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The Frölicher–Nijenhuis bracket on some functional spaces

by Ivan Kolář (Brno) and Marco Modugno (Florence)

Abstract. Two fiber bundles E_1 and E_2 over the same base space M yield the fibered set $\mathcal{F}(E_1, E_2) \to M$, whose fibers are defined as $C^\infty(E_{1x}, E_{2x})$, for each $x \in M$. This fibered set can be regarded as a smooth space in the sense of Frölicher and we construct its tangent prolongation. Then we extend the Frölicher–Nijenhuis bracket to projectable tangent valued forms on $\mathcal{F}(E_1, E_2)$. These forms turn out to be a kind of differential operators. In particular, we consider a general connection on $\mathcal{F}(E_1, E_2)$ and study the associated covariant differential and curvature.

1. Introduction. The idea of the Schrödinger connection on a double fibered manifold by A. Jadczyk and the second author ([4], [5]) inspired the study of differential geometric properties of certain bundles with infinite-dimensional fibers. In particular, let E_1 and E_2 be two classical fiber bundles over the same base space M and $\mathcal{F}(E_1, E_2)$ denote the set of all smooth maps of a fiber of E_1 into the fiber of E_2 over the same base point. In [1], A. Cabras and the first author studied the connections on $\mathcal{F}(E_1, E_2) \to M$ by means of some procedures which are based directly on the concept of fiber jet [7]. However, essential progress in the theory of general connections on classical fiber bundles has recently been achieved by using the Frölicher–Nijenhuis bracket. That is why we devote the present paper to projectable tangent valued forms on $\mathcal{F}(E_1, E_2)$ and their Frölicher–Nijenhuis bracket. Our main tool is a formula by P. W. Michor [7] and the second author [9] which expresses the classical Frölicher–Nijenhuis bracket in terms of the bracket of vector fields. This enables us to develop a generalization which preserves the most important properties of the classical case. In the last section we present the first applications of our general results to connections on $\mathcal{F}(E_1, E_2)$, but we hope there will be many others, similarly to the classical case.

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Since the objects we study represent a kind of differential operators, their basic properties can be treated systematically within the framework of the theory of smooth spaces by A. Frölicher [2]. However, we use a simplified version of that theory, which is sufficient for our purposes. We define a *smooth space* as a set S together with a set of curves $\mathcal{C}(S) = \{c : \mathbb{R} \to S\}$, which is called the set of *smooth curves*. A map $f : S \to S'$ between two smooth spaces is called *smooth* if, for each smooth curve $c : \mathbb{R} \to S$, the composition $f \circ c : \mathbb{R} \to S'$ is also a smooth curve.

In particular, each classical manifold becomes a smooth space by taking as smooth curves, in the sense of Frölicher, just the smooth curves in the classical sense. Moreover, a map between classical manifolds turns out to be smooth in the classical sense if and only if it is smooth in the sense of Frölicher (see e.g. [7], p. 172). We observe that each subset $\iota : S' \hookrightarrow S$ of a smooth space S inherits naturally a structure of a smooth space, by taking as smooth curves the curves $c' : \mathbb{R} \to S'$ such that $\iota \circ c' : \mathbb{R} \to S$ is smooth. Actually, these curves constitute the largest set $\mathcal{C}(S')$ which makes ι smooth.

2. Preliminaries. Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be two classical fiber bundles over the same base space. Consider the fibered set

$$p : \mathcal{F}(E_1, E_2) = \bigcup_{x \in M} C^\infty(E_{1x}, E_{2x}) \to M$$

of all smooth maps of a fiber of E_1 into the fiber of E_2 over the same base point. We define no topology on $\mathcal{F}(E_1, E_2)$, but equip naturally this set with a structure of a smooth space by taking as *smooth curves* the curves $c : \mathbb{R} \to \mathcal{F}(E_1, E_2)$ such that $p \circ c : \mathbb{R} \to M$ is smooth and the induced map $\widehat{c} : (p \circ c)^* E_1 \to E_2$, given by $\widehat{c}(\tau, y) = c(\tau)(y)$, with $\tau \in \mathbb{R}, y \in E_{1p(c(\tau))}$, is also smooth.

