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Conformal mapping of the domain bounded by a circular polygon with zero angles onto the unit disc

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Abstract. The conformal mapping $\omega(z)$ of a domain D onto the unit disc must satisfy the condition $|\omega(t)| = 1$ on ∂D, the boundary of D. The last condition can be considered as a Dirichlet problem for the domain D. In the present paper this problem is reduced to a system of functional equations when ∂D is a circular polygon with zero angles. The mapping is given in terms of a Poincaré series.

1. Introduction. This paper is devoted to constructing conformal maps from circular polygons to the unit disc. The general theory [7] is based on differential equations. It allows us to construct conformal mappings in closed form for special polygons [4, 12–16] having five or less vertices.

In the present paper we use a boundary value problem [6, 9] to construct the conformal mapping for polygons with zero angles. The number of vertices can be arbitrary. As is well known [9], if for a certain simply connected domain D we know a solution of the Dirichlet problem for the Laplace equation, then it is possible to derive the function conformally transforming this domain onto the unit disc.

Let $\omega(z)$ be the unknown conformal mapping and $\omega(0) = 0$, where the point $z = 0$ belongs to D. The function $\omega(z)$ has to satisfy the boundary condition $|\omega(t)| = 1$, $t \in \partial D$, where ∂D is the boundary of D. Let us introduce the auxiliary function $\varphi(z) = \ln z^{-1}\omega(z)$ which is analytic and univalent in D, and continuous in $\overline{D}$. The branch of the logarithm is fixed in an arbitrary way. The function $\varphi(z)$ satisfies the boundary condition $\operatorname{Re}\varphi(t) = -\ln|t|$, $t \in \partial D$. The last problem has a unique solution up to an arbitrary additive purely imaginary constant $i\gamma$ (see [9]). Then the conformal mapping $\omega(z)$ is determined up to the factor $\exp i\gamma$, which

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corresponds to a rotation of the unit disc. This also applies to a circular polygon.

Let us consider mutually disjoint discs $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ with boundaries $\partial D_k := \{t \in \mathbb{C} : |t - a_k| = r_k\}$ $(k = 0, 1, \ldots, n)$. Let ∂D_i be tangent to ∂D_{i+1} at the point w_{i+1} for $i = 0, 1, \ldots, n$, where the subscripts are taken modulo $n + 1$. For instance, this means $w_{n+1} = w_0$. Let the circumferences divide the plane $\overline{\mathbb{C}}$ into disjoint discs D_k and two domains G_1 and $G_2 :=$ the complement of $\bigcup_{k=0}^{n} D_k \cup G_1$. Here $\partial G_1 \cup \partial G_2 = \bigcup_{k=0}^{n} \partial D_k$, $\partial G_1 \cap \partial G_2 =: W$ is the finite set $\{w_0, w_1, \ldots, w_n\}$. The circumferences ∂D_k are oriented in the positive sense. Suppose the point $z = 0$ belongs to G_1, and the point $z = \infty$ belongs to G_2.

We solve the following boundary value problem:

$$\operatorname{Re} \varphi(t) = f(t), \qquad t \in \partial G_1, \tag{1.1}$$

where $f(t)$ is a given function continuously differentiable on ∂G_1, $\varphi(z)$ is an unknown function analytic in G_1, continuous in $\overline{G}_1$ and continuously differentiable in $\overline{G}_1 \setminus W$. The problem (1.1) is a particular case of the Hilbert boundary value problem [6]

$$\operatorname{Re} \lambda(t)\varphi(t) = f(t), \qquad t \in \partial G_1. \tag{1.2}$$

Consider the inversions with respect to $|t - a_k| = r_k$:

$$z_k^* := \frac{r_k^2}{\overline{z - a_k}} + a_k,$$

and the Möbius transformations

$$\begin{aligned} &z_{k_1 k_2}^* := (z_{k_2}^*)_{k_1}^*, \quad \text{where } k_1 = 0, 1, \ldots, n,\ k_2 = 0, 1, \ldots, n; k_2 \neq k_1, \\ &z_{k_1 k_2 \ldots k_m}^* := (z_{k_2 k_3 \ldots k_m}^*)_{k_1}^*, \qquad \text{where } k_1, \ldots, k_m = 0, 1, \ldots, n \end{aligned} \tag{1.3}$$

and $k_j \neq k_{j+1}$ for $j = 1, \ldots, m - 1$. The number m is called the *level* of (1.3). The functions (1.3) generate a Kleinian group $\mathfrak{K}$ [1, 2].

