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Remark on the equisingularity of families of affine plane curves

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Abstract. We give some criteria for the equisingularity of families of affine plane curves.

1. Introduction. Let $f_\alpha : \mathbb{C}^2 \to \mathbb{C}$ be a family of polynomials whose coefficients are polynomial functions of $\alpha \in \mathbb{C}^n$. Consider the family of affine curves $V_\alpha := \{(x,y) \in \mathbb{C}^2 \mid f_\alpha(x,y) = 0\}$, $\alpha \in \mathbb{C}^n$. The aim of this paper is to give certain necessary and sufficient conditions for the family $\{V_\alpha\}_{\alpha \in \mathbb{C}^n}$ to be *equisingular*. These conditions read as follows: apart from the requirement that the curves V_α satisfy Whitney's conditions at each common critical point (or equivalently, $\mu_a(V_\alpha) = \text{const}$ at such a point a, where $\mu_a(V_\alpha)$ denotes the Milnor number of the curve V_α at a) they need to have *good behavior at infinity* (i.e., in a sense, V_α satisfy the so-called *Whitney's affine conditions at infinity*).

We now suppose that the affine curves V_α, $\alpha \in \mathbb{C}^n$, all have the same critical points, say $a_i = (x_i, y_i) \in \mathbb{C}^2$, $i = 1, \ldots, s$.

1.1. Definition. The family of affine curves V_α is said to be *equisingular* if for all $\alpha^0 \in \mathbb{C}^n$ there exist a neighborhood U_{α^0} of α^0 and a diffeomorphism h such that $h(a_i, \alpha) = (a_i, \alpha)$, $i = 1, \ldots, s$, and the diagram

$$\begin{array}{ccc} \{(x,y) \mid f_{\alpha^0}(x,y) = 0\} \times U_{\alpha^0} & \xrightarrow{h} & \{(x,y,\alpha) \mid f_\alpha(x,y) = 0\} \cap \pi^{-1}(U_{\alpha^0}) \\ \downarrow \pi & & \downarrow \pi \\ U_{\alpha^0} & \xrightarrow{\text{id}} & U_{\alpha^0} \end{array}$$

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is commutative, where π is the second projection. Let $\Gamma := \{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0\}$.

1.2. Definition. The family of affine curves V_α is said to be *good at infinity* if for each $\alpha^0 \in \mathbb{C}^n$, there exist $c > 0$ and a neighborhood U_{α^0} of α^0 such that the tangent hyperplane $T_{(u,v,\beta)}(\Gamma \cap \{\alpha = \beta\})$ is transverse within the plane $\{\alpha = \beta\} \subset \mathbb{C}^2 \times \mathbb{C}^n$ to the line $\{x = u,\ \alpha = \beta\}$ for all $(u, v, \beta) \in \Gamma$, $\beta \in U_{\alpha^0}$, $|u| \geq c$.

1.3. Remark. (a) Although the above definition is based on a specific (and explicit) choice of the line $\{x = u,\ \alpha = \beta\}$, it is easily seen that we can choose $\{l_1 x + l_2 y = l_1 u + l_2 v,\ \alpha = \beta\}$, $(l_1 : l_2) \in \mathbb{CP}^1$, instead.

(b) By definition, the family $\{V_\alpha\}_{\alpha \in \mathbb{C}^n}$ is good at infinity if and only if there exist $c > 0$ and a neighborhood U_{α^0} of α^0 such that

$$\frac{\partial f_\alpha}{\partial y} \neq 0 \quad \text{for all } \alpha \in U_{\alpha^0},\ x, y \in \mathbb{C} \text{ with } |x| \geq c,\ f_\alpha(x, y) = 0.$$

(c) The following assumptions will be made throughout this paper:

- the curves $\{f_\alpha(x, y) = 0\}$ are all reduced;
- $d := \deg(f_\alpha) = \deg_y(f_\alpha)$.

