbolic and disc-convex domain S in $\mathbb{C}^2$ which is not taut (Proposition 2.2). We prove that every pseudoconvex and Brody hyperbolic (Banach) manifold is disc-convex and has the Hartogs extension property (Theorem 2.3 and Proposition 2.5). Let $f : X \to Y$ be a proper holomorphic map into a Banach analytic space Y. If all the f-fibres are hyperbolic and Y is disc-convex, then so is X (Theorem 2.6); an example is given which shows the necessity of hyperbolicity of all the f-fibres (Remarks 2.7.1, 2.7.2).

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For a complex space X of finite dimension define $S^0X = X$, $S^1X = \operatorname{sing} X, \ldots, S^iX = \operatorname{sing} S^{i-1}X$. Then X is disc-convex iff $\widetilde{S^iX}$ is for all $i \geq 0$ (Theorem 2.8).

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1. Existence of hyperbolic neighbourhoods of compact subsets which contain no complex lines. We first give the following.

1.1. Definition. Let X be a Banach C^k-manifold. We say that X *has C^k-partitions of unity* if for every open cover $\{U_i\}_{i\in I}$ of X there exists a family of functions $\{\alpha_i\}_{i\in I} \subset C^k(X)$ satisfying:

(i) $\operatorname{supp}\alpha_i \subset U_i$ for every $i \in I$ and the family $\{\operatorname{supp}\alpha_i\}_{i\in I}$ is locally finite.

(ii) $\sum_{i\in I} \alpha_i(x) = 1$ for every $x \in X$.

The family $\{\alpha_i\}_{i\in I}$ is called a C^k-partition of unity subordinate to the cover $\{U_i\}_{i\in I}$. For details concerning smooth partitions of unity on Banach analytic manifolds we refer the readers to [15], [19]. Let X be a Banach analytic space in the sense of Mazet [8]. We denote by d_X the Kobayashi pseudodistance on X. In contrast to the finite-dimensional case, there exists a Banach manifold X on which the Kobayashi pseudodistance is a distance but it does not define the topology of X. We say that X is *hyperbolic* if d_X is a distance defining the topology of X.

We now give the main result of this section.

1.2. Proposition. *Let Z be a compact subset in a Banach manifold X having C^1-partitions of unity such that Z contains no complex lines, i.e. every holomorphic map φ from $\mathbb{C}$ into X with $\varphi(\mathbb{C}) \subset Z$ is constant. Then there exists a hyperbolic neighbourhood of Z in X.*

Let $\{(U_i, \varphi_i)\}_{i\in I}$ be an atlas of X such that φ_i is an isomorphism from U_i onto an open ball in Banach space for every $i \in I$. By hypothesis, there exists a C^1-partition of unity $\{h_i\}_{i\in I}$ subordinate to the cover $\{U_i\}_{i\in I}$.

Let $\pi : TX \to X$ be the tangent bundle of X. For each $u \in TX$, put

$$\|u\| = \sum h_i(\pi u)\|D\varphi_i(\pi u)(u)\|.$$

Denote by ϱ_X the integral distance on X associated with $\|\cdot\|$.

1.3. Lemma. *ϱ_U defines the topology in U, where U is the unit ball in a Banach space B.*

Proof. By [12] we have

$$\|x - y\| = \sup\{|x^*(x) - x^*(y)| : x^* \in B^*,\ \|x^*\| \leq 1\}$$

$$\begin{aligned} &\le \sup\{|f(x-y)| : f \in H^\infty(U),\ f(0)=0,\ \|f\| \le 1\} \\ &= \frac{e^{2c_U(x,y)}-1}{e^{2c_U(x,y)}+1}, \end{aligned}$$

where B^* denotes the dual space of B and c_U denotes the Carathéodory distance on U. On the other hand, for the differential Carathéodory metric γ_U of U we have by [19],

$$\begin{aligned} c_U(x,y) &\le \inf\Big\{\int_0^1 \gamma_U(\dot\sigma(t))\,dt : \sigma \in \Omega_{x,y}(U)\Big\} \\ &\le \inf\Big\{\int_0^1 \|\dot\sigma(t)\|\,dt : \sigma \in \Omega_{x,y}(U)\Big\} = \varrho_U(x,y) \end{aligned}$$

for all $x, y \in U$, where $\Omega_{x,y}$ is the set of C^1-paths joining x and y in U. Thus ϱ_U defines the topology of U. ■

