ANNALES
POLONICI MATHEMATICI
LXVIII.3 (1998)

Commutators of diffeomorphisms of a manifold with boundary

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Abstract. A well known theorem of Herman–Thurston states that the identity component of the group of diffeomorphisms of a boundaryless manifold is perfect and simple. We generalize this result to manifolds with boundary. Remarks on C^r-diffeomorphisms are included.

1. Introduction. The aim of this paper is to extend a well known theorem of M. Herman and W. Thurston to manifolds with boundary. Let us fix the notation. Let M be an n-dimensional smooth manifold, and $\mathrm{Diff}^r(M)_0$ denote the totality of C^r-diffeomorphisms of M which are isotopic to the identity through a compactly supported isotopy. It is clear that (as a result of local contractibility) $\mathrm{Diff}^r(M)_0$ is the identity component in the C^r topology iff M is compact.

Theorem 1 (Herman, Thurston, Mather). *If M is a boundaryless manifold, and $1 \le r \le \infty$, $r \ne n+1$, then $\mathrm{Diff}^r(M)_0$ is a simple group.*

D. B. A. Epstein [2] demonstrated for a large class of transitive groups of homeomorphisms that the perfectness yields the simplicity (the converse statement is trivial). By appealing to a difficult K.A.M. theory Herman [5] proved that $\mathrm{Diff}^\infty(T^n)_0$ is perfect, T^n being the n-dimensional torus. Next, Thurston announced in [12] (for the proof, see [1]) that the result of Herman can be extended to an arbitrary manifold by making use of Kan simplices. Finally, J. N. Mather in [7] showed the assertion for any positive integer r not equal to $n+1$ by a completely different argument.

The case of manifolds with boundary has been considered by A. Masson [6] who extended the results of F. Sergeraert [11]. By making use of a different method than the two above they proved

1991 *Mathematics Subject Classification*: Primary 57R50; Secondary 58D05.
Key words and phrases: Group of diffeomorphisms, simplicity, perfectness, manifold with boundary, fixed point theory.

THEOREM 2. *If M is a manifold with boundary, and $\mathrm{Diff}^{\infty}_{\infty}(M)$ is the group of all C^∞ diffeomorphisms which are infinitely tangent to the identity on the boundary, then $\mathrm{Diff}^{\infty}_{\infty}(M)_0$ (defined as in the previous theorem) is a perfect group.*

It is interesting to consider the group of all diffeomorphisms of a manifold with boundary. Such a group cannot be simple for obvious reasons, but the problem of its perfectness is still meaningful. The following result can be viewed as an extension of the above theorems.

THEOREM 3. *Let M be an n-dimensional manifold with boundary, $n \geq 2$. Then $\mathrm{Diff}^{\infty}(M)_0$ is perfect.*

The proof consists in a modification and a slight correction of an argument from Epstein [3] which, in turn, extends Mather [7, I]. The case of C^r diffeomorphisms is considered in the last section, and a partial analogue of Theorem 3 is announced.

Throughout, all manifolds are supposed to be C^∞, connected and second countable.

2. Notation and preliminary results

2.1. *Factorization property.* Let us recall the following

PROPOSITION 1. *Let $1 \leq r \leq \infty$. If $f \in \mathrm{Diff}^r(M)_0$ is sufficiently near the identity, and $\mathrm{supp}(f) \subset U_1 \cup \ldots \cup U_r$, where U_i are open balls or open half-balls, then there is a factorization $f = f_s \ldots f_1$ such that $\mathrm{supp}(f_j) \subset U_{i(j)}$ for $j = 1, \ldots, s$.*

For the proof see [8, Lemma 3.1]. The proof is still valid in the case of manifolds with boundary.

The factorization property enables us to reduce our considerations to the case $M = \mathbb{R}^n$ or $M = \mathbb{R}^n_+ = \{x_n \geq 0\}$. We shall deal with the case $M = \mathbb{R}^n_+$ exclusively as the case $M = \mathbb{R}^n$ has been solved in [3]. From now on we adopt the notation

$$\mathrm{Diff}(n)_0 = \mathrm{Diff}^{\infty}(\mathbb{R}^n_+)_0.$$

Next for any finite interval U in $\mathbb{R}^n_+$ let $\mathrm{Diff}_U(n)_0$ be the totality of elements of $\mathrm{Diff}(n)_0$ compactly supported in U.