Next, we define the tangent space $T\mathcal{F}(E_1, E_2)$ in the following way. For every smooth curve $c : \mathbb{R} \to \mathcal{F}(E_1, E_2)$ and $\tau \in \mathbb{R}$, we first construct the tangent vector $X = \frac{\partial}{\partial t}\big|_\tau (p \circ c) \in T_x M$, where $x = p(c(\tau))$, and consider the subsets $T_X E_1 = (Tp_1)^{-1}(X) \subset TE_1$ and $T_X E_2 = (Tp_2)^{-1}(X) \subset TE_2$. It can be easily seen that $T_X E_1$ and $T_X E_2$ are affine bundles over E_{1x} and E_{2x} with derived vector bundles $T(E_{1x}) = V_x E_1$ and $T(E_{2x}) = V_x E_2$, respectively. Then we obtain a map $T_\tau c : T_X E_1 \to T_X E_2$ which is well defined by the formula

$$T_\tau c\left(\frac{\partial}{\partial t}\bigg|_\tau h(t)\right) = \frac{\partial}{\partial t}\bigg|_\tau c(t)(h(t)), \tag{1}$$

where $h : \mathbb{R} \to E_1$ is a smooth curve such that $p \circ c = p_1 \circ h$. Having another smooth curve $c' : \mathbb{R} \to \mathcal{F}(E_1, E_2)$ and $\tau' \in \mathbb{R}$ satisfying $\frac{\partial}{\partial t}\big|_{\tau'} (p \circ c') = X$, the equivalence relation $T_\tau c = T_{\tau'} c' : T_X E_1 \to T_X E_2$ defines a *tangent vector* of $\mathcal{F}(E_1, E_2)$ which is denoted by $\frac{\partial}{\partial t}\big|_\tau c$. Then the *tangent space* $T\mathcal{F}(E_1, E_2)$ is defined as the set consisting of all tangent vectors of $\mathcal{F}(E_1, E_2)$.

By the way, we can interpret the tangent vectors also in the following way. If $A \in T\mathcal{F}(E_1, E_2)$, then denote by $\widehat{A} : T_X E_1 \to T_X E_2$ the associated map (1). One sees easily that $T_\tau c : T_X E_1 \to T_X E_2$ is an affine bundle morphism over the base map $\phi = c(\tau) : E_{1x} \to E_{2x}$ with derived linear morphism $T\phi : T(E_{1x}) \to T(E_{2x})$. In [1] the converse assertion is proved: if $C : T_X E_1 \to T_X E_2$ is an affine bundle morphism over $\phi : E_{1x} \to E_{2x}$ with derived linear morphism $T\phi : T(E_{1x}) \to T(E_{2x})$, then there exists a smooth curve $c : \mathbb{R} \to \mathcal{F}(E_1, E_2)$ and a $\tau \in \mathbb{R}$ such that $C = \widehat{\frac{\partial}{\partial t}\big|_\tau c}$.

The tangent space is naturally equipped with the following structures. By (1) we have defined an inclusion

$$T\mathcal{F}(E_1, E_2) \subset \mathcal{F}(TE_1 \to TM,\ TE_2 \to TM).$$

Hence, $T\mathcal{F}(E_1, E_2)$ inherits a structure of a smooth space. Moreover, we have two canonical projections $\pi : T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ and $Tp : T\mathcal{F}(E_1, E_2) \to TM$, which turn out to be smooth. Even the tangent prolongation $dc = \frac{\partial}{\partial t} c : \mathbb{R} \to T\mathcal{F}(E_1, E_2) : \tau \mapsto \frac{\partial}{\partial t}\big|_\tau c$ of a smooth curve $c : \mathbb{R} \to \mathcal{F}(E_1, E_2)$ is smooth.

