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Estimations of the second coefficient of a univalent, bounded, symmetric and non-vanishing function by means of Loewner's parametric method

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Abstract. Let $\mathcal{B}_0^{(R)}(b)$ denote the class of functions $F(z) = b + A_1 z + A_2 z^2 + \ldots$ analytic and univalent in the unit disk U which satisfy the conditions: $F(U) \subset U$, $0 \notin F(U)$, $\operatorname{Im} F^{(n)}(0) = 0$. Using Loewner's parametric method we obtain lower and upper bounds of A_2 in $\mathcal{B}_0^{(R)}(b)$ and functions for which these bounds are realized. The class $\mathcal{B}_0^{(R)}(b)$, introduced in [6], is a subclass of the class $\mathcal{B}_u$ of bounded, non-vanishing univalent functions in the unit disk. This last class and closely related ones have been studied by various authors in [1]–[4]. We mention in particular the paper of D. V. Prokhorov and J. Szynal [5], where a sharp upper bound for the second coefficient in $\mathcal{B}_u$ is given.

1. Introduction. Let $\mathcal{B}_0^{(R)}(b)$, $0 < b < 1$, denote the class of all functions F that are analytic, univalent in the unit disk U and satisfy the conditions

$$F(U) \subset U,\ \ F(0) = b,\ \ 0 \notin F(U),\ \ \operatorname{Im} F^{(n)}(0) = 0,\ \ n = 0, 1, \ldots,\ \ F'(0) > 0.$$

Let

$$F(z) = b + A_1 z + A_2 z^2 + \ldots, \qquad A_1 > 0, \tag{1}$$

and

$$L(z) = K^{-1}\left(\frac{4b}{(1-b)^2}\left(K(z) + \frac{1}{4}\right)\right) = b + B_1 z + B_2 z^2 + \ldots, \tag{2}$$

where $K(z) = z/(1-z)^2$,

$$B_1 = \frac{4b(1-b)}{1+b}, \qquad B_2 = \frac{-8b(1-b)(b^2+2b-1)}{(1+b)^3}. \tag{2'}$$

The function (2) maps U onto $U \setminus (-1, 0]$, is univalent and symmetric in U, $L(0) = b$, and therefore $L \in \mathcal{B}_0^{(R)}(b)$. Let further $S_1^{(R)}$ denote the family of

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all functions f which are analytic, univalent and symmetric in U and satisfy $f(U) \subset U$, $f(0) = 0$. It is obvious that if $f \in S_1^{(R)}$ then $L \circ f \in \mathcal{B}_0^{(R)}(b)$. But also conversely, if $F \in \mathcal{B}_0^{(R)}(b)$ then $F(U) \cap (-1, 0] = \emptyset$, hence $L^{-1} \circ F \in S_1^{(R)}$. Moreover, any $F \in \mathcal{B}_0^{(R)}(b)$ is subordinate to L. The above relations allow the application of Loewner's theory, adapted to the class $S_1^{(R)}$ by O. Tammi [7], pp. 61–77, to functions of the class $\mathcal{B}_0^{(R)}(b)$. It turns out that in this manner it is possible not only to obtain estimates of A_2 in the class $\mathcal{B}_0^{(R)}(b)$ in an easier way than using the variational method as in [6], but also to obtain all the extremal functions.

2. Loewner's theory applied to $\mathcal{B}_0^{(R)}(b)$. D is called a *symmetric* 2-*slit disk* if it is obtained from the disk U by removing two Jordan arcs not containing 0, symmetric about the real axis and such that D is a simply connected domain. It is known that each simply connected domain, included in the disk U, symmetric about the real axis and containing 0, can be approximated, in the sense of convergence towards a kernel, by domains like ones considered above, and hence on account of the Carathéodory Convergence Theorem, each function in $S_1^{(R)}$ can be approximated in the topology of uniform convergence on compact sets by $S_1^{(R)}$ functions that map U onto symmetric 2-slit disks. Hence the set of all such functions—denote it by $\mathcal{S}$—is dense in $S_1^{(R)}$ and the infimum and supremum in $S_1^{(R)}$ of any functional (real and continuous) are the same in $S_1^{(R)}$ as in $\mathcal{S}$.