2.2. *Fixed point theory.* We shall appeal to the following well-known theorem.

THEOREM 4 (Schauder–Tychonoff). *Let C be a convex and compact set in a locally convex topological vector space E. Then every continuous map $F : C \to C$ has a fixed point.*

This will be applied to the space of C^∞ mappings.

2.3. *The space of C^∞ mappings*. For any map $u : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ of class C^∞ we have the sequence of seminorms

$$\|u\|_r = \sup_x \|D^r u(x)\|, \qquad r = 1, 2, \ldots,$$

where $D^r u(x)$ denotes the rth differential of u at x, and $\|\cdot\|$ is the usual norm on the space of r-linear mappings between normed vector spaces. These seminorms may be infinite in general. However, we restrict ourselves to the space of all C^∞ maps $u : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ having support contained in a fixed finite interval U. Then $\|\cdot\|_r$ become norms. Notice that the C^r topology on this space is defined by the norm $\|\cdot\|_r$, and the C^∞ topology is defined by the sequence of these norms. Of course $\|u + \mathrm{id}\|_r = \|u\|_r$ for $r \geq 2$.

For each C^∞ map u of $\mathbb{R}^n_+$ we define

$$M_r(u + \mathrm{id}) = \max(\|u\|_1, \ldots, \|u\|_r)$$

for any $r \geq 1$.

Proposition 2. *Let U be a finite interval of $\mathbb{R}^n_+$, and $\mathcal{C} \subset \mathrm{Diff}_U(n)_0$ with the C^∞ topology. Then $\mathcal{C}$ is compact if and only if it is bounded and closed.*

This fact is well known for $\mathbb{R}^n$. It is still true for the half-space $\mathbb{R}^n_+$. Indeed, the proofs of Corollaries 2 and 3 of Theorem 51, Ch. VII in [10] are exactly the same in this case.

Let us recall the formulae for the differential of composed maps. Let $u, v : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be of class C^∞. Then we have (cf. [3])

$$D(u \circ v) = Du \circ v.Dv \tag{2.3.1}$$

and

$$\begin{aligned} D^r(u \circ v) = {} & (D^r u \circ v)(Dv \times \ldots \times Dv) + (Du \circ v)(D^r v) \\ & + \sum C(i, j_1, \ldots, j_i)(D^i u \circ v)(D^{j_1} v \times \ldots \times D^{j_i} v), \end{aligned} \tag{2.3.2}$$

where $C(i, j_1, \ldots, j_i)$ is an integer independent of n, $1 < i < r$, $j_l > 0$, and $j_1 + \ldots + j_i = r$. It follows that there is $s \in \{1, \ldots, i\}$ with $j_s \geq 2$.

2.4. *Rolling-up operators $\Psi_{i,A}$*. First we give preparatory definitions. For a given integer A greater than 1 denote by $D_i = D_{i,A}$ the closed interval $[-2, 2]^i \times [-2A, 2A]^{n-1-i} \times [0, 2A] \subset \mathbb{R}^n_+$ for $i = 0, \ldots, n-1$, and

$$D_n = [-2, 2]^{n-1} \times [0, 2].$$

Then we have

$$D_n \subset D_{n-1} = [-2, 2]^{n-1} \times [0, 2A] \subset \ldots \subset D_0 = [-2A, 2A]^{n-1} \times [0, 2A].$$

We shall make use of Mather's operator $\Psi_{i,A}$ which rolls up a diffeomorphism along the x_i-axis, $i = 1, \ldots, n-1$. Since in our case $\Psi_{i,A}$ does not act transversely to the boundary, the operation of rolling up diffeomorphisms will be applied $n-1$ times only while Mather applies it n times.

In our case we just take the restriction of Mather's original operator $\Psi_{i,A}$ to (a sufficiently small C^1 neighbourhood of the identity in) $\mathrm{Diff}(n)_0$. This restriction will be still denoted by $\Psi_{i,A}$. Basic features of $\Psi_{i,A}$ are the following.

(2.4.1) The domain of $\Psi_{i,A}$ is a sufficiently small C^1 neighbourhood of the identity in $\mathrm{Diff}_{D_{i-1}}(n)_0$, and the range of $\Psi_{i,A}$ is a C^1 neighbourhood of the identity in $\mathrm{Diff}_{D_i}(n)_0$.