1.4. Lemma. *Assume that X is a Banach manifold having C^1-partitions of unity. Then ϱ_X defines the topology of X.*

P r o o f. Let $\{x_n\} \subset X$ and $\varrho_X(x_n,x) \to 0$. Take j_0 such that $x \in U_{j_0}$ and $h_{j_0}(x) \ne 0$. For each $n \ge 1$ take $\sigma_n \in \Omega_{x_n,x}(X)$ such that

$$\begin{aligned} \varrho_X(x_n,x) &\ge \int_0^1 \|\dot\sigma_n(t)\|\,dt - 1/n \\ &= \int_0^1 \sum_j h_j(\sigma_n(t))\|D\varphi_j(\sigma_n(t))(\dot\sigma_n(t))\|\,dt - 1/n \\ &\ge \int_0^1 h_{j_0}(\sigma_n(t))\|D\varphi_{j_0}(\sigma_n(t))(\dot\sigma_n(t))\|\,dt - 1/n. \end{aligned}$$

Assume that $x_n \not\to x$. Then there exists an open neighbourhood V of x in U_{j_0} such that $x_n \notin V$ for all $n \ge 1$ and $\inf\{h_{j_0}(y) : y \in V\} = \varepsilon > 0$. For every $n \ge 1$, put $\varepsilon_n = \sup\{r > 0 : \sigma_n([0,r]) \subset V\} > 0$. We have

$$\begin{aligned} \varrho_X(x_n,x) &\ge \int_0^1 h_{j_0}(\sigma_n(t))\|D\varphi_{j_0}(\sigma_n(t))(\dot\sigma_n(t))\|\,dt - 1/n \\ &\ge \varepsilon \int_0^{\varepsilon_n} h_{j_0}(\sigma_n(t))\|D\varphi_{j_0}(\sigma_n(t))(\dot\sigma_n(t))\|\,dt - 1/n \\ &\ge \varepsilon \int_0^{\varepsilon_n} \|D\varphi_{j_0}(\sigma_n(t))(\dot\sigma_n(t))\|\,dt - 1/n \end{aligned}$$

$$= \varepsilon \int_0^1 \|D\varphi_{j_0}(\beta_n(s))(\dot\beta_n(s))\|\,ds - 1/n$$
$$\geq \varrho_{\varphi_{j_0}(U_{j_0})}(\varphi_{j_0}(\beta_n(1)), \varphi_{j_0}(x)) - 1/n$$

where $s = t/\varepsilon_n$ and $\beta_n(s) = \sigma_n(\varepsilon_n s)$ for $s \in [0,1]$. It follows that $\varrho_{\varphi_{j_0}(U_{j_0})}(\varphi_{j_0}(\beta_n(1)), \varphi_{j_0}(x))) \to 0$. By Lemma 1.3, $\varphi_{j_0}(\beta_n(1)) \to \varphi_{j_0}(x)$. Hence $\beta_n(1) \to x$. This is a contradiction because $\beta_n(1) = \sigma_n(\varepsilon_n) \in \partial V$ for every $n \geq 1$. ∎

1.5. Lemma. *Let X be a Banach manifold having C^1-partitions of unity such that*

$$\sup\{\|f'(0)\| : f \in \mathrm{Hol}(\Delta, X)\} < \infty,$$

where $\mathrm{Hol}(\Delta, X)$ *denotes the space of holomorphic maps from the open unit disc Δ in $\mathbb{C}$ into X. Then X is hyperbolic.*