Tammi [7], p. 68, proved the following theorem for functions of class $\mathcal{S}$.

THEOREM I. *For each symmetric* 2-*slit domain* D *there exists a function* $\vartheta = \vartheta(u)$, *continuous in* $[u_0, 1]$, $u_0 > 0$, *which determines a differential equation*

$$u \frac{\partial f(z,u)}{\partial u} = \frac{f(z,u) - f^3(z,u)}{1 - 2\cos\vartheta(u) f(z,u) + f^2(z,u)}, \tag{3}$$

so that its solution $f(z, u_0)$ *with the initial condition* $f(z, 1) = z$ *is the mapping function of* U *onto* D *with* $f(0, u_0) = 0$.

Conversely, *if* ϑ *is continuous in* $[u_0, 1]$ *for some* $u_0 > 0$ *and* (3) *is integrated with the initial condition* $f(z, 1) = z$, *then the solution satisfies* $f(z, u) \in S_1^{(R)}$, $f_z'(0, u) = u$.

Denoting by $\mathcal{S}_1$ the set of all solutions of the equations (3) with the functions ϑ continuous in $[u_0, 1]$ for some $u_0 > 0$ and with the initial condition $f(z, 1) = z$, we have $\mathcal{S} \subset \mathcal{S}_1 \subset S_1^{(R)}$. The continuity of the function L implies that the family $\mathcal{L} = \{F : F = L \circ f \text{ for some } f \in \mathcal{S}_1\}$ is dense in the class $\mathcal{B}_0^{(R)}(b)$, and hence if $\mathcal{F}$ is a functional real, continuous and bounded in $\mathcal{B}_0^{(R)}(b)$, then

$$\inf_{\mathcal{B}_0^{(R)}(b)} \mathcal{F} = \inf_{\mathcal{L}} \mathcal{F}, \qquad \sup_{\mathcal{B}_0^{(R)}(b)} \mathcal{F} = \sup_{\mathcal{L}} \mathcal{F}.$$

3. Lower and upper bounds of A_2. Let

$$f(z,u) = u(z + a_2(u)z^2 + a_3(u)z^3 + \ldots)$$

satisfy the equation (3) and the initial condition $f(z,1) = z$ with some ϑ continuous in $[u_0, 1]$ for some $u_0 > 0$. Let

$$F(z,u) = L(f(z,u)) = b + A_1(u)z + A_2(u)z^2 + \ldots \tag{4}$$

By (3), $a_2'(u) = 2\cos\vartheta(u)$, and hence

$$a_2(u) = -2\int_u^1 \cos\vartheta(t)\,dt, \qquad u_0 \le u \le 1.$$

From (4), (2) and (2′) it follows that

$$\begin{aligned} A_1(u) &= B_1 u = \frac{4b(1-b)}{1+b}u, \\ A_2(u) &= B_1 u a_2(u) + B_2 u^2 \\ &= \frac{-8b(1-b)}{1+b}\left(u\int_u^1 \cos\vartheta(t)\,dt + \frac{b^2+2b-1}{(1+b)^2}u^2\right). \end{aligned}$$

It is obvious that $A_2(u)$ is maximal if $\cos\vartheta(t) = -1$ and it is minimal if $\cos\vartheta(t) = 1$ for $u \le t \le 1$. Thus we obtain the following inequality:

$$\begin{aligned} &\frac{-8b(1-b)}{(1+b)}\left(u - \frac{2}{(1+b)^2}u^2\right) \\ &\qquad \le A_2(u) \le \frac{-8b(1-b)}{(1+b)}\left(u - \frac{2b^2+4b}{(1+b)^2}u^2\right), \qquad 0 \le u \le 1. \end{aligned} \tag{5}$$

Both inequalities are sharp. The right-hand side of (5) attains its maximal value for $u^* = (1+b)^2/(4b(b+2))$. If $u^* \le 1$, that is, if $2/\sqrt{3} - 1 \le b < 1$, then

$$\max_{u\in[0,1]} A_2(u) = A_2(u^*) = \frac{1-b^2}{b+2}. \tag{6}$$

If $u^* > 1$, that is, if $0 < b \le 2/\sqrt{3} - 1$, then

$$\max_{u\in[0,1]} A_2(u) = A_2(1) = -\frac{8b(1-b)}{(1+b)^3}(b^2+2b-1) = B_2. \tag{7}$$

The left-hand side of (5) attains its minimal value for $u^{**} = (1+b)^2/4 \le 1$, hence for every $0 < b < 1$,

$$\min_{u\in[0,1]} A_2(u) = A_2(u^{**}) = -b(1-b^2). \tag{8}$$

Exactly the same results were obtained in [6] by means of the variational method.