(2.4.2) It is continuous with respect to the C^∞ topology, and it preserves the identity.

(2.4.3) There is $\tau_i \in \mathrm{Diff}(n)_0$ such that for any $u \in \mathrm{dom}(\Psi_{i,A})$, $\tau_i u$ and $\tau_i \Psi_{i,A}(u)$ are conjugate.

(2.4.4) There are a universal constant $K > 1$ and a constant $\delta > 0$ depending on A such that $M_1(u) < \delta$ yields

$$M_1(\Psi_{i,A}(u)) \le KAM_1(u).$$

(2.4.5) Let $r \ge 2$. There are constants $\delta > 0$, $K > 1$, $C_r > 1$ such that

$$\|\Psi_{i,A}(u)\|_r \le KA\|u\|_r + C_r(M_{r-1}(u)),$$

whenever $u \in \mathrm{dom}(\Psi_{i,A})$ and u satisfies $M_1(u) < \delta$. Here K is a universal constant, δ depends on A but not on r, and C_r depends on A, r.

(2.4.6) For $r \ge 2$ there are constants $\delta > 0$, $K'_r > 1$, and an admissible polynomial F_r such that

$$\|\Psi_{i,A}(u)\|_r \le K'_r A\|u\|_r + F_r(M_{r-1}(u))$$

for any $u \in \mathrm{dom}(\Psi_{i,A})$ with $M_1(u) < \delta$, where δ depends on A but not on r, K'_r depends on r but not on A, and F_r depends on A, r.

The constants and polynomials in (2.4.4)–(2.4.6) do not depend on i.

A polynomial of one variable is said to be *admissible* iff it has no constant or linear term and its coefficients are non-negative integers. The explicit definition of the $\Psi_{i,A}$ and τ_i will be given in Section 4 in order to check that they are applicable to our case.

Remark. The condition (2.4.5) has been omitted in [3], and the constant K'_r in (2.4.6) has been claimed there to be independent of r. This seems to be incorrect (see Section 4). However, the conditions (2.4.5) and (2.4.6) are sufficient to have the proof in the boundaryless case [3] as well as in our case (see Section 3).

3. Proof of Theorem 3. The basic idea of the proof is to apply the operators $\Psi_{i,A}$ only tangently to the boundary. Next, by making use of an additional operation we replace a diffeomorphism in question by another one with support away from the boundary.

Let $f \in \mathrm{Diff}(n)_0$ and, without loss of generality, we assume that f is supported in $D' = [-2,2]^{n-1} \times [0,1]$. This may be accomplished by making use of a conjugation. Moreover, we may and do suppose that f is sufficiently near the identity in the C^∞ topology.

We set

$$\mathcal{C} = \{u - \mathrm{id} \in \mathrm{Diff}_{D_n}(n) - \mathrm{id} : \|u\|_r \le a_r,\ r \ge r_0\},$$

where $r_0 \ge n+1$ is a fixed integer, and the sequence a_r will be specified in due course. Then $\mathcal{C}$ is convex and, in view of Proposition 2, compact.

Note that choosing the constant a_{r_0} small enough we may assume that each diffeomorphism u such that $u - \mathrm{id} \in \mathcal{C}$ is in a sufficiently small C^1 neighbourhood of id. In fact, by integrating over a finite interval we have, for such u,

$$M_1(u) \le C_1 \|u\|_{r_0}, \qquad \|u\|_s \le C_s \|u\|_{r_0} \tag{3.1}$$

for any $s = 2, \ldots, r_0$, with some universal constants C_i.

For any $u - \mathrm{id} \in \mathcal{C}$ we define

$$u_0 = \widetilde{A} f u \widetilde{A}^{-1},$$

where $\widetilde{A} \in \mathrm{Diff}(n)_0$ such that $\widetilde{A}$ is the multiplication by A in a neighbourhood of D_n and $\mathrm{supp}\,\widetilde{A} \subset [-2A-1, 2A+1]^{n-1} \times [0, 2A+1]$.