Proof. Let $d_X(x_n, x) \to 0$. For each $n \geq 1$ there exists a holomorphic chain $(f_1^n, \ldots, f_{k_n}^n, a_1^n, \ldots, a_{k_n}^n)$ joining x_n and x such that

$$\sum_{j=1}^{k_n} d_\Delta(0, a_j^n) \to 0.$$

By the hypothesis we have

$$a = \sup\{\|f'(z)\| : f \in \mathrm{Hol}(\Delta, X),\ |z| < r\} < \infty,$$

where $0 < r < 1$ is chosen such that $|a_j^n| < r$ for $j = 1, \ldots, k_n$. Then

$$\varrho_X(p_i^n, p_{i-1}^n) \leq \int_0^1 \|(f_j^n \sigma_i^n)'(t)\|\,dt \leq a \int_0^1 \|\dot\sigma_i^n(t)\|\,dt = a d_\Delta(0, a_i^n),$$

where $p_i^n = f_i^n(a_i^n)$ and $\sigma_i^n(z) = a_i^n z$. Thus $\varrho_X(x_n, x) \to 0$. By Lemma 1.4 we have $x_n \to x$. ∎

Proof of Proposition 1.2. By Lemma 1.5 it suffices to show that there exists a neighbourhood W of Z in X such that

$$\sup\{\|f'(0)\| : f \in \mathrm{Hol}(\Delta, W)\} < \infty.$$

If not, for each n we can find $f_n \in \mathrm{Hol}(\Delta, W_n)$ such that $\|f_n'(0)\| = r_n \uparrow \infty$, where $\{W_n\}$ is decreasing neighbourhood basis of Z in X. By the parametrization lemma of Brody [1] there exists for each n a holomorphic map φ_n from $(r_n/2)\Delta$ into W_n such that $\|\varphi_n'(0)\| = 1$ and

$$\|\varphi_n'(z)\| \leq \frac{r_n^2}{r_n^2 - |z|^2} \quad \text{for } |z| < r_n/2.$$

This yields $\|\varphi_n'(z)\| \leq 4/3$ for $|z| < r_n/2$ and hence $\{\varphi_n\}$ is equicontinuous. By the compactness of Z and since $Z = \bigcap W_n$, it follows that $\{\varphi_n\}$ contains a subsequence $\{\psi_n\}$ converging to $\psi \in \mathrm{Hol}(\mathbb{C}, X)$ with $\psi(\mathbb{C}) \subset Z$. Obviously, $\psi \neq \mathrm{const}$. This is impossible because Z contains no complex lines.

1.6. Proposition. *Let $\theta : X \to Y$ be a proper holomorphic map from a Banach manifold X having C^1-partitions of unity into a Banach analytic space Y. Assume that $\theta^{-1}(y_0)$ is hyperbolic for some $y_0 \in Y$. Then there exists an open neighbourhood U of y_0 in Y such that $\theta^{-1}(U)$ is Banach hyperbolic.*

Proof. By Proposition 1.2, there exists a hyperbolic neighbourhood W of $\theta^{-1}(y_0)$ in X.

Suppose that there is no neighbourhood U of y_0 in Y such that $\theta^{-1}(U) \subset W$. Consider a decreasing sequence $\{U_n\}$ of neighbourhoods of y_0 which is convergent to y_0. Then for each $n \geq 1$ there are $y_n \in U_n$ and $x_n \in \theta^{-1}(y_n)$ such that $x_n \notin W$. Since the set $K = \{y_n : n \geq 1\} \cup \{y_0\}$ is compact, so is

$$\theta^{-1}(K) = \bigcup_{n \geq 1} \theta^{-1}(y_n) \cup \theta^{-1}(y_0).$$

Thus $\{x_n\}$ contains subsequence $\{x_{n_k}\}$ convergent to x_0. It follows that $\theta x_{n_k} \to \theta x_0$, i.e. $y_{n_k} \to \theta x_0$. Hence $\theta x_0 = y_0$, i.e. $x_0 \in \theta^{-1}(y_0) \subset W$. Thus $x_{n_k} \in W$ for all $k \geq N$. This is impossible, because $x_n \notin W$ for each $n \geq 1$. ■

1.7. Remark. From a result of Ramis [9, Théorème II.2.1.1, p. 36], we can deduce the existence of finite proper holomorphic maps between Banach analytic spaces.

