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A singular initial value problem for the equation $u^{(n)}(x) = g(u(x))$

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Abstract. We consider the problem of the existence of positive solutions u to the problem

$$u^{(n)}(x) = g(u(x)),$$
$$u(0) = u'(0) = \ldots = u^{(n-1)}(0) = 0 \ (g \geq 0,\ x > 0,\ n \geq 2).$$

It is known that if g is nondecreasing then the Osgood condition

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{g(s)}\right]^{1/n} ds < \infty$$

is necessary and sufficient for the existence of nontrivial solutions to the above problem. We give a similar condition for other classes of functions g.

1. Introduction. In this paper we consider the equation

$$u^{(n)}(x) = g(u(x)) \quad (x > 0), \tag{1.1}$$

where $g : (0,\infty) \to (0,\infty)$, $n \in \mathbb{N}$, with initial condition

$$u(0) = u'(0) = \ldots = u^{(n-1)}(0) = 0. \tag{1.2}$$

If $g(0) = 0$, then $u \equiv 0$ is a solution to the problem (1.1), (1.2). We are interested in the existence of solutions $u \in C[0,M] \cap C^{(n)}(0,M)$, $0 < M \leq \infty$, such that $u(x) > 0$ for $x > 0$, which we call *nontrivial solutions*. For $n = 1$ this problem is classical and leads to the well-known Osgood condition, for $n = 2$ it is also standard. The case of $n = 3$ was considered in [5]. When g is a nondecreasing continuous function, the problem has been solved for any n (see [2], [4]). In that case, a necessary and sufficient condition for the

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existence of nontrivial continuous solutions is

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{g(s)}\right]^{1/n} ds < \infty \quad (\delta > 0).$$

We are going to obtain a similar condition for some other classes of functions g satisfying the following conditions:

(1.3) $g \in C(0,\infty)$, $g \ge 0$;

(1.4) $x^m g(x)$ is bounded as $x \to 0+$ for some $m \ge 0$.

We will rather deal with an integral formulation of the original problem which reads

$$u(x) = \frac{1}{(n-1)!}\int_0^x (x-s)^{n-1} g(u(s))\, ds, \tag{1.5}$$

and we will seek for nontrivial continuous solutions $u \ge 0$ of this integral equation. We now present our main results which will be proved in Section 4.

THEOREM 1.1. *Let g satisfy* (1.3), (1.4). *Then the condition*

$$\int_0^\delta g(s) s^{-(n-2)/(n-1)}\, ds < \infty \tag{1.6}$$

is necessary for the existence of nontrivial solutions of the equation (1.5).

Before stating our further results we introduce some auxiliary definitions and notations.

Let g satisfy (1.3), (1.4). We put

$$g^\star(x) = x^{-m} \sup_{0<s<x} s^m g(s) \quad \text{for } x > 0.$$

We easily see that $g(x) \le g^\star(x)$ for $x > 0$ and $x^m g^\star(x)$ is nondecreasing. We define two function classes K_n and $K_n^\star$ ($n \ge 2$) as follows:

$$K_n = \{g : g \text{ satisfies (1.3), (1.4), (1.6) and } x^m g(x) \text{ is nondecreasing}\},$$

$$K_n^\star = \left\{g : g \text{ satisfies (1.3), (1.4), (1.6) and } \sup_{0<x} \frac{G^\star(x)}{G(x)} < \infty\right\},$$

where

$$G(x) = \int_0^x g(s) s^{-(n-2)/(n-1)}\, ds, \quad G^\star(x) = \int_0^x g^\star(s) s^{-(n-2)/(n-1)}\, ds.$$

We easily observe that K_n contains nondecreasing functions and that $K_n \subset K_n^\star$. In contrast to K_n the class $K_n^\star$ admits functions which can oscillate at the origin like $|\sin(1/x)|$ (see [5]).

Let u be a nontrivial solution of (1.5). We define

$$v(x) = u'(u^{-1}(x)) = \frac{1}{(u^{-1})'(x)} \quad (x > 0),$$

for which we establish some a priori estimates.