In [11] the exact solution of the Hilbert problem (1.2) has been constructed for each multiply connected circular domain. In the present paper the results of [11] are applied to the problem (1.1). As a conclusion we obtain the conformal mapping of the domain G_1 onto the unit disc. This mapping is given by the series (3.6) corresponding to the group $\mathfrak{K}$ [1, 2]. Necessary and sufficient conditions of the absolute convergence of the Poincaré series have been found in [1, 2]. We prove the uniform convergence of the series (3.6) which is closely related to the Poincaré series. This does not contradict [1, 2] because there is a difference between absolute and uniform convergence.

The Schwarz problem [6, 9] for the disc D_k (k fixed) consists in finding a function $\psi(z)$ analytic in D_k and continuous in $\overline{D}_k$ satisfying the boundary condition $\operatorname{Re}\psi(t) = g(t)$, $|t-a_k| = r_k$, where $g(t)$ is a given function Hölder continuous on $|t-a_k| = r_k$.

The Schwarz problem can be viewed as the classical Dirichlet problem with respect to the function $\operatorname{Re}\psi(z)$ harmonic in D_k. A solution of the Schwarz problem can be represented in the form [6, 9]

$$\psi(z) = \frac{1}{\pi i}\int_{\partial D_k}\frac{g(\tau)}{\tau - z}\,d\tau - \frac{1}{2\pi i}\int_{\partial D_k}\frac{g(\tau)}{\tau - a_k}\,d\tau.$$

If $g(t)$ is Hölder continuous on $\partial D_k \setminus \{w\}$, and the limit values $g(\pm w)$ exist, then the last formula holds, but $\psi(z)$ is almost bounded at w (see [6]), i.e. we have the representation

$$\psi(z) = \psi^0(z) - \frac{1}{\pi i}[g(w+0) - g(w-0)]\ln(z-w).$$

Here the function $\psi^0(z)$ is analytic in D_k and continuous in $\overline{D}_k$.

The $\mathbb{R}$-*linear problem* [5, 6, 10] for the contour $\bigcup_{k=0}^{n}\partial D_k$ consists in finding $\varphi(z)$ analytic in G_1 and G_2, and $\varphi_k(z)$ analytic in D_k with the boundary condition

$$\varphi(t) = \varphi_k(t) - \lambda\overline{\varphi_k(t)} + g(t), \quad |t-a_k| = r_k,\ k = 0,1,\ldots,n. \tag{1.4}$$

We assume that $\varphi(z)$ is continuous in $\overline{G}_1$ and $\overline{G}_2$ separately ([1]), and continuously differentiable in $(\overline{G}_1 \cup \overline{G}_2)\setminus W$ and $\varphi(0) = \varphi(\infty) = 0$. The function $\varphi_k(z)$ is continuously differentiable in $\overline{D}_k \setminus W$ and almost bounded at w_k and w_{k+1}. We shall consider the $\mathbb{R}$-linear problem with λ constant and $g(t)$ Hölder continuous in $\partial G_1 \cup \partial G_2$. If $|\lambda| < 1$ then the $\mathbb{R}$-linear problem has a unique solution [5].

2. Reducing to functional equations. Let us continue the given function $f(t)$ to ∂D_k $(k = 0,1,\ldots,n)$ in such a way that $f(t)$ is continuously differentiable in $\bigcup_{k=0}^{n}\partial D_k$. Consider in G_2 the auxiliary problem

$$\operatorname{Re}\varphi(t) = f(t), \quad t \in \partial G_2, \tag{2.1}$$

where $\varphi(z)$ is analytic in G_2, continuous in $\overline{G}_2$ and continuously differentiable in $\overline{G}_2 \setminus W$. We consider the equalities (1.1) and (2.1) as a boundary value problem with respect to $\varphi(z)$ for $z \in \overline{G}_1$ and $z \in \overline{G}_2$. Consider the problem (1.4) with $\lambda = 1$:

$$\varphi(t) = \varphi_k(t) - \overline{\varphi_k(t)} + f_k(t), \quad |t-a_k| = r_k,\ k = 0,1,\ldots,n, \tag{2.2}$$

with respect to φ and φ_k. Here $f_k(t)$ is a solution of the Schwarz problem $\operatorname{Re} f_k(t) = f(t)$, $|t-a_k| = r_k$, for the fixed disc D_k.