Let S be an analytic subset of codimension p in a Banach space E. Then we can decompose $E = B \oplus \mathbb{C}^p$ so that the restriction $\pi|_S$ of the canonical projection $\pi : E \to B$ is a finite proper holomorphic map from S onto B (at least locally).

2. Disc-convexity of Banach analytic spaces. We first give the following.

2.1. Definitions. For every $0 < r < 1$ and $s > 0$ we define

$$\Delta_s = \{z \in \mathbb{C} : |z| < s\}, \qquad \Delta_1 = \Delta, \qquad \Delta_{r1} = \{z \in \mathbb{C} : r < |z| < 1\}.$$

We say that a Banach analytic space X is *disc-convex* if every sequence $\{f_n\} \subset \mathrm{Hol}(\Delta, X)$ converges in $\mathrm{Hol}(\Delta, X)$ whenever $\{f_n|_{\Delta_{r1}}\}$ converges in $\mathrm{Hol}(\Delta_{r1}, X)$ for some $r < 1$, where $\mathrm{Hol}(X, Y)$ denotes the space of all holomorphic maps from X into Y with the open-compact topology. It is well known that, by the Montel theorem, a taut (finite-dimensional) complex space is disc-convex and hyperbolic. The converse is not true in general.

Let $u(z)$ be a negative subharmonic function on Δ. In addition, suppose $u(z)$ is bounded from below and discontinuous at 0 and $\lim_{z \to 0} u(z) < u(0)$.

In $\mathbb{C}^2$ we consider the Hartogs domain

$$S = \{(z, w) \in \mathbb{C}^2 : |z| < 1,\ |w| < e^{-u(z)}\}$$

(see Shabat [10] or Diederich and Sibony [2]).

We have the following.

2.2. Proposition. *The domain S is hyperbolic and disc-convex but not taut.*

Proof. Clearly S is bounded and by the Hartogs theorem, S is a domain of holomorphy. Then S is a hyperbolic disc-convex manifold.

Now assume that S is taut. Put $-\lim_{z\to 0} u(z) = R > -u(0)$. Take $\beta \in \mathbb{R}$ and a sequence $\{z_n\} \subset \Delta$ converging to 0 such that

$$\lim_{n\to\infty} (-u(z_n)) = R > \ln\beta > -u(0).$$

Without loss of generality we can suppose that

$$-u(z_n) > \ln\beta > -u(0) \quad \text{for all } n \geq 1.$$

Let $\theta : \Delta_\beta \to \Delta_\beta$ be a biholomorphic map such that $\theta(0) = e^{-u(0)}$. Then the map $f : \Delta \to \Delta_\beta$, where $f(z) = \theta(\beta z)$ for all $z \in \Delta$, is biholomorphic and $f(\Delta) \subset \Delta_{e^{-u(z_n)}}$ for all $n \geq 1$.

Consider holomorphic maps $g_n : \Delta \to S,\ z \mapsto (z_n, f(z))$, for all $n \geq 1$. We have $\lim_{n\to\infty} g_n(z) = (0, f(z))$ for all $z \in \Delta$ and $\lim_{n\to\infty} g_n(0) = (0, f(0)) \notin S$. Since S is taut, $(0, f(z)) \notin S$ for all $z \in \Delta$. Hence $|f(z)| \geq e^{-u(0)} = f(0)$ for all $z \in \Delta$. This is impossible. ■

2.3. Theorem. *Let X be a pseudoconvex Banach manifold having C^1-partitions of unity. Suppose that X contains no complex lines. Then X is disc-convex.*

Proof. Assume that a sequence $\{f_n\} \subset \mathrm{Hol}(\Delta, X)$ is such that $\{f_n|_{\Delta_{r1}}\}$ converges, uniformly on compact sets, to a map f in $\mathrm{Hol}(\Delta_{r1}, X)$. Let $\{f_{n_k}\}$ be any subsequence of $\{f_n\}$. Put $K = \bigcup_{k=1}^{\infty} f_{n_k}(\partial\Delta_s)$, where $r < s < 1$.