Theorem 1.2. *Let $g \in K_n^\star$ and $n \geq 2$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 x^{n-2} \left(\frac{v(x)^{n-1}}{x^{n-2}} \right)^{n/(n-1)} \leq \int_0^x (x-s)^{n-2} g(s) s^{-(n-2)/(n-1)} \, ds \leq c_2 x^{n-2} \left(\frac{v(x)^{n-1}}{x^{n-2}} \right)^{n/(n-1)}$$

for $x > 0$.

As a consequence of the above estimates we obtain the existence result for (1.1), (1.2).

Theorem 1.3. *Let $g \in K_n^\star$ and $n \geq 2$. Then the problem* (1.1), (1.2) *has a continuous solution u such that $u(x) > 0$ for $x > 0$ if and only if*

$$\int_0^\delta \phi(s)^{-1/(n-1)} \, ds < \infty \quad (0 < \delta), \tag{1.7}$$

where

$$\phi(x) = x^{n-2} \left\{ \frac{\int_0^x (x-s)^{n-2} g(s) s^{-(n-2)/(n-1)} \, ds}{x^{n-2}} \right\}^{(n-1)/n} \quad (x > 0). \tag{1.8}$$

Remark 1.1. Observe that the existence of nontrivial solutions to (1.1), (1.2) depends only on the behaviour of g in a neighbourhood of zero. Therefore the assumptions on g could be reformulated to take this fact into account.

We also give a condition for the blow-up of solutions, which means that there exists $0 < M < \infty$ such that $\lim_{x \to M-} u(x) = \infty$.

Theorem 1.4. *Let $g \in K_n^\star$ and $n \geq 2$. A continuous solution u to* (1.1), (1.2) *positive for $x > 0$ blows up if and only if*

$$\int_0^\infty \phi(s)^{-1/(n-1)} \, ds < \infty$$

where ϕ is given in (1.8).

We call the condition (1.7) the *generalized Osgood condition* for the problem (1.1), (1.2). Such conditions for convolution type integral equations $u(x) = \int_0^x k(x-s) g(u(s)) \, ds$ have been widely studied (see [1], [6]). Unfortunately, only the case of nondecreasing functions g was considered.

2. Auxiliary lemmas. Let $f:(0,\infty)\to(0,\infty)$ be a continuous locally integrable function. We will use some properties of the functions

$$w(x)=cx^{k-1}+\int_0^x (x-s)^{k-1}f(s)\,ds \quad (x>0),$$

where $k\ge 2$ and $c\ge 0$ is a constant.

LEMMA 2.1. *For any $x>0$,*

$$(k-1)^{-k}w'(x)^{k-1}\le cw(x)^{k-2}+\int_0^x (w(x)-w(s))^{k-2}f(s)\,ds \le (k-1)^{-1}w'(x)^{k-1}.$$

Proof. We notice first that w' is nondecreasing. So the mean value theorem gives the right inequality immediately.

To prove the left inequality we first introduce the Borel measure $d\mu(s)=f(s)ds+c\delta_0$ $(s\ge 0)$. Thus w can be rewritten in the form

$$w(x)=\int_0^x (x-s)^{k-1}\,d\mu(s).$$

Moreover, we see that

$$w(x)-w(s)\ge\int_0^s \{(x-t)^{k-1}-(s-t)^{k-1}\}\,d\mu(t).$$

Since

$$(x-t)^{k-1}-(s-t)^{k-1}\ge (x-s)(x-t)^{k-2} \quad \text{for } 0\le s\le x,$$

we get

$$w(x)-w(s)\ge (x-s)I(s), \quad \text{where} \quad I(s)=\int_0^s (x-t)^{k-2}\,d\mu(t).$$

Noting that $I'(s)=(x-s)^{k-2}f(s)$ and $w(x)\ge cx^{k-1}$, $I(0)=cx^{k-2}$, we obtain

$$\begin{aligned}cw(x)^{k-2}&+\int_0^x (w(x)-w(s))^{k-2}f(s)\,ds\\ &\ge cw(x)^{k-2}+\int_0^x I(s)^{k-2}(x-s)^{k-2}f(s)\,ds\\ &\ge cw(x)^{k-2}+\frac{1}{k-1}(I(x)^{k-1}-I(0)^{k-1})\ge\frac{1}{k-1}I(x)^{k-1}.\end{aligned}$$

Finally, since $I(x)=\frac{1}{k-1}w'(x)$, we get our assertion.