([1]) The limit values of $\varphi(z)$ in W for $\overline{G}_1$ and $\overline{G}_2$ can be different.

LEMMA 2.1. *The function $\varphi(z)$ is a solution of the problem* (1.1), (2.1), *if and only if $\varphi(z)$ satisfies* (2.2) *with some $\varphi_k(z)$.*

Proof. If $\varphi(z)$ is a solution of (2.2), then $\varphi(z)$ satisfies (1.1) and (2.1). Conversely, let $\varphi(z)$ be a solution of the problem (1.1), (2.1). Then the real part of (2.2) is valid. We have to construct a function $\varphi_k(z)$ such that the imaginary part of (2.2),

$$2\,\mathrm{Im}\,\varphi_k(t) = \mathrm{Im}(\varphi(t) - f_k(t)), \quad |t-a_k| = r_k, \tag{2.3}$$

also holds. Consider (2.3) as a Schwarz problem in the disc D_k with respect to $-2i\varphi_k(z)$. It follows from the general theory that the function $\varphi(z)$ contains a purely imaginary additive constant $i\gamma_1$ in G_1 and, generally speaking, another constant $i\gamma_2$ in G_2. Hence the right side of (2.3) is discontinuous at w_k and w_{k+1}. Therefore we have [6]

$$\begin{aligned}\varphi_k(z) &= \frac{1}{2\pi}\int_{\partial D_k}\frac{\mathrm{Im}(\varphi(\tau) - f_k(\tau))}{\tau - z}\,d\tau - \frac{1}{4\pi}\int_{\partial D_k}\frac{\mathrm{Im}(\varphi(\tau) - f_k(\tau))}{\tau - a_k}\,d\tau\\ &= -\frac{1}{2\pi}\,\mathrm{Im}\,\Delta\varphi(w_k)\ln(z-w_k) + \varphi_k^0(z)\end{aligned}$$

near $z = w_k$. Here $\varphi_k^0(z)$ is analytic in D_k and continuous in $\overline{D}_k$, and

$$\Delta\varphi(w_k) := \lim_{z\to w_k,\, z\in G_1}\varphi(z) - \lim_{z\to w_k,\, z\in G_2}\varphi(z).$$

An analogous representation holds near $z = w_{k+1}$. So, the function $\varphi_k(z)$ is represented in the form

$$\varphi_k(z) = \Phi_k(z) + p_k\ln(z-w_k) + q_k\ln(z-w_{k+1}), \quad z\in D_k, \tag{2.4}$$

where p_k, q_k are real constants. The branch of the logarithm is fixed in such a way that the cut connecting the points $z = w_k$ and $z = \infty$ lies in $G_2\cup W$. The function $\Phi_k(z)$ is analytic in D_k, continuous in $\overline{D}_k$ and continuously differentiable in $\overline{D}_k\setminus W$. So, assuming that $\varphi(z)$ satisfies (1.1) and (2.1) we have constructed $\varphi_k(z)$ in such a way that (2.2) holds.

The lemma is proved.

In order to reduce (2.2) to a system of functional equations we introduce the function

$$\Omega(z) = \begin{cases}\varphi_k(z) + \sum\limits_{\substack{m=0\\ m\neq k}}^{n}\overline{\varphi_m(z_m^*)} + f_k(z), & z\in D_k,\\[2ex] \varphi(z) + \sum\limits_{m=0}^{n}\overline{\varphi_m(z_m^*)}, & z\in G_1\cup G_2,\end{cases}$$

where z_m^* is the inversion with respect to the circumference $|t-a_m| = r_m$. Using (2.2) one can see that $\Omega(z)$ is continuous in $\overline{\mathbb{C}}$ except in the set W, where it is almost bounded. By Liouville's theorem, $\Omega(z) = \text{constant}$. Let us calculate this constant:

$$\Omega(z) = \Omega(0) = \varphi(0) + \sum_{m=0}^{n} \overline{\varphi_m(0_m^*)}, \quad z \in \overline{\mathbb{C}}.$$