Now, consider the vector fields on $\mathcal{F}(E_1, E_2)$. As usual, a *vector field* on $\mathcal{F}(E_1, E_2)$ is defined to be a smooth map $A : \mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2)$ satisfying $\pi \circ A = \mathrm{id}$. We say that A is *projectable* if there exists a classical smooth vector field $A^0 : M \to TM$ such that $A^0 \circ p = Tp \circ A$. A vector field A on $\mathcal{F}(E_1, E_2)$ projectable over A^0 is said to be *of order* r if the condition $j^r_y \phi = j^r_y \psi$, with $\phi, \psi \in C^\infty(E_{1x}, E_{2x})$ and $y \in E_{1x}$, implies

$$\widehat{A\phi} | (T_{A^0(x)} E_1)_y = \widehat{A\psi} | (T_{A^0(x)} E_1)_y. \tag{2}$$

Let $S(TE_1, TE_2) \to E_1 \times_M E_2 \times_M TM$ be the fiber bundle of all affine morphisms $(T_X E_1)_y \to (T_X E_2)_z$ with $p_1(y) = p_2(z) = \pi_M(X)$, where $\pi_M : TM \to M$ is the bundle projection. Write

$$\mathcal{F}J^r(E_1, E_2) = \bigcup_{x \in M} J^r(E_{1x}, E_{2x}),$$

which is also a classical manifold. An rth order vector field A over A^0 defines the associated map $\mathcal{A} : \mathcal{F}J^r(E_1, E_2) \to S(TE_1, TE_2)$ by

$$\mathcal{A}(j^r_y \phi) = \widehat{A\phi} | (T_{A^0(x)} E_1)_y. \tag{3}$$

In [1] it is proved that $\mathcal{A}$ is a classical C^∞-map. The derived linear map of each element of $S(TE_1, TE_2)$ is identified with an element of $\mathcal{F}J^1(E_1, E_2)$. This defines a map $D : S(TE_1, TE_2) \to \mathcal{F}J^1(E_1, E_2)$ and the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F}J^1(E_1,E_2) & = = = = = & \mathcal{F}J^1(E_1,E_2) \\
{\scriptstyle \beta_r}\big\uparrow & & \big\uparrow{\scriptstyle D} \\
\mathcal{F}J^r(E_1,E_2) & \xrightarrow{\ \mathcal{A}\ } & S(TE_1,TE_2) \\
\big\downarrow & & \big\downarrow \\
E_1 \times_M E_2 & \xrightarrow[(\mathrm{id},A^0)]{} & (E_1 \times_M E_2) \times_M TM
\end{array}
\tag{4}
$$

where β_r is the jet projection. Conversely, let $\mathcal{A} : \mathcal{F}(E_1,E_2) \to S(TE_1,TE_2)$ be a C^∞-map with underlying vector field $A^0 : M \to TM$ such that (4) commutes. Then the rule

$$\widehat{A(\phi)} = \bigcup_{y \in E_{1x}} \mathcal{A}(j^r_y \phi) \tag{5}$$

defines an rth order vector field A on $\mathcal{F}(E_1,E_2)$ over A^0.

Let (x^i) be a local chart on M, (y^p) and (z^a) be additional fiber coordinates on E_1 and E_2, respectively, and

$$x^i = c^i(t), \qquad z^a = c^a(y^p,t) \tag{6}$$

be the coordinate expression of $c : \mathbb{R} \to \mathcal{F}(E_1,E_2)$. Write $Y^p = dy^p, Z^a = dz^a$, $\phi^a(y,\tau) = c^a(y,\tau)$, $\varPhi^a(y,\tau) = \frac{\partial c^a}{\partial t}(y,\tau)$. Then the coordinate expression of $dc(\tau)$ is

$$Z^a(\tau) = \frac{\partial \phi^a}{\partial y^p}(y,\tau) Y^p + \varPhi^a(y,\tau). \tag{7}$$

Hence, the tangent vector to (6) at τ is locally characterized by two systems of numbers $x^i = c^i(\tau)$, $X^i = \frac{dc^i}{dt}(\tau)$ and two systems of functions $\phi^a(y^p,\tau)$, $\varPhi^a(y^p,\tau)$. In this sense, the coordinate form of the map $\mathcal{A}$ associated with an rth order vector field is given by $X^i(x^j)$ and

$$\varPhi^a = \varPhi^a(x^i, y^p, z^a_\alpha), \qquad 0 \le |\alpha| \le r, \tag{8}$$

where α is a multi-index whose range is the dimension of the fiber of E_1.