LEMMA 2.2. *Let μ be a Borel measure on $[0,\infty)$. Then the function*

$$\Phi_{k,n}(x)=\frac{(\int_0^x (x-s)^n\,d\mu(s))^{n+k}}{(\int_0^x (x-s)^{n+k}\,d\mu(s))^n}\quad (x>0),$$

where $k,n\in\mathbb{N}$, is nondecreasing.

Proof. By differentiation we verify that for $k=1$ and any $n\in\mathbb{N}$,

$$\operatorname{sign}\Phi'_{1,n}(x)=\operatorname{sign}\Big(\int_0^x (x-s)^{n-1}\,d\mu(s)\cdot\int_0^x (x-s)^{n+1}\,d\mu(s)-\Big(\int_0^x (x-s)^n\,d\mu(s)\Big)^2\Big).$$

Hence the Schwarz inequality yields the required assertion in that case. Now by an inductive argument based on the relation

$$\Phi_{k+1,n}(x)=[\Phi_{k,n}(x)]^{(n+k+1)/(n+k)}[\Phi_{1,n+k}(x)]^{n/(n+k)}$$

we obtain the required assertion for any $k,n\in\mathbb{N}$.

We set

$$z(x)=\int_0^x (x-s)^{n-2}g(s)s^{-(n-2)/(n-1)}\,ds\qquad (x>0,\ n\ge 2).\tag{2.1}$$

LEMMA 2.3. *Let $g\in K_n$ and $w(x)=xz^{(n-1)}(x)+(m+1)z^{(n-2)}(x)$, $w(0)=0$. Then w is nondecreasing and continuous. Moreover, there exist constants $c_1,c_2>0$ such that*

$$\frac{c_1}{(n-k-1)!}\int_0^x (x-s)^{n-k-1}\,dw(s)\le (xz)^{(k)}(x)\le\frac{c_2}{(n-k-1)!}\int_0^x (x-s)^{n-k-1}\,dw(s)\qquad (x>0)\tag{2.2}$$

for $k=0,1,\ldots,n-1$.

Proof. Define $h(x)=x^{m+2}z^{(n-1)}(x)$ for $x>0$ and $h(0)=0$. By our assumptions on g the function h is continuous and nondecreasing. Note also that

$$z^{(n-2)}(x_2)-z^{(n-2)}(x_1)=\int_{x_1}^{x_2} s^{-m-2}h(s)\,ds$$
$$=-\frac{1}{m+1}(x_2z^{(n-1)}(x_2)-x_1z^{(n-1)}(x_1))+\frac{1}{m+1}\int_{x_1}^{x_2} s^{-m-1}\,dh(s)$$

for any $0 < x_1 < x_2$, from which it follows immediately that w is nondecreasing. Let

$$\gamma = \lim_{x\to 0+} w(x) = \lim_{x\to 0+} xz^{(n-1)}(x).$$

Then we easily see that γ must be 0. Thus w is continuous at 0 and everywhere else. To get (2.2) we first notice that using the Leibniz rule we can find some constants $c_1, c_2 > 0$ such that

$$c_1 w(x) \le (xz)^{(n-1)}(x) \le c_2 w(x)$$

for $x > 0$. This gives the required assertion immediately if we just observe that $w(x) = \int_0^x dw(s)$ for $x > 0$.

LEMMA 2.4. *Let $g \in K_n^\star$. Then there exists a constant $c > 0$ such that*

$$\int_0^x (x-s)^{n-2} g(s)\phi(s)^{-1/(n-1)}\,ds \le c\phi(x) \quad (x > 0), \tag{2.3}$$

where ϕ is defined in (1.8).