From the definition of $\Omega(z)$ in D_k and (2.4) we obtain

$$\Phi_k(z) = -\sum_{\substack{m=0\\ m\neq k}}^{n} [\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(0_m^*)}] - f_k(z) + \varphi(0) + \overline{\Phi_k(0_k^*)} - \alpha_k(z), \tag{2.5}$$

where

$$\begin{aligned} \alpha_k(z) &= p_k \ln(z-w_k)/\overline{0_k^* - w_k} + q_k \ln(z - w_{k+1})/\overline{0_k^* - w_{k+1}} \\ &\quad - \sum_{\substack{m=0\\ m\neq k}}^{n} \left[p_m \ln\left[\frac{\overline{z_m^* - w_m}}{\overline{0_m^* - w_m}}\right] + q_m \ln\left[\frac{\overline{z_m^* - w_{m+1}}}{\overline{0_m^* - w_{m+1}}}\right]\right]. \end{aligned} \tag{2.6}$$

The points w_m and w_{m+1} belong to ∂D_m. Hence $(w_m)_m^* = w_m$ and $(w_{m+1})_m^* = w_{m+1}$. Let us transform the expression appearing in (2.6)

$$\ln(\overline{z_m^* - w_m}) = \ln(\overline{z_m^* - (w_m)_m^*}) = \ln \frac{r_m^2(z-w_m)}{(z-a_m)(a_m - w_m)}.$$

Similarly,

$$\ln(\overline{z_m^* - w_{m+1}}) = \ln \frac{r_m^2(z-w_{m+1})}{(z-a_m)(a_m - w_{m+1})}.$$

It follows from (2.5) that the function $\alpha_k(z)$ has to be continuous in $\overline{D}_k$. On the other hand, the logarithms appearing in (2.6) have jumps along the curve connecting the points $z = w_k$, $z = a_k$ and $z = w_{k+1}$. This contradiction can be overcome only if $p_k = q_{k-1}$, $k = 0, 1, \ldots, n$. Write the functions $\alpha_k(z)$ in the form $\alpha_k(z) = \sum_{j=0}^{n} p_j H_k^j(z)$, where

$$\begin{aligned} H_k^k(z) &= \ln \frac{a_k(a_k - w_k)(z - a_{k-1})}{r_k^2 a_{k-1}}, \\ H_k^{k+1}(z) &= \ln \frac{a_k(a_k - w_{k+1})(z - a_{k+1})}{r_k^2 a_{k+1}}, \\ H_k^j(z) &= \ln \frac{\overline{0_j^* - w_j^*} \cdot \overline{0_{j-1}^* - w_j^*}}{\overline{z_j^* - w_j^*} \cdot \overline{z_{j-1}^* - w_j^*}}, \quad j \neq k, k+1. \end{aligned} \tag{2.7}$$

The function $\varphi(z)$ is analytic in G_1 and G_2, continuous in $\overline{G}_1$ and $\overline{G}_2$. Hence Cauchy's theorem can be applied: $\int_{\partial G_1} \varphi(z)\,dz + \int_{\partial G_2} \varphi(z)\,dz = 0$. Let us rewrite the last equality in the form

$$\sum_{k=0}^{n} \int_{\partial D_k} \varphi(z)\,dz = 0. \tag{2.8}$$

From the definition of $\Omega(z)$ and the representation (2.4) we have

$$(2.9)\quad \varphi(z) = \varphi(0) - \sum_{m=0}^{n} [\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(0_m^*)}] - \sum_{m=0}^{n} [p_m \ln(\overline{z_m^* - w_m}) + p_{m+1} \ln(\overline{z_m^* - w_{m+1}})], \qquad z \in \overline{G}_1 \cup \overline{G}_2.$$