The second assumption implies that the restriction map

$$l|_{V_\alpha} : V_\alpha \to \mathbb{C}, \qquad (x, y) \mapsto x,$$

is proper. Let $\delta(x, \alpha) := \mathrm{disc}_y(f_\alpha(x, y))$ be the discriminant of f_α with respect to y. Then we can write

$$\delta(x, \alpha) = q_k(\alpha) x^k + q_{k-1}(\alpha) x^{k-1} + \ldots + q_0(\alpha), \qquad q_k \not\equiv 0,$$

where $q_i(\alpha)$, $i = 0, \ldots, k$, are polynomials of α. Therefore, by the properties of resultants, the family $\{V_\alpha\}_{\alpha \in \mathbb{C}^n}$ is good at infinity if and only if $q_k(\alpha) = \text{const} \neq 0$.

2. The main result of this paper is the following theorem.

2.1. Theorem. *Suppose that the affine curves V_α, $\alpha \in \mathbb{C}^n$, have the same critical points, say $a_i = (x_i, y_i) \in \mathbb{C}^2$, $i = 1, \ldots, s$. Then the following two conditions are equivalent*:

(a) *the family $\{V_\alpha\}_{\alpha \in \mathbb{C}^n}$ is equisingular*;

(b) $\mu_{a_i}(V_\alpha) = \text{const}$, $i = 1, \ldots, s$; *and the family $\{V_\alpha\}_{\alpha \in \mathbb{C}^n}$ is good at infinity.*

For the proof we need the below lemma.

2.2. Lemma ([4]). *Let there be given a polynomial F of two complex variables and the map*

$$l:\mathbb{C}^2\to\mathbb{C},\qquad (x,y)\mapsto x,$$

such that the restriction map $l|_V$, $V:=F^{-1}(0)$, is proper. Moreover, suppose that the curve V is reduced. Then

$$\chi(F^{-1}(0)) = d - \deg \operatorname{disc}_y F(x,y),$$

where $d:=\deg_y(F)$ and $\chi(F^{-1}(0))$ is the Euler characteristic of $F^{-1}(0)$.

Proof of Theorem 2.1. (a)⇒(b) is easy. Indeed, the Milnor number is a topological invariant for isolated curve singularities [5]; hence,

$$\mu_{a_i}(V_\alpha)=\text{const},\qquad i=1,\dots,s.$$

Moreover, by the definition of equisingularity, $\chi(V_\alpha)=\text{const}$. Therefore, according to Lemma 2.2, $q_k(\alpha)=\text{const}$. So the family $\{V_\alpha\}_{\alpha\in\mathbb{C}^n}$ is good at infinity by Remark 1.3(c).

(b)⇒(a). We denote by $\operatorname{grad} f$ the vector $\operatorname{grad} f := \overline{(\partial f/\partial x, \partial f/\partial y)}$, so the chain rule may be expressed by the inner product $\partial f/\partial v = \langle v, \operatorname{grad} f\rangle$. Assume that $\alpha^0\in\mathbb{C}^n$. Since $\mu_{a_i}(V_\alpha)=\text{const}$, $i=1,\dots,s$, there exists a neighborhood U_{α^0} of α^0 such that the family $\{V_\alpha\}_{\alpha\in\mathbb{C}^n}$ satisfies Whitney's conditions along $\{a_i\}\times U_{\alpha^0}$ at a_i, $i=1,\dots,s$ (see [6], [2]). Thus there exist closed balls D_i small enough centered at a_i such that $D_i\cap D_j=\emptyset$ $(i\neq j)$ and there exist integrable vector fields $\overline{\xi}^{ij}(x,y,\alpha)$, $\overline{\eta}^{ij}(x,y,\alpha)$, $i=1,\dots,s$, $j=1,\dots,n$, nowhere zero on the set

$$\Gamma^i_{\alpha^0} := \{(x,y,\alpha)\in D_i\times U_{\alpha^0} \mid f_\alpha(x,y)=0\}$$