By the hypothesis and by the maximum principle it follows that $(\overline{K})^{\wedge}_{\mathrm{PSH}(X)}$ is compact and

$$\bigcup_{k=1}^{\infty} f_{n_k}(\Delta_s) \subset (\overline{K})^{\wedge}_{\mathrm{PSH}(X)}.$$

Again by Proposition 1.2, there is a hyperbolic neighbourhood W of $(\overline{K})^{\wedge}_{\mathrm{PSH}(X)}$ in X. This implies that the family $\{f_{n_k}\}$ is equicontinuous. On the other hand, since $\{f_{n_k}(\lambda)\}$ is relatively compact for each $\lambda \in \Delta_s$, by the Ascoli theorem the family $\{f_{n_k} : k \geq 1\}$ is relatively compact in $\mathrm{Hol}(\Delta_s, X)$. Thus there exists a subsequence $\{f_{n_{k_l}}\}$ of $\{f_{n_k}\}_{k=1}^{\infty}$ which converges, uniformly on compact sets, to the map F in $\mathrm{Hol}(\Delta, X)$. The equality $F|_{\Delta_{r1}} = f$ determines F uniquely, hence independently of the choices of the subsequences $\{f_{n_k}\}$. It follows that $\{f_n\}$ converges, uniformly on compact sets, to the map F in $\mathrm{Hol}(\Delta, X)$. ∎

We now recall the following definition.

2.4. Definition. Let X be a Banach analytic space. We say that X has the *Hartogs extension property* (briefly HEP) if every holomorphic map from a Riemann domain over a Banach space having a Schauder basis into X can be extended to the envelope of holomorphy of that map.

The following assertion is deduced immediately from Theorem 2.3 and the result of B. D. Tac [13].

2.5. Proposition. *Let X be a pseudoconvex Banach manifold having C^1-partitions of unity. If X contains no complex line then X has HEP.*

2.6. Theorem. *Let $\pi : X \to Y$ be a proper holomorphic map from a Banach manifold X having C^1-partitions of unity onto a Banach analytic space Y such that the fibre $\pi^{-1}(y)$ is hyperbolic for all $y \in Y$. If Y is disc-convex, then so is X.*

Proof. Assume that $\{f_n\} \subset \mathrm{Hol}(\Delta, X)$ is a sequence such that $\{f_n|_{\Delta_{r1}}\}$ converges, uniformly on compact sets, to a map f in $\mathrm{Hol}(\Delta_{r1}, X)$. Let $\{f_{n_k}\}$ be any subsequence of $\{f_n\}$. Put $g_k = \pi \circ f_{n_k}$ for all $k \geq 1$. Since Y is disc-convex, $\{g_k\}$ converges uniformly to a map G in $\mathrm{Hol}(\Delta, X)$.

Consider the family $\mathcal{V}$ of all pairs (V, F), where V is an open subset of Δ containing Δ_{r1} and $F \in \mathrm{Hol}(V, X)$ is such that there exists a subsequence $\{f_{n_{k_l}}|_V\}$ of $\{f_{n_k}|_V\}$ converging, uniformly on compact sets, to F in $\mathrm{Hol}(V, X)$. We define an order relation in the family $\mathcal{V}$ by $(V_1, F_1) \leq (V_2, F_2)$ if $V_1 \subset V_2$ and for every subsequence $\{f_{n_{k_l}}|_{V_1}\}$ of $\{f_{n_k}|_{V_1}\}$ converging, uniformly on compact sets, to F_1 in $\mathrm{Hol}(V_1, X)$, the sequence $\{f_{n_{k_l}}|_{V_2}\}$ contains a subsequence converging, uniformly on compact sets, to F_2 in $\mathrm{Hol}(V_2, X)$.