Observe that in view of (2.3.1) one has

$$\begin{aligned} M_1(u_0) &= \|u_0 - \mathrm{id}\|_1 = \|fu - \mathrm{id}\|_1 \\ &\le \|(f - \mathrm{id})u\|_1 + \|u - \mathrm{id}\|_1 \\ &\le M_1(f)(1 + M_1(u)) + M_1(u) \\ &= M_1(f) + M_1(u) + M_1(f)M_1(u). \end{aligned} \tag{3.2}$$

Next we let

$$u_1 = \Psi_{1,A}(u_0), \ldots, u_{n-1} = \Psi_{n-1,A}(u_{n-2}),$$

so that u_i is supported in D_i.

Let $\xi : [0,\infty) \to [0,1]$ be an arbitrary C^∞ function such that $\xi = 1$ on $[0,1]$ and $\xi = 0$ on $[2,\infty)$. We define u_n by

$$u_n - \mathrm{id} = \xi(x_n)(u_{n-1} - \mathrm{id}).$$

By shrinking the initial C^1 neighbourhood if necessary, u_n is a diffeomorphism. By definition $\mathrm{supp}\, u_n \subset D_n$. Observe that $u_{n-1} = u_n$, that is, $u_{n-1}u_n^{-1} = \mathrm{id}$, in a neighbourhood of the boundary.

Suppose that for some universal choice of the a_r we have $u_n - \mathrm{id} \in \mathcal{C}$. Then we can conclude the proof as follows. Due to (2.4.2) and Theorem 4 there exists $u - \mathrm{id} \in \mathcal{C}$ such that $u_n - \mathrm{id} = u - \mathrm{id}$, i.e. $u_n = u$. Now by the definition of u_0 and (2.4.3),

$$[fu] = [u_0] = [u_1] = \ldots = [u_{n-1}]$$

and, consequently,

$$[f] = [u_{n-1}u^{-1}]$$

in the abelianization of $\mathrm{Diff}(n)_0$. But $[u_{n-1}u^{-1}] = e$ by Theorem 1, and $[f] = e$. Thus f can be written as a product of commutators.

Now we have to choose a suitable sequence a_r.

By the definition of the norm $\|\cdot\|_r$ and (2.3.2) we have

$$\text{(3.3)} \quad \begin{aligned} \|u_0\|_r &= A^{1-r}\|fu\|_r \\ &\le A^{1-r}(\|f\|_r + \|u\|_r)(1 + M_1(f) + M_1(u))^r + F_1(M_{r-1}(u)), \end{aligned}$$

where F_1 is an admissible polynomial which depends on A, r and f.

Next, proceeding by induction, and making use of (2.4.4) and (2.4.6) we get

$$\text{(3.4)} \qquad \|u_{n-1}\|_r \le K_r A^{n-1}\|u_0\|_r + F_2(M_{r-1}(u_0))$$

for each $u - \mathrm{id} \in \mathcal{C}$ with $M_1(u_0) < \delta$. Here K_r depends on r and is independent of A, and F_2 is an admissible polynomial depending on r, A and f. By choosing f close enough to id, by using (3.2), and by taking $\delta/3$ instead of δ, we may and do assume that (3.4) is fulfilled whenever $M_1(u) < \delta$.

Likewise, from (2.4.4) and (2.4.5) we obtain the existence of a universal constant K', and a constant C'_r depending on A, r such that

$$\text{(3.5)} \qquad \|u_{n-1}\|_r \le K' A^{n-1}\|u_0\|_r + C'_r M_{r-1}(u)$$

whenever $M_1(u) < \delta$.

Finally, we can estimate u_n by

$$\text{(3.6)} \qquad \|u_n\|_r \le K''\|u_{n-1}\|_r + C''_r(M_{r-1}(u_{n-1})),$$

where K'' is a universal constant, and C''_r is a constant depending on r and ξ. This inequality is an immediate consequence of the Leibniz formula. Next, from (3.6) follows the existence of a constant K''_r which depends on r and ξ but not on A such that

$$\text{(3.7)} \qquad \|u_n\|_r \le K''_r\|u_{n-1}\|_r.$$

In fact, (3.7) follows from (3.6) and the estimates

$$M_1(u_{n-1}) \le C'''_1\|u_{n-1}\|_r, \qquad \|u_{n-1}\|_s \le C'''_s\|u_{n-1}\|_r$$

for $s = 2, \ldots, r-1$, where C'''_i are independent of A. This is easily seen by repeated integration with respect to x_i, $i = 1, \ldots, n-1$, as the interval D_{n-1} is of length 4 in the directions of these x_i.