Proof. First we consider $g \in K_n$ and define

$$I_k(x) = \frac{1}{k!}\int_0^x (x-s)^k g(s)\phi(s)^{-1/(n-1)}\,ds \quad (x \ge 0)$$

for $k = 0, 1, \ldots, n-2$.

For z defined in (2.1) we have

$$\phi(x)^{-1/(n-1)} = x^{-(n-2)/(n-1)} z(x)^{-1/n} x^{(n-2)/n}$$

and

$$(n-2)!I_k(x) = \frac{1}{k!}\int_0^x (x-s)^k z^{(n-1)}(s) z(s)^{-1/n} s^{(n-2)/n}\,ds \quad (x > 0)$$

for $k = 0, 1, \ldots, n-2$.

We shall prove that there exist constants $c_0, c_1, \ldots, c_{n-2} > 0$ such that

$$I_k(x) \le c_k z^{(n-k-2)}(x) z(x)^{-1/n} x^{(n-2)/n} \quad (x > 0) \tag{2.4}$$

for $k = 0, 1, \ldots, n-2$.

Our assertion will follow from (2.4) with $k = n-2$. Set

$$H_k(x) = (xz^{(n-k-2)}(x))^{n-1}(xz(x))^{-k-1},$$
$$J_k(x) = [(xz)^{(n-k-2)}(x)]^{n-1}(xz(x))^{-k-1} \quad (x > 0),$$

$k = 0, 1, \ldots, n-2$. Using the Leibniz rule and monotonicity properties of the derivatives of z, we can observe that

$$xz^{(k)}(x) \le (xz)^{(k)}(x) \le (k+1)xz^{(k)}(x) \quad (x > 0)$$

for $k = 0, 1, \ldots, n-2$. Hence

$$(n-k-1)^{-(n-1)} J_k(x) \le H_k(x) \le J_k(x) \quad (x > 0) \tag{2.5}$$

for $k = 0, 1, \ldots, n-2$.

Lemmas 2.2 and 2.3 yield the following monotonicity property of the functions J_k:

there exist constants $c_0, c_1, \ldots, c_{n-2}$ *such that*

$$J_k(s) \le c_k J_k(x) \quad \textit{for } k = 0, 1, \ldots, n-2 \textit{ and } 0 < s < x.$$

It follows from (2.5) that the functions H_k have the same property. Now, we are ready to prove (2.4) by induction. Using the above property for H_0 we obtain

$$\begin{aligned} I_0(x) &= \frac{1}{(n-2)!} \int_0^x z^{(n-1)}(s) z(s)^{-1/n} s^{(n-2)/n}\, ds \\ &\le \frac{1}{(n-2)!} \int_0^x z^{(n-1)}(s) (z^{(n-2)}(s))^{-(n-1)/n} H_0(s)^{1/n}\, ds \\ &\le nc_0 \frac{1}{(n-2)!} H_0(x)^{1/n} (z^{(n-2)}(x))^{1/n} \\ &= nc_0 \frac{1}{(n-2)!} z^{(n-2)}(x) z(x)^{-1/n} x^{(n-2)/n}. \end{aligned}$$

Applying the inductive assumption and the relation

$$(xz(x))^{-1/n} = (z^{(n-3-k)}(x))^{-\frac{n-1}{n(k+2)}} x^{-\frac{n-1}{n(k+2)}} H_{k+1}(x)^{\frac{1}{n(k+2)}},$$

where $k = 0, 1, \ldots, n-3$ and $x > 0$, we get

$$\begin{aligned} I_{k+1}(x) &= \int_0^x I_k(s)\, ds \le c_k \int_0^x z^{(n-2-k)}(s) (sz(s))^{-1/n} s^{(n-1)/n}\, ds \\ &\le c_k H_{k+1}(x)^{\frac{1}{n(k+2)}} x^{\frac{n-1}{n}(1-\frac{1}{k+2})} \\ &\quad \times \int_0^x z^{(n-2-k)}(s) (z^{(n-3-k)}(s))^{-\frac{n-1}{n(k+2)}}\, ds \\ &\le \frac{n(k+2)}{nk+n+1} c_k z^{(n-3-k)}(x) z(x)^{-1/n} x^{(n-2)/n}, \end{aligned}$$

which ends the proof of (2.4).