such that the following equations are satisfied:

$$\langle\overline{\xi}^{ij}(x,y,\alpha), \operatorname{grad} f_\alpha(x,y)\rangle + \frac{\partial f_\alpha}{\partial\alpha_j}(x,y)=0,$$
$$\langle\overline{\eta}^{ij}(x,y,\alpha), \operatorname{grad} f_\alpha(x,y)\rangle + \sqrt{-1}\,\frac{\partial f_\alpha}{\partial\alpha_j}(x,y)=0,$$

for all $(x,y,\alpha)\in\{(x,y,\alpha)\in\partial D_i\times U_{\alpha^0}\mid f_\alpha(x,y)=0\}$. Moreover, by the same method as in [4], integrating the above vector fields, we can get the diffeomorphisms h_i, $i=1,\dots,s$, such that the diagrams

$$\begin{array}{ccc}
\{(x,y)\in D_i \mid f_{\alpha^0}(x,y)=0\}\times U_{\alpha^0} & \xrightarrow{h_i} & \{(x,y,\alpha)\in D_i\times U_{\alpha^0}\mid f_\alpha(x,y)=0\} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
U_{\alpha^0} & \xrightarrow{\text{id}} & U_{\alpha^0}
\end{array}$$

are commutative and $h_i(a_i,\alpha)=(a_i,\alpha)$.

Further, by Remark 1.3(b), there exist $c > 0$ and a neighborhood of α^0, which we may assume to be precisely U_{α^0}, such that

$$(1) \qquad \frac{\partial f_\alpha}{\partial y} \neq 0 \quad \text{for all } (x, y, \alpha) \in \Gamma_{\alpha^0} \cup \Omega,$$

where

$$\Gamma_{\alpha^0} := \{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0},\ |x| \geq c\}$$

and Ω is an open neighborhood of the set

$$\{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0},\ |x| = c\}.$$

On the other hand, one has

$$(2) \qquad \operatorname{grad} f_\alpha \neq 0 \quad \text{for all } (x, y, \alpha) \in \Big(\mathbb{C}^2 \setminus \bigcup_{i=1}^{s} \mathring{D}_i\Big) \times U_{\alpha^0} \quad \text{with } f_\alpha(x, y) = 0,\ |x| \leq c.$$

From (1) and (2) we conclude that there exist smooth vector fields

$$\xi^j(x, y, \alpha) = (\xi_1^j(x, y, \alpha), \xi_2^j(x, y, \alpha)),$$
$$\eta^j(x, y, \alpha) = (\eta_1^j(x, y, \alpha), \eta_2^j(x, y, \alpha)), \qquad j = 1, \ldots, n,$$

such that

$$\langle \xi^j(x, y, \alpha), \operatorname{grad} f_\alpha(x, y)\rangle + \frac{\partial f_\alpha}{\partial \alpha_j}(x, y) = 0,$$
$$\langle \eta^j(x, y, \alpha), \operatorname{grad} f_\alpha(x, y)\rangle + \sqrt{-1}\,\frac{\partial f_\alpha}{\partial \alpha_j}(x, y) = 0,$$

on the set $X := \{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0}\}$, and

$$\xi_1^j(x, y, \alpha) = 0, \qquad \eta_1^j(x, y, \alpha) = 0,$$

for all $(x, y, \alpha) \in \Gamma_{\alpha^0} \cup \Omega$.