Assume that $\{(V_\alpha, F_\alpha)\}_{\alpha \in \Lambda}$ is a well-ordered subset of $\mathcal{V}$. Put $V_0 = \bigcup_{\alpha \in \Lambda} V_\alpha$. Define a map $F_0 \in \mathrm{Hol}(V_0, X)$ by $F_0|_{V_\alpha} = F_\alpha$ for all $\alpha \in \Lambda$. Take a sequence $\{(V_i, F_i)\}_{i=1}^{\infty} \subset \{(V_\alpha, F_\alpha)\}_{\alpha \in \Lambda}$ such that

$$(V_1, F_1) \leq (V_2, F_2) \leq \ldots \quad \text{and} \quad \bigcup_{i=1}^{\infty} V_i = V_0.$$

By assumption, there is a subsequence $\{f_k^1|_{V_1}\}$ of $\{f_{n_k}|_{V_1}\}$ converging to F_1 in $\mathrm{Hol}(V_1, X)$. Consider the sequence $\{f_k^1|_{V_2}\}$. As above, it contains a subsequence $\{f_k^2|_{V_2}\}$ converging to F_2 in $\mathrm{Hol}(V_2, X)$. Continuing this process we can find sequences $\{f_k^i\}$ such that $\{f_k^i\} \subset \{f_k^{i-1}\}$ for all $i \geq 2$ and $\{f_k^i|_{V_i}\}$ converges to F_i in $\mathrm{Hol}(V_i, X)$. Then the sequence $\{f_i^i\}$ converges to F_0 in

$\mathrm{Hol}(V_0, X)$. Thus $(V_0, F_0) \in \mathcal{V}$ and hence, the subset $\{(V_\alpha, F_\alpha)\}_{\alpha\in\Lambda}$ has an upper bound.

By the Zorn lemma, the family $\mathcal{V}$ has a maximal element (V, F). Let $\{F_{n_{k_l}}|_V\}$ be a subsequence of $\{f_{n_k}|_V\}$ converging uniformly to F in $\mathrm{Hol}(V, X)$. We now prove that V is closed in Δ.

Indeed, take $z_0 \in V$. By Proposition 1.6, there is an open neighbourhood U of $G(z_0)$ in Y such that $\pi^{-1}(U)$ is hyperbolic. Since $\{g_k\}$ converges uniformly to G in $\mathrm{Hol}(\Delta, Y)$, there is an open neighbourhood W of z_0 in Δ such that $g_k(W) \subset U$ for all $k \geq N$. Hence $f_{n_k}(W) \subset \pi^{-1}(U)$ for all $k \geq N$. Since π is a proper map, $\{f_{n_{k_l}}(z) : l \geq 1\}$ is relatively compact in $\pi^{-1}(U)$ for all $z \in W$. By the equicontinuity of $\{f_{n_{k_l}}\}$ for $d_{\pi^{-1}(U)}$, the family $\{f_{n_{k_l}} : l \geq 1\}$ is relatively compact in $\mathrm{Hol}(W, \pi^{-1}(U))$. By the maximality of the element (V, F), we have $W \subset V$ and hence, $V = \Delta$.

Thus the sequence $\{f_{n_{k_l}}\}$ converges, uniformly on compact sets, to the map F in $\mathrm{Hol}(\Delta, X)$. The equality $F|_{\Delta_{r1}} = f$ determines F uniquely, hence independently of the choices of the subsequences $\{f_{n_k}\}$. It follows that $\{f_n\}$ converges, uniformly on compact sets, to F in $\mathrm{Hol}(\Delta, X)$. ∎

2.7. Remark. 1. *The following counterexample shows that the condition of hyperbolicity of all fibres in Theorem* 2.6 *cannot be replaced by the condition of disc-convexity of all fibres*. Consider the canonical holomorphic map θ from the Hopf surface $X = \mathbb{C} \setminus \{0\}/(z \sim 2z)$ onto $\mathbb{C}P^1$. Then $\theta^{-1}(y) \cong \mathbb{C} \setminus \{0\}/(z \sim 2z)$. Since the universal cover of $\mathbb{C} \setminus \{0\}/(z \sim 2z)$ is a Stein manifold, every fibre $\theta^{-1}(y) \cong \mathbb{C} \setminus \{0\}/(z \sim 2z)$, which is an elliptic curve, satisfies the condition of disc-convexity.