The inclusion $T\mathcal{F}(E_1,E_2) \subset \mathcal{F}(TE_1 \to TM,\ TE_2 \to TM)$ defines the second tangent bundle $TT\mathcal{F}(E_1,E_2)$ of $\mathcal{F}(E_1,E_2)$ (see [1]), which turns out to be a smooth space. We can easily see that the usual geometric structures of the classical second tangent bundle (such as the canonical involution and vertical inclusion) can be naturally extended to our smooth spaces $\mathcal{F}(E_1,E_2)$.

A projectable vector field A on $\mathcal{F}(E_1,E_2)$ is said to be *differentiable* if the rule $TA\big(\frac{\partial}{\partial t}\big|c\big) = \frac{\partial}{\partial t}\big|(A \circ c)$ defines a smooth map $TA : T\mathcal{F}(E_1,E_2) \to TT\mathcal{F}(E_1,E_2)$. One sees easily that every rth order vector field is differentiable [1].

Given two differentiable vector fields A, B on $T\mathcal{F}(E_1, E_2)$ projectable over A^0, B^0, their bracket $[A, B] : \mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2)$ is a projectable vector field over $[A^0, B^0]$ defined by means of the so-called strong difference:

$$[A, B] = TB \circ A \div TA \circ B, \tag{9}$$

where $\div$ is a special operation based on the geometry of the second tangent bundle ([1], [6], [8]).

Let B be an sth order vector field with associated map $Y^i(x^j)$ and

$$\Psi^a(x^i, y^p, z^a_\beta), \qquad 0 \le |\beta| \le s. \tag{10}$$

In [1] it is deduced that the bracket of (8) and (10) has order $r + s$. To express its associated map, we need the concept of *formal derivatives* of a smooth function $f : \mathcal{F}J^r(E_1, E_2) \to \mathbb{R}$. The rule

$$Df(j^{r+1}_y \phi) = (df(j^r \phi))_y$$

defines a map

$$Df : \mathcal{F}J^{r+1}(E_1, E_2) \to V^* E_1$$

called the *formal differential* of f. For the coordinate vector fields $\partial/\partial y^p$ we obtain the formal derivatives

$$D_p f = \left\langle Df, \frac{\partial}{\partial y^p} \right\rangle = \frac{\partial f}{\partial y^p} + \frac{\partial f}{\partial z^a} z^a_p + \ldots + \frac{\partial f}{\partial z^a_\alpha} z^a_{\alpha+p}. \tag{11}$$

By iteration, we introduce $D_\alpha : \mathcal{F}J^{r+|\alpha|}(E_1, E_2) \to \mathbb{R}$ for every multi-index α. According to [1], the map associated with $[A, B]$ is given by $[A^0, B^0]$ and

$$\begin{aligned} \frac{\partial \Psi^a}{\partial x^i} X^i + \frac{\partial \Psi^a}{\partial z^b} \Phi^b + \ldots + \frac{\partial \Psi^a}{\partial z^b_\beta} D_\beta \Phi^b \\ - \frac{\partial \Phi^a}{\partial x^i} Y^i - \frac{\partial \Phi^a}{\partial z^b} \Psi^b - \ldots - \frac{\partial \Phi^a}{\partial z^b_\alpha} D_\alpha \Psi^b. \end{aligned} \tag{12}$$

We conclude this section by a lemma. Consider two differentiable vector fields A, B over A^0, B^0 and a function $f \in C^\infty(M, \mathbb{R})$. Then fA is a differentiable vector field as well.