It follows from the original definition of $\ln(z - w_m)$ that the cut connecting the points $z = w_m$ and $z = a_m$ corresponds to the function $\ln(\overline{z_m^* - w_m})$. The cut from $z = w_{m+1}$ to $z = a_m$ corresponds to $\ln(\overline{z_m^* - w_{m+1}})$. Using (2.9) let us calculate the integral (2.8). Since the terms with $m = k$ under the logarithm sign in (2.9) are equal to πi, we have $\sum_{k=0}^{n}(p_k + p_{k+1}) = 0$, or simply

$$(2.10)\qquad \sum_{k=0}^{n} p_k = 0.$$

Consider the Banach space C of functions continuous in $\partial G_1 \cup \partial G_2 = \bigcup_{k=0}^{n} \partial D_k$ with norm $\|f\| := \max_{0\le k\le n} \max_{\partial D_k} |f(t)|$ corresponding to uniform convergence. Let us introduce the closed subspace $C^+ \subset C$ consisting of the functions analytically continuable to all discs D_k. For one disc we obtain the classical space of functions analytic in the disc and continuous in its closure [7].

Consider the system (2.5) as an equation

$$(2.11)\qquad \Phi = A\Phi + F$$

in C^+, where the operator A is defined by

$$A\Phi(z) := - \sum_{\substack{m=0\\ m\ne k}}^{n} [\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(0_m^*)}], \qquad z \in D_k\ (k = 0, 1, \ldots, n).$$

Here $\Phi \in C^+$ and $\Phi(z) := \Phi_k(z)$ in $\overline{D}_k$.

3. Solution of functional equations

Lemma 3.1. *The homogeneous equation* (2.11), $\Phi = A\Phi$, *has only the zero solution.*

Lemma 3.2. *The non-homogeneous equation* (2.11), $\Phi = A\Phi + F$, *has a unique solution in* C^+ *for each* $F \in C^+$.

Proof (of Lemmas 3.1 and 3.2). Let $\Phi(z)$ be a solution of (2.11) in C^+. Hence the functions $\Phi_k(z) := \Phi(z)$ are analytic in D_k and continuous in $\overline{D}_k$. Let us introduce the function

$$(3.1)\qquad \psi(z) := - \sum_{m=0}^{n} [\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(0_m^*)}],$$

analytic in $G_1 \cup G_2$ and continuous in $\overline{G_1 \cup G_2}$. From (2.11) we have

$$\psi(t) = \Phi_k(t) - \overline{\Phi_k(t)} - F(t) + \overline{\Phi_k(0_k^*)}, \qquad |t - a_k| = r_k.$$

Let us rewrite the last equality in the form

$$\operatorname{Re}\psi(t) = -\operatorname{Re}F(t) + \operatorname{Re}\Phi_k(0_k^*), \qquad |t - a_k| = r_k, \tag{3.2}$$

$$2\operatorname{Im}\Phi_k(t) = \operatorname{Im}[\psi(t) + F(t) + \Phi_k(0_k^*)], \qquad |t - a_k| = r_k. \tag{3.3}$$

Since the functions $F(t)$ and $\psi(t)$ are continuous in ∂D_k, from (3.2) we get $\operatorname{Re}\Phi_k(0_k^*) = \operatorname{Re}\Phi_{k+1}(0_{k+1}^*)$. Therefore $\operatorname{Re}\Phi_k(0_k^*) =$ constant for each k. If $F(t) \equiv 0$ then from (3.2) we have $\psi(z) \equiv$ constant. But $\psi(0) = 0$, hence $\psi(z) \equiv 0$. It follows from the definition (3.1) and the decomposition theorem [3] that each function $\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(0_m^*)}$ is a constant. Therefore, using the relation $\Phi = A\Phi$ we have $\Phi_m(z) \equiv 0$ for each m. Lemma 3.1 is proved.

By [6] and Lemma 3.1 the non-homogeneous problem (3.2) with respect to $\psi(z)$ analytic in $G_1 \cup G_2$ has a unique solution up to an additive constant $c + i\gamma$. Hence the problem (3.3) with respect to $\Phi_k(z)$ has a unique solution up to an arbitrary additive constant which vanishes in (2.11).

Lemma 3.2 is proved.