We check that there exists a non-empty open subset V of $\mathbb{C}P^1$ ($V \neq \mathbb{C}P^1$) such that $\theta^{-1}(V)$ is not disc-convex.

Otherwise consider the commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathbb{C}^2 & \xrightarrow{g} & \mathbb{C}P^1 \end{array}$$

in which $\Omega = \mathbb{C}^2 \setminus \{0\}$; $f : \Omega \to X$ is the canonical map; $\widetilde{\Omega}$ is the local envelope of biholomorphy of f over $\mathbb{C}^2$, the envelope of holomorphy of $\mathbb{C}^2 \setminus \{0\}$; $g : \mathbb{C}^2 \to \mathbb{C}P^1$ is the meromorphic extension of θf; and j, θ are canonical maps.

It is easy to see that $\widetilde{f} : \widetilde{\Omega} \to X$ is locally pseudoconvex, i.e. for every $x \in X$ there is a pseudoconvex neighbourhood U of x in X such that $\widetilde{f}^{-1}(U)$ is pseudoconvex. By hypothesis, $\theta\widetilde{f} : \widetilde{\Omega} \to \mathbb{C}P^1$ is locally pseudoconvex.

Since X is homogeneous compact, as follows from [4], $e : \Omega \to \widetilde{\Omega}$ and hence $f : \Omega \to X$ extends holomorphically to $\mathbb{C}^2$. This is impossible.

2. *The disc-convexity is not closed under proper holomorphic maps*. Indeed, let $Z = \{(z, [w]) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}/(z \sim 2z) : z_i w_j = z_j w_i,\ 1 \le i, j \le 2\}$ and $\theta : Z \to \mathbb{C}^2$ be the canonical projection. We have

$$\theta^{-1}(z) = \begin{cases} \mathbb{C}^2 \setminus \{0\}/(z \sim 2z) & \text{if } z = 0, \\ \mathbb{C} \setminus \{0\}/(z \sim 2z) & \text{if } z \ne 0. \end{cases}$$

Clearly, $\theta^{-1}(z)$ is disc-convex for each $z \ne 0$ but $\theta^{-1}(0)$ is not disc-convex.

We now investigate the disc-convexity of the normalizations of complex spaces.

2.8. Theorem. *Let X be a (finite-dimensional) complex space. Then X is disc-convex if and only if $\widetilde{S^i X}$ is disc-convex for all $i \ge 0$, where $S^0 X = X$, $S^1 X = S(X)$ is the singular locus of X, and $S^i X = S(S^{i-1} X)$ for all $i \ge 2$.*

Proof. Let X be a (finite-dimensional) disc-convex space. Then $S^i X$ is disc-convex for every $i \ge 0$. By Theorem 2.6, $\widetilde{S^i X}$, the normalization of $S^i X$, is also disc-convex for every $i \ge 0$.

Now assume that $\widetilde{S^i X}$ is disc-convex for every $i \ge 0$. Given $g \in \mathrm{Hol}(\Delta, X)$, take $i \ge 0$ such that $g(\Delta) \subset S^i X$ but $g(\Delta) \not\subset S^{i+1} X$. Consider the diagram

$$\begin{array}{ccc} \Delta \times_{S^i X} \widetilde{S^i X} & \xrightarrow{\ \widetilde{g}\ } & \widetilde{S^i X} \\ {\scriptstyle \theta_i}\big\downarrow & & \big\downarrow{\scriptstyle \pi_i} \\ \Delta & \xrightarrow{\ g\ } & S^i X \end{array}$$

Since π_i is finite and proper, so is θ_i. By the normality of Δ and by the integrity lemma [3], it follows that $\theta_i : \Delta \times_{S^i X} S^i X \to \Delta$ is an analytic covering map. This yields that $\operatorname{card} \theta_i^{-1}(z) = 1$ for every $z \in \Delta$. Hence from the normality of Δ we deduce that $\widetilde{g} \circ \theta_i^{-1} : \Delta \to \widetilde{S^i X}$ is holomorphic.