By combining (3.2), (3.3), (3.5) and (3.6) one can deduce that there are a universal constant $K > 1$ and a constant C_r which depends on r, A, f and ξ such that

$$\|u_n\|_r \le KA^{n-r}(\|f\|_r + \|u\|_r) + C_r(M_{r-1}(u)) \tag{3.8}$$

whenever $M_1(u) < \delta$. Likewise, (3.1), (3.3), (3.4) and (3.7) imply the existence of a constant K_r depending on r, f and ξ but independent of A such that

$$\|u_n\|_r \le K_r A^{n-r}(\|f\|_r + \|u\|_r) \tag{3.9}$$

if $M_1(u) < \delta$.

Now let us fix $r_0 \ge n+1$ and choose A so large that

$$K_{r_0} A^{n-r_0} \le 1/2.$$

Then setting $a_{r_0} = \|f\|_{r_0}$ we deduce from (3.9) that $\|u\|_{r_0} \le a_{r_0}$ implies $\|u_n\|_{r_0} \le a_{r_0}$. We stress that $\|f\|_{r_0}$ may be chosen sufficiently small. Next we proceed by induction. Suppose a_s are defined for $s < r$. By enlarging A if necessary we have $KA^{n-r} \le 1/4$ for any $r \ge r_0$, and (3.8) assumes the form

$$\|u_n\|_r \le \frac{1}{4}(\|f\|_r + \|u\|_r) + b_r$$

for some constant b_r which depends on $a_s, s<r$, and C_r. Therefore it suffices to put $a_r = \|f\|_r + 2b_r$, and the induction follows.

This completes the proof.

Remarks. (1) The original method of proof of Theorem 1 (in case $r = \infty$) is much more difficult and longer than the method of Mather–Epstein presented above. Furthermore, it seems impossible to make use of the Herman–Thurston method in case of manifolds with boundary as the diffeomorphism group of the torus is a starting point. On the other hand, in the case of the leaf preserving diffeomorphism group of a foliated manifold (which is also a nontransitive group of diffeomorphisms) the first method works after some essential modifications (cf. [9]) while it is unclear how to apply the second method.

(2) Our proof breaks down in case $\dim M = 1$. Indeed, we had to use the rolling-up operators Ψ_i *along* the boundary. Moreover, due to results of K. Fukui [4] one cannot expect that Theorem 3 holds in this case. Namely,

$$H_1(\mathrm{Diff}^\infty([0,1])_0) = \text{abelianization of } \mathrm{Diff}^\infty([0,1])_0 = \mathbb{R}^2$$

and

$$H_1(\mathrm{Diff}^\infty_r(\mathbb{R},0)_0) = \text{abelianization of } \mathrm{Diff}^\infty_r(\mathbb{R},0)_0 = \mathbb{R}^{r+1},$$

where $\mathrm{Diff}^\infty_r(\mathbb{R},0)$ is the group of all C^∞ diffeomorphisms of $\mathbb{R}$ which are r-tangent to the identity at 0.

4. Properties of the operators $\Psi_{i,A}$. The construction of $\Psi_{i,A}$ is the following (cf. [7, I]). For $i = 1, \ldots, n-1$ we write C_i for $\mathbb{R}^{i-1} \times S^1 \times \mathbb{R}_+^{n-i}$, i.e. C_i is the set of n-tuples $(x_1, \ldots, \vartheta_i, \ldots, x_n)$ where ϑ_i is a real defined mod 1, and $x_n \geq 0$. Let $\pi : \mathbb{R}^n \to C_i$ be the canonical projection. Notice that there is the obvious action of S^1 on C_i, and denote by $\mathcal{G}_i$ the group of equivariant diffeomorphisms with respect to this action.