Lemma 2.1. *We have* $[fA, B] = f[A, B] - (B^0 f)A$.

Proof. Let $B(\phi) = \frac{\partial}{\partial t}\big|_0 g(t), \phi \in \mathcal{F}(E_1, E_2)$. Then we have

$$T(fA)\left(\frac{\partial}{\partial t}\bigg|_0 g(t) \right) = \frac{\partial}{\partial t}\bigg|_0 f((p \circ g)(t)) A(g(t)) = (B^0 f)A + fTA(B(\phi)).$$

So, our claim follows from (9). ∎

3. Projectable tangent valued forms on $\mathcal{F}(E_1, E_2)$. We start with a more general concept. Let $V \to M$ be a vector bundle.

Definition 3.1. A *tangent valued form of type* V on $\mathcal{F}(E_1,E_2)$ is defined to be a map $A:\mathcal{F}(E_1,E_2)\times_M V\to T\mathcal{F}(E_1,E_2)$ which is linear in the second factor and satisfies the following three conditions:

(i) $\pi\circ A=\mathrm{pr}_1$,

(ii) there exists a linear base-preserving morphism $A^0:V\to TM$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{F}(E_1,E_2)\times_M V & \xrightarrow{A} & T\mathcal{F}(E_1,E_2) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle Tp} \\ V & \xrightarrow[A^0]{} & TM \end{array} \tag{13}$$

(iii) for every C^∞-section $s:M\to V$, the induced vector field As in $\mathcal{F}(E_1,E_2)$ is smooth.

Clearly, the lifting $s\mapsto As$ is $C^\infty(M,\mathbb{R})$-linear, i.e., $A(fs)=fAs$ for every smooth function $f:M\to\mathbb{R}$. We say that A is *differentiable* if As is a differentiable vector field for every C^∞-section s. Furthermore, A is said to have *order* r if As is an rth order vector field for every C^∞-section s. Such a form A defines the associated map $\mathcal{A}:\mathcal{F}J^r(E_1,E_2)\times_M V\to S(TE_1,TE_2)$ by

$$\mathcal{A}(j^r_y\phi,v)=A\widehat{(\phi,v)}(T_{A^0(v)}E_1)_y. \tag{14}$$

Let w^λ be some additional linear fiber coordinate on V. Then the coordinate form of $\mathcal{A}$ is given by $A^i_\lambda(x)v^\lambda$ and

$$A^a_\lambda(x^i,y^p,z^a_\alpha)v^\lambda,\qquad 0\le|\alpha|\le r. \tag{15}$$

Obviously, every rth order tangent valued form of type V is differentiable.

Consider an arbitrary mapping Φ transforming C^∞-sections of V into vector fields on $\mathcal{F}(E_1,E_2)$. The following lemma will be of fundamental importance in §4.

Lemma 3.1. *If* Φ *is* $C^\infty(M,\mathbb{R})$*-linear, then there exists a tangent valued form* A *of type* V *such that* $\Phi s=As$ *for all* $s\in C^\infty V$.

Proof. We first deduce that $s|U=0$, for an open subset $U\subset M$, implies $\Phi(s)|p^{-1}(U)=0$. Indeed, for every $x\in U$ we take a function $f\in C^\infty(M,\mathbb{R})$ such that $f(x)=0$ and $f|M\setminus U=1$. Then $s=fs$, so that $\Phi(s)(\phi)=f(x)\Phi(s)(\phi)=0$ for $\phi\in p^{-1}(x)$. By linearity, we now find that $s_1|U=s_2|U$ implies $\phi(s_1)|p^{-1}(U)=\phi(s_2)|p^{-1}(U))$. Further, let $s(x_0)=0$. On a neighbourhood U of x_0 we have $s(x)=s^\lambda(x)e_\lambda(x)$, $s^\lambda(x_0)=0$, where e_λ are C^∞-sections of V over U such that $e_\lambda(x)$ constitute a basis of each V_x. Take a neighbourhood W of x_0 satisfying $\overline{W}\subset U$ and $g\in C^\infty(M,\mathbb{R})$ with $g|W=1$, supp $g\subset U$. Then $g^2s|W=s|W$ and gs^λ and ge_λ are globally defined. Hence $\Phi(s)(\phi) = (gs^\lambda(x_0))\Phi(ge_\lambda)(\phi) = 0$. By linearity, $\Phi(s)(\phi)$ depends on $s(x)$ only. This induces the required map $\mathcal{F}(E_1, E_2) \times_M V \to T\mathcal{F}(E_1, E_2)$.