Let us now find functions whose second coefficient satisfies the equalities in (6), (7) and (8).

Putting in (3) $\cos\vartheta(u) = -1$ for $u \in [u_0, 1]$, $u_0 > 0$ arbitrary, we get the identity

$$\frac{f(z,u)+1}{f(z,u)(1-f(z,u))}\frac{\partial f(z,u)}{\partial u} = \frac{1}{u}, \tag{9}$$

where f is the function from the second part of Theorem I.

Integrating (9) from u_1 to 1, where $u_1 = u^*$ for $0 < b \le -1 + \frac{2}{3}\sqrt{2}$ and $u_1 = 1$ for $2/\sqrt{3} - 1 < b < 1$, we obtain

$$\frac{f(z,u_1)}{(1-f(z,u))^2} = u_1\frac{z}{(1-z)^2}. \tag{10}$$

If $u_1 = 1$ then $f(z,u_1) = f(z,1) = z$ and $F(z) = L(z)$, hence for $0 < b \le 2/\sqrt{3}-1$ the function (2) maximizes the second coefficient A_2. If $2/\sqrt{3}-1 < b < 1$ then A_2 is maximal, by (10), for the function

$$F(z) = L(f(z,u^*)) = K^{-1}\left(\frac{4b}{(1-b)^2}\left(\frac{(1+b)^2}{4b(b+2)}\frac{z}{(1-z)^2}+\frac{1}{4}\right)\right), \tag{11}$$

which maps the disk U on $U \setminus (-1, c]$, where

$$c = \frac{(2b^3+3b^2+3)\sqrt{2+b}-2(2+b)(1-b^2)\sqrt{1+b}}{\sqrt{2+b}(3b^2+6b-1)}.$$

We see that c tends to 0 as $b \to (2/\sqrt{3}-1)-0$.

Putting now in (3) $\cos\vartheta(u) = 1$ for $u \in [u_0, 1]$, $u_0 > 0$ arbitrary, we get for the function f satisfying (3) the identity

$$\frac{1-f(z,u)}{f(z,u)(1+f(z,u))}\frac{\partial f(z,u)}{\partial u} = \frac{1}{u}. \tag{12}$$

Integrating (12) from u^{**} to 1 we obtain

$$\frac{f(z,u^{**})}{(1+f(z,u^{**}))^2} = u^{**}\frac{z}{(1+z)^2}.$$

The coefficient A_2 is minimized by the function

$$F(z) = L(f(z,u^{**})) = K^{-1}\left(\frac{b}{(1-b)^2}\frac{1}{1-(1+b)^2z/(1+z)^2}\right). \tag{13}$$

This function maps the disk U onto $U \setminus ((-1,0] \cup [d,1))$, where

$$d = \frac{3 + 6b^2 - b^4 - (1-b^2)\sqrt{9 - 10b^2 + b^4}}{8b}.$$

We see that d tends to 1 as b tends to 1.

We now restate the result obtained above:

THEOREM. *If* $F \in \mathcal{B}_0^{(R)}(b)$, $0 < b < 1$, *then*

$$-b(1-b^2) \le A_2 \le \begin{cases} -\dfrac{8b(1-b)}{(1+b)^3}(b^2 + 2b - 1) & \text{for } 0 < b \le \dfrac{2}{\sqrt{3}} - 1, \\ \dfrac{1-b^2}{b+2} & \text{for } \dfrac{2}{\sqrt{3}} - 1 < b < 1. \end{cases}$$

The left-hand bound is realized by the function (13) *and the right-hand bounds by the functions* (11) *and* (2).

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