Now let $u \in \mathrm{Diff}_{D_{i-1}}(n)_0$ and $\|u - \mathrm{id}\|_0 < 1/2$. We define a diffeomorphism $\Gamma_i(u) : C_i \to C_i$ in the following way. For $\vartheta \in C_i$ we choose $x \in \mathbb{R}_+^n$ such that $\pi(x) = \vartheta$ and $x_i < -2A$. Next we choose an integer $N > 0$ so large that

$$(T_i u)^N(x)_i > 2A.$$

Here $T_i : \mathbb{R}_+^n \to \mathbb{R}_+^n$ denotes the time-one transformation of the vector field $\partial_i = \partial/\partial x_i$, that is, the unit translation along the x_i-axis. It is visible that we can arrange so that $N = [8A + 4]$. We define

$$\Gamma_i(u)(\vartheta) = \pi(T_i u)^N(x).$$

In particular, if $|x_j| > 2A$ for some $j \neq i$ then $\Gamma_i(u)(\vartheta) = \vartheta$. One can also check that the operation $u \mapsto \Gamma_i(u)$ is continuous.

For $\Gamma_i(u)$ sufficiently small there is a unique S^1-equivariant diffeomorphism $\widetilde{\Gamma_i(u)} : C_i \to C_i$ such that

$$\widetilde{\Gamma_i(u)}|\{\vartheta_i = 0\} = \Gamma_i(u)|\{\vartheta_i = 0\}.$$

Let $g = \widetilde{\Gamma_i(u)}^{-1}\Gamma_i(u)$.

On C_i we introduce a bump function $\zeta : C_i \to \mathbb{R}$ such that $\zeta = 1$ on a neighbourhood of 0, and $\zeta = 0$ on a neighbourhood of 1/2. Shrinking the neighbourhood of the identity in $\mathrm{Diff}(n)_0$ if necessary, one can lift $\widetilde{\Gamma_i(u)} - \mathrm{id} : C_i \to C_i$ to $\gamma : C_i \to \mathbb{R}^n$ such that $\pi\gamma = \widetilde{\Gamma_i(u)} - \mathrm{id}$ and $\|\gamma\|_0 < 1/2$. Then we set

$$g_0(\vartheta) = \pi(\zeta(\vartheta)\gamma(\vartheta)) + \vartheta \tag{4.1}$$

for $\vartheta \in C_i$. Again, by reducing the C^1 neighbourhood one can assume that $g_0 : C_i \to C_i$ is a diffeomorphism. We set $g_1 = g_0^{-1}g$.

Finally, putting $E_i^- = \{x \in \mathbb{R}_+^n : -1 < x_i < 0\}$ and $E_i^+ = \{x \in \mathbb{R}_+^n : 1/2 < x_i < 3/2\}$ we define $v = \Psi_{i,A}(u)$ as the unique diffeomorphism of $\mathbb{R}_+^n$ characterized by the equalities:

$$v|\mathbb{R}_+^n - (E_i^- \cup E_i^+) = \mathrm{id},$$
$$\pi v|E_i^+ = g_0\pi|E_i^+, \qquad \pi v|E_i^- = g_1\pi|E_i^-,$$
$$v(E_i^+) = E_i^+, \quad v(E_i^-) = E_i^-.$$

It is visible that $\Gamma_i(v) = g_0 g_1 = g$. Therefore

$$\Gamma_i(u)\Gamma_i(v)^{-1} = \widetilde{\Gamma_i(u)} \in \mathcal{G}_i,$$

and this is a key relation in order to obtain the property (2.4.3) (see Prop. 3). Furthermore, it is easily checked that (2.4.1) and (2.4.2) are satisfied.

Following [7] we give the definition of τ_i, $i \le n-1$.

Choose a function $\varrho \in C^\infty(\mathbb{R})$ such that $\varrho = 1$ on $[-2A, 2A]$ and $\operatorname{supp}(\varrho) \subset [-2A-1, 2A+1]$. Next, define $\widetilde{\varrho} \in C^\infty(\mathbb{R}^n_+)$ by $\widetilde{\varrho}(x) = \varrho(x_1) \ldots \ldots \varrho(x_n)$. We set

$$\tau_i(x) = \exp(\widetilde{\varrho}(x)\partial_i), \quad i = 1, \ldots, n-1.$$

Next let ϕ_i, $i = 1, \ldots, n-1$, be diffeomorphisms characterized by the following properties:

(a) $\operatorname{dom}(\phi_i) = \{x \in \mathbb{R}^n_+ : |x_j| \le 2A, i \ne j\}$,
(b) $\phi_{i*}(\partial_i) = \widetilde{\varrho}\partial_i$,
(c) $\phi|D_0 = \mathrm{id}$.