If $g \in K_n^\star$, then we employ the fact that $g^\star \in K_n$. From the definitions of $g^\star$ and ϕ it follows that there exists a constant $c > 0$ such that for $\phi^\star$ corresponding to $g^\star$ we have

$$\phi(x) \le \phi^\star(x) \le c\phi(x) \quad (x > 0).$$

Hence

$$I_{n-2}(x)=\int_0^x (x-s)^{n-2}g(s)\phi(s)^{-1/(n-1)}\,ds \le c^{1/(n-1)}\int_0^x (x-s)^{n-2}g^\star(s)\phi^\star(s)^{-1/(n-1)}\,ds$$

for $x>0$. Therefore our assertion follows from the inequality in (2.3) just proved.

3. A perturbed integral equation. Since g admits a singularity at 0, we are going to obtain a solution u of (1.1), (1.2) as a limit of solutions u_ε of more regular problems. We perturb the equation (1.5) to

$$u_\varepsilon(x)=\varepsilon x^{n-1}+\int_0^x (x-s)^{n-1}g(u_\varepsilon(s))\,ds \quad (x>0), \tag{3.1}$$

where $\varepsilon\ge 0$ ($n\ge 2$). Let $u_\varepsilon\ge 0$ ($\varepsilon\ge 0$) be a continuous solution of (3.1) such that $u_\varepsilon>0$ for $x>0$. To give some a priori estimates for u_ε we introduce an auxiliary function

$$v_\varepsilon(x)=u_\varepsilon'(u_\varepsilon^{-1}(x))=\frac{1}{(u_\varepsilon^{-1})'(x)} \quad (x>0)$$

and show that it satisfies a useful integral inequality stated in the following lemma.

LEMMA 3.1. *Let g satisfy* (1.3), (1.4). *Then for any* $\varepsilon\ge 0$,

$$(n-1)^{-n}v_\varepsilon(x)^{n-1}\le \varepsilon x^{n-2}+\int_0^x (x-s)^{n-2}g(s)\frac{1}{v_\varepsilon(s)}\,ds \le (n-1)^{-1}v_\varepsilon(x)^{n-1} \quad (x>0).$$

Proof. This follows from Lemma 2.1 if we take $f(s)=g(u_\varepsilon(s))$ ($s>0$) and then substitute $\tau=u_\varepsilon(s)$.

From this lemma we obtain the following a priori estimates for v_ε.

LEMMA 3.2. *Let $g\in K_n^\star$. Then there exist constants $c_1,c_2>0$ such that for any* $\varepsilon\ge 0$,

$$c_1(\varepsilon x^{n-2}+\phi(x))^{1/(n-1)}\le v_\varepsilon(x)\le c_2(\varepsilon x^{n-2}+\phi(x))^{1/(n-1)} \quad (x>0). \tag{3.2}$$

Proof. Define

$$w(x)=\varepsilon x^{n-2}+\int_0^x (x-s)^{n-2}g(s)\frac{1}{v_\varepsilon(s)}\,ds.$$

Since $w(x)/x^{n-2}$ is nondecreasing, it follows from Lemma 3.1 that

$$\frac{v_\varepsilon(s)^{n-1}}{s^{n-2}} \le (n-1)^{n-1}\frac{v_\varepsilon(x)^{n-1}}{x^{n-2}} \qquad (0 < s \le x).$$