THEOREM 3.1 ([8]). *If the equation*

$$U = \lambda AU + F \tag{3.4}$$

has a unique solution for each $|\lambda| \le 1$, *then the series*

$$U = \sum_{m=0}^{\infty} A^m F \tag{3.5}$$

converges in C^+.

THEOREM 3.2. *The equation* (2.11) *has a unique solution in* C^+ *which can be found by the method of successive approximations converging in* C^+.

Proof. The proof is based on Theorem 3.1. Let $|\lambda| = 1$. Then equation (3.4) reduces to the same equation with $\lambda = 1$ by the change of variable $z = \sqrt{\lambda}\,Z$, $a_k = \sqrt{\lambda}\,A_k$, $\Omega_k(Z) = \Psi_k(z)$. According to Lemma 3.2 the last problem has a unique solution.

Let $|\lambda| < 1$. We introduce the function $\psi(z) := -\lambda \sum_{m=0}^{n} [\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(0_m^*)}]$. By (3.4) we have the following $\mathbb{R}$-linear problem:

$$\psi(t) = \Phi_k(t) - \lambda\overline{\Phi_k(t)} - f_k(t), \qquad |t - a_k| = r_k, \ k = 0, 1, \ldots, n,$$

with respect to $\psi(z)$ analytic in $G_1 \cup G_2$ and continuous in $\overline{G_1 \cup G_2}$, and $\Phi_k(z)$ analytic in D_k and continuous in $\overline{D}_k$. The last problem has a unique solution, since $|\lambda| < 1$ (see [5]). Using Theorem 3.1 we conclude the proof.

Applying the method of successive approximations to the system (2.5) let us find $\overline{\Phi_k(z_k^*)} - \overline{\Phi_k(0_k^*)} = \mathfrak{B}_k h(z)$, where $h(z) := h_k(z)$ for $|z - a_k| \le r_k$,

$h_k(z) := f_k(z) + \alpha_k(z)$, the operator $\mathfrak{B}_k$ has the form

$$(3.6)\quad \mathfrak{B}_k h(z) := -[\overline{h_k(z_k^*)} - \overline{h_k(0_k^*)}] + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} [h_{k_1}(z_{k_1 k}^*) - h_{k_1}(0_{k_1 k}^*)]$$

$$- \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} [\overline{h_{k_2}(z_{k_2 k_1 k}^*)} - \overline{h_{k_2}(0_{k_2 k_1 k}^*)}] + \ldots,$$

$$|z - a_k| > r_k.$$

It follows from the definitions of $h_k(z)$ and $\alpha_k(z)$ that

$$(3.7)\qquad \overline{\Phi_k(z_k^*)} - \overline{\Phi_k(0_k^*)} = \mathfrak{B}_k f(z) + \sum_{j=0}^{n} p_j D_k^j(z),$$

where $D_k^j(z) := \mathfrak{B}_k H_j(z)$, $H_j(z) := H_k^j(z)$ for $|z - a_k| \leq r_k$. The functions $H_k^j(z)$ have the form (2.7).

Substituting $z = 0_k^*$ into (2.5) we obtain

$$(3.8)\quad \Phi_k(0_k^*) = \sum_{\substack{m=0 \\ m \neq k}}^{n} \Big[(\mathfrak{B}_m f)(0_k^*) + \sum_{j=0}^{n} p_j D_m^j(0_k^*)\Big] - f_k(0_k^*) + \varphi(0)$$

$$+ \overline{\Phi_k(0_k^*)} - \sum_{j=0}^{n} p_j H_k^j(0_k^*).$$

The real parts of the relations (3.8) together with (2.10) constitute a real system of $n+2$ linear algebraic equations with respect to $n+2$ unknown values $\operatorname{Re}\varphi(0)$, $p_0, p_1, \ldots, p_n$. After solving the system using (2.9) we arrive at the formula

$$(3.9)\quad \varphi(z) = \operatorname{Re}\varphi(0) + i\operatorname{Im}\varphi(0) - \sum_{m=0}^{n} \Big(\mathfrak{B}_m f(z) + \sum_{j=0}^{n} p_j \mathfrak{B}_m H_j(z)\Big)$$