Note that

$$\tau_i = \phi_i T_i \phi_i^{-1}, \quad i = 1, \ldots, n-1, \tag{4.2}$$

since $T_i = \exp(\partial_i)$, $\tau_i = \exp(\widetilde{\varrho}\partial_i)$.

The following is a repetition of [7, I] and [3], and we reproduce only a sketch of the proof.

PROPOSITION 3. *Let $i = 1, \ldots, n-1$ and let $u, v \in \operatorname{Diff}_{D_0}(n)_0$. If u and v are sufficiently C^1-close to the identity and $\Gamma_i(v)\Gamma_i(u)^{-1} \in \mathcal{G}_i$ then $\lambda\tau_i u\lambda^{-1} = \tau_i v$, where $\lambda \in \operatorname{Diff}(n)_0$.*

Proof. Following Mather we put

$$\Lambda(x) = (T_i v)^N (T_i u)^{-N}(x),$$

where N is a positive integer so large that $(T_i u)^{-N}(x)_i < 2A$. For any $x \in \mathbb{R}^n_+$ such an N exists, and the definition is independent of N (large). Observe that $\operatorname{supp}\Lambda \subset \{x \in \mathbb{R}^n_+ : x_i > -2A, |x_j| < 2A,\ j \ne i\}$. It follows from the definition

$$\Lambda T_i u \Lambda^{-1} = T_i v \tag{4.3}$$

and

$$\Gamma_i(v)\Gamma_i(u)^{-1}\pi = \pi\Lambda. \tag{4.4}$$

The assumption $\Gamma_i(v)\Gamma_i(u)^{-1} \in \mathcal{G}_i$ and (4.4) yield the existence of $\Lambda_j : \mathbb{R}^{n-1}_+ \to \mathbb{R}$, $j \ne i$, satisfying

$$\Lambda_j(x_i') = \Lambda(x)_j$$

for $x_i > 2A$, where $x_i' = (x_1, \ldots, \widehat{x}_i, \ldots, x_n)$. Let

$$\Lambda'(x_i') = (\Lambda_1(x_i'), \ldots, \Lambda_n(x_i')),$$

and let Λ_t' be an isotopy from Λ' to id supported in $\{|x_j| < 2A, j \ne i\}$.

Now the definition of λ is the following (in the second and third line the ith coordinate is written last)

$$\lambda(x) = \begin{cases} \phi_i \Lambda \phi_i^{-1}, & x \in \operatorname{im}(\phi_i), \\ (\Lambda'(x_i'), \exp(\widetilde{\varrho}(\Lambda(x)_i - x_i)\partial_i)(x)_i), & 2A \le x_i \le 2A+1, \\ (\Lambda_t'(x_i'), x_i), & 2A+1 \le x_i \le 2A+2, \\ x & \text{otherwise.} \end{cases}$$

The proof is completed by the observation that conjugating by ϕ_i the equality (4.3) and making use of (4.2) imply the desired equality $\lambda \tau_i u \lambda^{-1} = \tau_i v$.

Proof of (2.4.5) *and* (2.4.6). As in [3] for each $u \in \operatorname{dom}(\Psi_{i,A})$ in a sufficiently small C^1 neighbourhood of id there are a universal constant K and an admissible polynomial F_r which depends on A, r such that

$$\|g\|_r \le K^r A \|u\|_r + F_r(M_{r-1}(u)).$$

Next we can estimate g_0 (defined by (4.1)) by means of the Leibniz formula. As the coefficients of this formula cannot be estimated by K^r, and due to the fact that we know nothing about $\|\zeta\|_r$, we only have

$$\|g_0\|_r \le K_1 \|g\|_r + C_r(M_{r-1}(g)),$$

where K_1 is a universal constant, and the large constant C_r depends on r but not on A. The last term cannot be included in any admissible polynomial. Bearing in mind that $g = id$ in a neighbourhood of $\{\theta_i = 1/2\}$, and making use of analogues of (3.1) for g one can transform the above inequality into

$$\|g_0\|_r \le K_r' \|g\|_r,$$

where the constant K_r' depends on r and is independent of A. The rest of the proof is the same as in [7, I] or [3].