Therefore,

$$\begin{aligned}(3.3)\qquad w(x) &\ge \int_0^x (x-s)^{n-2} g(s)\frac{1}{v_\varepsilon(s)}\,ds \\ &\ge \frac{1}{n-1} v_\varepsilon(x)^{-1} x^{(n-2)/(n-1)} \int_0^x (x-s)^{n-2} g(s) s^{-(n-2)/(n-1)}\,ds.\end{aligned}$$

Since $\varepsilon x^{n-2} \le w(x) \le (n-1)^{-1} v_\varepsilon(x)^{n-1}$, the left inequality in (3.2) follows from (3.3). Now, by the left inequality and the definition of w we have

$$w(x) \le c\Big(\varepsilon x^{n-2} + \int_0^x (x-s)^{n-2} g(s)\phi(s)^{-1/(n-1)}\,ds\Big),$$

where $c > 0$ is some constant. Thus the right inequality is a consequence of Lemmas 2.2 and 3.1.

As an immediate consequence of Lemma 3.2 we obtain the following estimates for u_ε^{-1}.

Corollary 3.3. *Let $g \in K_n^\star$. Then there exist constants $c_1, c_2 > 0$ such that for any $\varepsilon \ge 0$,*

$$\begin{aligned}(3.4)\qquad c_1 \int_0^x (\varepsilon s^{n-2} + \phi(s))^{-1/(n-1)}\,ds &\le u_\varepsilon^{-1}(x) \\ &\le c_2 \int_0^x (\varepsilon s^{n-2} + \phi(s))^{-1/(n-1)}\,ds \qquad (x > 0).\end{aligned}$$

Now we study the local existence of solutions to the original problem. We begin with the consideration of the perturbed equation (3.1) with $\varepsilon > 0$, for which we prove the following existence result.

Lemma 3.4. *Let $g \in K_n^\star$. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the perturbed equation* (3.1) *has a continuous solution $u_\varepsilon(x) > 0$ for $x > 0$ defined locally on $[0, \delta_\varepsilon]$.*

Proof. We introduce the operator

$$Tw(x) = (n-1)\varepsilon x^{n-2} + (n-1)\int_0^x (x-s)^{n-2} g(\widetilde{w}(s))\,ds,$$

$$\widetilde{w}(s) = \int_0^s w(t)\,dt,$$

considered in the cone $(n-1)\varepsilon x^{n-2} \le w(x) \le 2(n-1)\varepsilon x^{n-2}$ $(x>0)$. Since for $\widetilde{w}$ and its inverse $\widetilde{w}^{-1}$ we have the estimates

$$\varepsilon x^{n-1} \le \widetilde{w}(x) \le 2\varepsilon x^{n-1} \quad (x>0),$$

$$\left(\frac{y}{2\varepsilon}\right)^{1/(n-1)} \le \widetilde{w}^{-1}(y) \le \left(\frac{y}{\varepsilon}\right)^{1/(n-1)} \quad (y>0),$$

we can find $\delta_\varepsilon > 0$ such that for any $0 < x < \delta_\varepsilon$,

$$\int_0^x g(\widetilde{w}(s))\,ds \le \int_0^\delta g(s)\frac{1}{w(\widetilde{w}^{-1}(s))}\,ds \le c_\varepsilon \int_0^\delta g(s) s^{-(n-2)/(n-1)}\,ds < \varepsilon, \tag{3.5}$$

where

$$\delta = \widetilde{w}(\delta_\varepsilon) \quad \text{and} \quad c_\varepsilon = \frac{1}{n-1}2^{(n-2)/(n-1)}\varepsilon^{-1/(n-1)}.$$

Thus T maps the cone $K_\varepsilon = \{w : (n-1)\varepsilon x^{n-2} \le w(x) \le 2(n-1)\varepsilon x^{n-2}, 0 < x < \delta_\varepsilon\}$ into itself. We can also verify that all the functions of the family $\{Tw : w \in K_\varepsilon\}$ are equicontinuous. So $T : K_\varepsilon \to K_\varepsilon$ is compact in $C[0,\delta_\varepsilon]$ topology. Now, by the Schauder fixed point theorem, T has a fixed point w_ε. Taking $u_\varepsilon'(x) = w_\varepsilon(x)$ $(0 < x < \delta_\varepsilon)$, we obtain the required solution as $u_\varepsilon(x) = \int_0^x w_\varepsilon(s)\,ds$.