Furthermore, the restrictions of ξ^j (resp., η^j), $j = 1, \ldots, n$, on $\Gamma^i_{\alpha^0}$, $i = 1, \ldots, s$, are precisely $\overline{\xi}^{ij}$ (resp., $\overline{\eta}^{ij}$). (We can construct such vector fields locally and then extend them over X by a smooth partition of unity.) The resulting vector fields on X are the ones we are looking for. Using them, we may follow again the method of [4] to obtain the global diffeomorphism h such that the diagram

$$\begin{array}{ccc}
\{(x, y) \mid f_{\alpha^0}(x, y) = 0\} \times U_{\alpha^0} & \xrightarrow{h} & \{(x, y, \alpha) \mid f_\alpha(x, y) = 0\} \cap \pi^{-1}(U_{\alpha^0}) \\
\downarrow \pi & & \downarrow \pi \\
U_{\alpha^0} & \xrightarrow{\text{id}} & U_{\alpha^0}
\end{array}$$

is commutative and $h|_{D_i \times U_{\alpha^0}} = h_i,\ i = 1, \ldots, s$. The theorem is proved. ∎

2.3. Remark. By definition and the above theorem, it is reasonable to say that the family $\{V_\alpha\}_{\alpha\in\mathbb{C}^n}$ satisfies *Whitney's affine conditions at infinity* provided it is good at infinity.

3. In $\mathbb{CP}^2$ we consider the family of curves

$$\overline{V}_\alpha := \{(x:y:z)\in\mathbb{CP}^2 \mid z^d f_\alpha(x/z,y/z)=0\}, \qquad \alpha\in\mathbb{C}^n.$$

Clearly, $\overline{V}_\alpha$ is the compactification of V_α. From now on we make the assumption that $B := \overline{V}_\alpha \cap \{z=0\} \subset \mathbb{CP}^2$ is a finite set, say $\{b_1,\ldots,b_m\}$, i.e., the degree d homogeneous parts of f_α are independent of α.

3.1. Lemma. *Suppose that the Milnor numbers of* V_α *are independent of* α, *i.e.*, $\sum_{a\in V_\alpha}\mu_a(V_\alpha)=\text{const}$. *Then*

$$\chi(V_\alpha)-\chi(V_{\alpha^0})=\sum_{j=1}^m[\mu_{b_j}(\overline{V}_\alpha)-\mu_{b_j}(\overline{V}_{\alpha^0})].$$

Proof. According to [3],

$$\begin{aligned}\chi(V_\alpha)&=\chi(\overline{V}_\alpha)-m\\ &=2-(d-1)(d-2)+\sum_{a\in V_\alpha}\mu_a(V_\alpha)+\sum_{j=1}^m\mu_{b_j}(\overline{V}_\alpha)-m,\end{aligned}$$

which completes the proof. ∎

From Theorem 2.1, we obtain the following corollary.

3.2. Corollary. *Under the hypotheses of Theorem* 2.1, (a), *hence* (b), *is equivalent to*

(c) $\mu_{a_i}(V_\alpha)=\text{const}$, $i=1,\ldots,s$, *and* $\mu_{b_j}(\overline{V}_\alpha)=\text{const}$, $j=1,\ldots,m$.

Proof. We need only prove (b)⇔(c). From Lemma 2.2 and Remark 1.3(c), the family $\{V_\alpha\}_{\alpha\in\mathbb{C}^n}$ is good at infinity if and only if $\chi(V_\alpha)=\text{const}$. But, by Lemma 3.1, this is equivalent to

$$\sum_{j=1}^m\mu_{b_j}(\overline{V}_\alpha)=\text{const},$$

or equivalently (using the semicontinuity of the Milnor number),

$$\mu_{b_j}(\overline{V}_\alpha)=\text{const},\qquad j=1,\ldots,m. \ ∎$$

3.3. Remark. According to [6], [1] and [2], $\mu_{b_j}(\overline{V}_\alpha)=\text{const}$, $j=1,\ldots,m$, if and only if the family $\{\overline{V}_\alpha\}_{\alpha\in\mathbb{C}^n}$ satisfies Whitney's conditions along $\{b_j\}\times\mathbb{C}^n$ at b_j. In the case of a family of affine plane curves, the equisingularity is therefore equivalent to Whitney's conditions at each common singular point of the curves (including such singular points at infinity).

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