5. Commutators of C^r diffeomorphisms. In this case we have the following partial analogue of Theorem 1.

THEOREM 5. *Let M be an n-dimensional manifold with boundary*, $n \ge 2$, *and let a positive integer* $r \ne n$, $n+1$. *Then* $\operatorname{Diff}^r(M)_0$ *is a perfect group.*

The proof for $r < n$ follows closely the lines of [7, II]. Similarly to the proof of Theorem 3 we unroll and roll-up diffeomorphisms along the boundary, and not transversely to it. The estimations of [7, II] are still valid in our case, and suffice to prove the result under the assumption $r < n$ (in [7, II] one has $r \le n$). In contrast to Theorem 3 no modifications are needed.

On the other hand, in case $r > n+1$ the proof is similar to that of Theorem 3. In fact, it is a little simpler as the condition (2.4.5) is superfluous. Also the end of the proof follows [7, I] rather than [3]. Recall that the case $r = n+1$ is also unknown for boundaryless manifolds (cf. [7, III]). This is the only reason that our method breaks down for $r = n+1$. The case $r = n$ cannot be covered by our method because we use Mather's operators only $n - 1$ times.

In order to give some examples of non-perfectness we introduce the following notation. Denote by $\mathrm{Diff}^r_s(M)$ the group of all diffeomorphisms of class C^r on M which are s-tangent to the identity on the boundary, where $0 \le s \le r \le \infty$. The condition of s-tangency means that the s-jets of a diffeomorphism and the identity are equal at any point of the boundary. Suppose $1 \le s < r \le \infty$. We show that $\mathrm{Diff}^r_s(M)$ is not a perfect group. Indeed, let $(U, x_1, \ldots, x_n)$ be a local coordinate system at $p \in \partial M$ such that $U = \{x_n \ge 0\}$. For any diffeomorphisms $f, g \in \mathrm{Diff}^r_s(M)$ we have, in view of (2.3.2),

$$D^{s+1}(f \circ g)(p) = D^{s+1}f(p) + D^{s+1}g(p)$$

and

$$D^{s+1}f^{-1}(p) = -D^{s+1}f(p).$$

Therefore if we choose $h \in \mathrm{Diff}^r_s(M)$ such that $D^{s+1}h(p) \ne 0$, the above equalities yield that h cannot be in the commutator subgroup as

$$D^{s+1}[f, g](p) = 0.$$

The same reasoning is true for $\mathrm{Diff}^r_s(M)_0$, the identity component of the group $\mathrm{Diff}^r_s(M)$.

We have as well, due to (2.3.1) and a similar argument, that neither $\mathrm{Diff}^r_0(M)$ nor $\mathrm{Diff}^r_0(M)_0$ is perfect.

References

[1] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. 53 (1978), 174–227.

[2] D. B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. 22 (1970), 165–173.

[3] —, *Commutators of C^∞-diffeomorphisms*, Comment. Math. Helv. 59 (1984), 111–122.

[4] K. Fukui, *Homologies of the group* $\mathrm{Diff}^\infty(\mathbb{R}^n, 0)$ *and its subgroups*, J. Math. Kyoto Univ. 20 (1980), 475–487.

[5] M. R. Herman, *Sur le groupe des difféomorphismes du tore*, Ann. Inst. Fourier (Grenoble) 23 (2) (1973), 75–86.

[6] A. Masson, *Sur la perfection du groupe des difféomorphismes d'une variété à bord infiniment tangents à l'identité sur le bord*, C. R. Acad. Sci. Paris Sér. A 285 (1977), 837–839.

[7] J. N. Mather, *Commutators of diffeomorphisms*, Comment. Math. Helv. I 49 (1974), 512–528; II 50 (1975), 33–40; III 60 (1985), 122–124.

[8] J. Palis and S. Smale, *Structural stability theorems*, in: Proc. Sympos. Pure Math. 14, Amer. Math. Soc., 1970, 223–231.

[9] T. Rybicki, *The identity component of the leaf preserving diffeomorphism group is perfect*, Monatsh. Math. 120 (1995), 289–305.
[10] L. Schwartz, *Analyse Mathématique*, Hermann, Paris 1967.
[11] F. Sergeraert, *Feuilletages et difféomorphismes infiniment tangents à l'identité*, Invent. Math. 39 (1977), 253–275.
[12] W. Thurston, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. 80 (1974), 304–307.

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Reçu par la Rédaction le 4.12.1995