$$+ i \sum_{m=0}^{n} (p_m \ln(\overline{z_m^* - w_m}) + p_{m+1} \ln(\overline{z_m^* - w_{m+1}})),$$

$$z \in G_1 \cup G_2,$$

where $i\operatorname{Im}\varphi(0)$ is an arbitrary imaginary constant. We prove that the system (2.10), Re (3.8) always has a unique solution. The system (2.10), Re (3.8) corresponding to the homogeneous system ($f = 0$ and $i\operatorname{Im}\varphi(0) = 0$) has only the zero solution, because in the opposite case we would get non-zero functions (2.4) from which it is impossible to get $\varphi(z) \equiv 0$ using (2.9) and taking into account the equality $i\operatorname{Im}\varphi(0) = 0$. The last assertion is based on the decomposition theorem [3]. So we have proved the following

THEOREM 3.3. *The solution of the problem* (1.1), (2.1) *has the form* (3.9) *where* $i\,\mathrm{Im}\,\varphi(0)$ *is an arbitrary constant. The numbers* $\mathrm{Re}\,\varphi(0), p_0, p_1, \ldots, p_n$ *are defined from the system* (2.10), $\mathrm{Re}\,(3.8)$ *of linear algebraic equations, which always has a unique solution.*

Let us apply the last theorem to construct the conformal mapping $\omega(z)$ from Section 1. Assume that $f(t) = -\ln|t|$. Then $f_k(z) = -\ln z$, $z \in \overline{D}_k$, where the branch of the logarithm is such that the cut connecting $z = 0$ and $z = \infty$ lies in $G_1 \cup W \cup G_2$.

In (3.9) let us calculate

$$\begin{aligned}(3.10)\qquad \sum_{m=0}^{n} \mathfrak{B}_m f(z) &= \sum_{m=0}^{n} [\overline{\ln z_m^*} - \overline{\ln 0_m^*}] - \sum_{m=0}^{n} \sum_{\substack{k_1=0\\k_1\neq m}}^{n} [\overline{\ln z_{k_1m}^*} - \overline{\ln 0_{k_1m}^*}] + \ldots \\ &= \ln \prod_{m=0}^{n} \frac{\overline{z_m^*}}{\overline{0_m^*}} \prod_{m=0}^{n} \prod_{\substack{k_1=0\\k_1\neq m}}^{n} \frac{0_{k_1m}^*}{z_{k_1m}^*} \prod_{k=0}^{n} \prod_{\substack{k_1=0\\k_1\neq m}}^{n} \prod_{\substack{k_2=0\\k_2\neq k_1}}^{n} \frac{\overline{z_{k_2k_1m}^*}}{\overline{0_{k_2k_1m}^*}} \cdots\end{aligned}$$

Finally, let us describe a finite algorithm for constructing the conformal mapping in analytic form.

(i) Construct $H^j(z) := H_k^j(z)$ for $|z - a_k| \le r_k$, where the functions $H_k^j(z)$ have the form (2.7).

(ii) Construct $D_k^j(z) := (\mathfrak{B}_k H^j)(z)$ and $\mathfrak{B}_k f(z)$ according to (3.6) and (3.10).

(iii) Solve the real system of $n + 2$ linear algebraic equations

$$\begin{cases} \displaystyle\sum_{j=0}^{n} p_j \Big[\sum_{\substack{m=0\\m\neq k}}^{n} \mathrm{Re}(D_m^j(0_k^*) - H_k^j(0_k^*))\Big] + \mathrm{Re}\,\varphi(0) \\ \qquad = \mathrm{Re}\Big[f_k(0_k^*) - \displaystyle\sum_{\substack{m=0\\m\neq k}}^{n} (\mathfrak{B}_m f)(0_k^*)\Big], \quad k = 0, 1, \ldots, n, \\ \displaystyle\sum_{j=0}^{n} p_j = 0, \end{cases}$$

with respect to $p_0, p_1, \ldots, p_n$, $\mathrm{Re}\,\varphi(0)$.

(iv) The required conformal mapping is $\omega(z) = z \exp \varphi(z)$, where the function $\varphi(z)$ has the form (3.9), $i\,\mathrm{Im}\,\varphi(0)$ is an arbitrary pure imaginary constant.

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