4. Proofs of theorems. In this section we give the proofs of the theorems of Section 1.

Proof of Theorem 1.1. Let u be a nontrivial solution of (1.5). In view of Lemma 2.1 we have

$$(n-1)^{-n}u'(x)^{n-1} \le \int_0^x \{u(x)-u(s)\}^{n-2} g(u(s))\,ds \le (n-1)^{-1}u'(x)^{n-1} \quad (x>0),$$

which can be rewritten for $v(x) = u'(u^{-1}(x))$ as

$$(n-1)^{-n}v(x)^{n-1} \le \int_0^x (x-s)^{n-2} g(s)\frac{1}{v(s)}\,ds \le (n-1)^{-1}v(x)^{n-1} \quad (x>0). \tag{4.1}$$

Since

$$\int_0^\delta g(s)\frac{1}{v(s)}\,ds = \int_0^\delta g(s)s^{-(n-2)/(n-1)}\left(\frac{v(s)^{n-1}}{s^{n-2}}\right)^{-1/(n-1)}\,ds,$$

our result follows from the fact that $v(x)^{n-1}/x^{n-2} \to 0$ as $x \to 0$, easily obtained from (4.1).

Proof of Theorem 1.2. The required estimates follow from Lemma 3.2 immediately.

Proof of Theorem 1.3. Since $\int_0^x \frac{1}{v(s)}\,ds = u^{-1}(x) < \infty$, the necessity part follows immediately from the estimates given in Theorem 1.2.

Now, we prove the sufficiency. We first notice that if the condition (1.7) is satisfied then the a priori estimates for $u_\varepsilon^{-1}(x)$ given in Corollary 3.3 can be modified so as to be independent of ε. Therefore the local solutions u_ε $(0 < \varepsilon < \varepsilon_0)$ of the perturbed equation (3.1) obtained in Lemma 3.4 can be extended to a fixed interval $[0, M]$, independent of ε (see [3]).

Now, we consider the family $\{u_\varepsilon(x),\ 0<x<M\}$, $0 < \varepsilon < \varepsilon_0$, of solutions to (3.1). From (3.4) it follows that there exists a constant N such that

$$0 \le u_\varepsilon(x) \le N \quad \text{for } 0 < \varepsilon < \varepsilon_0,\ 0 < x < M.$$

Rewrite the perturbed equation (3.1) as follows:

$$u_\varepsilon(x) = \varepsilon x^{n-1} + (n-1)\int_0^x (x-s)^{n-2} \int_0^{u_\varepsilon(s)} g(t)\frac{1}{v_\varepsilon(t)}\,dt\,ds, \tag{4.2}$$

where $v_\varepsilon(t) = u_\varepsilon'(u_\varepsilon^{-1}(t))$. Since only $n \ge 3$ is of interest, we can study u_ε''. First we notice by the estimates of Lemma 3.2 that

$$0 \le \frac{1}{v_\varepsilon(t)} < c\phi(t)^{-1/(n-1)} \quad (t > 0),$$

where $c > 0$ is some constant. Since it follows from (2.4) that

$$\int_0^N g(t)\phi(t)^{-1/(n-1)} \le c,$$

where $c > 0$ is some constant, it is easy to deduce from (4.2) that $u_\varepsilon''(x)$ are uniformly bounded for $0 < \varepsilon < \varepsilon_0$ and $x \in [0, M]$. Therefore the Arzelà–Ascoli theorem shows that $\{u_\varepsilon\}$, $\{u_\varepsilon'\}$ and $\{u_\varepsilon^{-1}\}$, $0 < \varepsilon < \varepsilon_0$, are relatively compact families on $[0, M]$, possibly for a smaller M because of u_ε^{-1}. If we choose a sequence $\{u_{\varepsilon_n}\}$ such that $\{u_{\varepsilon_n}\}$, $\{u_{\varepsilon_n}'\}$, $\{u_{\varepsilon_n}^{-1}\}$ are simultaneously uniformly convergent on $[0, M]$ as $\varepsilon_n \to 0$ and put it into (4.2), then we can see that the limit function $u(x) = \lim_{n\to\infty} u_{\varepsilon_n}(x)$, $0 \le x < M$, is the required solution to the problem (1.1), (1.2).