Let $\{\varphi_j\} \subset \mathrm{Hol}(\Delta, X)$ be a sequence such that the sequence $\{\varphi_j|_{\Delta_{r1}}\}$ converges, uniformly on compact sets, to a map f in $\mathrm{Hol}(\Delta_{r1}, X)$ for some $0 < r < 1$. Let $\{\varphi_{j_n}\}$ be any subsequence of $\{\varphi_j\}$. Put $\varphi_{j_n} = f_n$ for all $n \ge 1$. Then we can find $i \ge 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(\Delta) \subset S^i X$ but $f_{n_k}(\Delta) \not\subset S^{i+1} X$ for all $k \ge 1$. Since $S^i X$ is closed in X, $f(\Delta_{r1}) \subset S^i X$.

Consider the commutative diagram

$$\begin{array}{ccc} \Delta \times_{S^iX} \widetilde{S^iX} & \xrightarrow{\widetilde{f}_{n_k}} & \widetilde{S^iX} \\ \theta_i \big\downarrow & & \big\downarrow \pi_i \\ \Delta & \xrightarrow{f_{n_k}} & S^iX \end{array}$$

Reasoning as above, we deduce that $g_k = \widetilde{f}_{n_k} \circ \theta_i^{-1} : \Delta \to \widetilde{S^iX}$ is holomorphic for every $k \ge 1$. Take any point $z \in \Delta_{r1}$. By the results of Brody [1], Urata [16] and Zaĭdenberg [18], there is a complete hyperbolic neighbourhood U of $f(z)$ in S^iX such that $\pi_i^{-1}(U)$ is complete hyperbolic. Since the sequence $\{f_{n_k}|_{\Delta_{r1}}\}$ converges uniformly to the map f in $\mathrm{Hol}(\Delta_{r1}, S^iX)$, there is an open neighbourhood V of z in Δ_{r1} such that $f_{n_k}(V) \subset U$ for all $k \ge N$. Hence $g_k(V) \subset \pi_i^{-1}(U)$ for all $k \ge N$.

Suppose that the sequence $\{g_k|_V\}$ contains a subsequence which is compactly divergent. Without loss of generality we may assume that $\{g_k|_V\}$ itself is compactly divergent. Let K and L be two compact subsets in V and U respectively. Since $\pi_i^{-1}(L)$ is compact, there is k_0 such that $g_k(K) \cap \pi_i^{-1}(L) = \emptyset$ for all $k > k_0$. This implies that $f_{n_k}(K) \cap L = \emptyset$ for all $k > k_0$, and hence the sequence $\{f_{n_k}|_V\}$ is compactly divergent. This is impossible.

Thus $\{g_k|_V\}$ contains a subsequence which is uniformly convergent in $\mathrm{Hol}(V, \widetilde{S^iX})$. Repeating the proof of Theorem 2.6 we can find a subsequence $\{g_{k_l}\}$ of $\{g_k\}$ such that $\{g_{k_l}|_{\Delta_{r1}}\}$ converges uniformly to a map G in $\mathrm{Hol}(\Delta_{r1}, \widetilde{S^iX})$. Since $\widetilde{S^iX}$ is disc-convex, $\{g_{k_l}\}$ converges uniformly to a map $\widetilde{G} \in \mathrm{Hol}(\Delta, \widetilde{S^iX})$ in $\mathrm{Hol}(\Delta, \widetilde{S^iX})$. Hence $\{f_{n_{k_l}}\}$ converges uniformly to $\pi_i \circ \widetilde{G} = F$ in $\mathrm{Hol}(\Delta, X)$. The equality $F|_{\Delta_{r1}} = f$ determines F uniquely, hence independently of the choices of the subsequences $\{\varphi_{j_n}\}$. It follows that $\{\varphi_j\}$ converges, uniformly on compact sets, to the map F in $\mathrm{Hol}(\Delta, X)$. ∎

Note that the analogous result for the tautness of normalizations of complex spaces was proved in [14].

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