Proof of Theorem 1.4. Since the solution u blows up if and only if $u^{-1}(x) \le M < \infty$ for any $x > 0$, our assertion follows from the estimates for $v(x) = u'(u^{-1}(x))$ given in Theorem 1.2.

Below we give some examples of functions g in the classes considered in this paper.

EXAMPLE 4.1. Let $g(s) = s^{-1/(n-1)}(-\ln s)^{-\beta}$ $(0 < s < \delta,\ n \ge 2)$. We easily verify that $g \in K_n$ provided $\beta > 1$. Since $\phi(s)$ behaves at 0 like $cs^{n-2}(-\ln s)^{\gamma}$, where $\gamma = -\frac{n-1}{n}(\beta - 1)$ and $c > 0$ is some constant, the condition of Theorem 1.2 is satisfied and the problem (1.1), (1.2) has a nontrivial solution.

EXAMPLE 4.2. Let $g(s) = s(-\ln s)^{\beta}$ $(\beta > 0,\ 0 < s < \delta)$. In this case $\phi(s)$ behaves at 0 like $cs^{n-1}(-\ln s)^{\beta(n-1)/n}$. Therefore the condition of Theorem 1.2 is satisfied if and only if $\beta > n$. In that case the problem (1.1), (1.2) has a nontrivial solution.

EXAMPLE 4.3. Let $\phi(x) = 1 - |x|$ for $-1 \le x \le 1$ and $\phi(x) = 0$ for $|x| > 1$. We consider the function $g(x) = \sum_{i=0}^{\infty} \phi_i(x)$, where $\phi_i(x) = \phi((x - \alpha_i)/\beta_i)$, $\alpha_i = 1/2^i$, $\beta_i = 1/(3 \cdot 2^i)$, $i = 0, 1, \ldots$, defined for $0 < x < 1$. We easily see that the supports of ϕ_i, $i = 0, 1, \ldots$, are pairwise disjoint and $g(\alpha_i) = 1$. We consider the function $g^\star$ corresponding to g with $m = 0$:

$$g^\star(x) = \sup_{0<s<x} g(s) = 1 \qquad (0 < x < 1).$$

We show that $g \in K_n^\star$ for any $n \in \mathbb{N}$. First we notice that the integrals

$$A_i = \int_{-\infty}^{\infty} \phi_i(s) s^{-(n-2)/(n-1)}\, ds, \qquad i = 0, 1, \ldots,$$

can be estimated as follows:

$$c_1 2^{-i/(n-1)} \le A_i \le c_2 2^{-i/(n-1)} \qquad i = 0, 1, \ldots,$$

where $c_1, c_2 > 0$ are some constants. Let $1/2^k < x \le 1/2^{k-1}$. Then

$$G(x) = \int_0^x g(s) s^{-(n-2)/(n-1)}\, ds = \sum_{i=0}^{\infty} \int_0^x \phi_i(s) s^{-(n-2)/(n-1)}\, ds \le \sum_{i=k-1}^{\infty} A_i.$$

Finally, we obtain

$$c_1 x^{1/(n-1)} \le G(x) \le c_2 x^{1/(n-1)} \quad (0 < x < 1),$$

where $c_1, c_2 > 0$ are some constants. Since

$$G^\star(x) = \int_0^x g^\star(s) s^{-(n-2)/(n-1)}\, ds = (n-1) x^{1/(n-1)},$$

we see that $g \in K_n^\star$. Now Theorem 1.3 shows that the problem (1.1), (1.2) has a nontrivial solution.

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