Now we restrict ourselves to the case $V = \bigwedge^k TM$, which is the main subject of the present paper.

DEFINITION 3.2. A *projectable tangent valued k-form* on $\mathcal{F}(E_1, E_2)$ is defined to be a tangent valued form of type $\bigwedge^k TM$.

To simplify the notation, we shall write $A(X_1, \ldots, X_k)$ instead of $A(X_1 \wedge \ldots \wedge X_k)$, for $X_1, \ldots, X_k \in C^\infty TM$. The map $\mathcal{A} : \mathcal{F}J^r(E_1, E_2) \times_M \bigwedge^k TM \to S(TE_1, TE_2)$ associated with an rth order projectable tangent valued k-form A will be expressed by $A^i_{j_1 \ldots j_k}(x)$ and

$$A^a_{j_1 \ldots j_k}(x^i, y^p, z^a_\alpha), \quad 0 \le |\alpha| \le r, \tag{16}$$

which are antisymmetric in all subscripts.

4. The Frölicher–Nijenhuis bracket. Let A and B be differentiable projectable tangent valued forms on $\mathcal{F}(E_1, E_2)$ of degree k and l, respectively. Starting from a formula for the classical Frölicher–Nijenhuis bracket [9], [6], we define a mapping $[A, B]$ of $C^\infty(\bigwedge^{k+l} TM)$ into vector fields on $\mathcal{F}(E_1, E_2)$ by

$$\begin{aligned}
&[A, B](X_1, \ldots, X_{k+l}) \\
&\quad = \frac{1}{k!l!} \sum_\sigma \operatorname{sign}\sigma \, [A(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), B(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})] \\
&\qquad + \frac{-1}{k!(l-1)!} \sum_\sigma \operatorname{sign}\sigma \\
&\qquad \times B([A^0(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \ldots, X_{\sigma(k+l)}) \\
&\qquad + \frac{(-1)^{kl}}{(k-1)!l!} \sum_\sigma \operatorname{sign}\sigma \\
&\qquad \times A([B^0(X_{\sigma(1)}, \ldots, X_{\sigma(l)}), X_{\sigma(l+1)}], X_{\sigma(l+2)}, \ldots, X_{\sigma(k+l)}) \\
&\qquad + \frac{(-1)^{k-1}}{(k-1)!(l-1)!2} \sum_\sigma \operatorname{sign}\sigma \\
&\qquad \times B(A^0([X_{\sigma(1)}, X_{\sigma(2)}], \ldots, X_{\sigma(k+1)}), X_{\sigma(k+2)}, \ldots, X_{\sigma(k+l)}) \\
&\qquad + \frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2} \sum_\sigma \operatorname{sign}\sigma \\
&\qquad \times A(B^0([X_{\sigma(1)}, X_{\sigma(2)}], \ldots, X_{\sigma(l+1)}), X_{\sigma(l+2)}, \ldots, X_{\sigma(k+l)}).
\end{aligned} \tag{17}$$

Proposition 4.1. *The map* (17) *is* $C^\infty(M,\mathbb{R})$*-linear.*

Proof. It suffices to deduce that if we multiply X_1 by $f \in C^\infty(M,\mathbb{R})$ on the left hand side of (17), then the right hand side is also multiplied by f. Using Lemma 2.1 we reduce it to a combinatorial question, which can be answered in the following way. In the case when E_1 is the trivial fibering $M \to M$, we have $\mathcal{F}(E_1,E_2) = E_2$ and formula (17) defines the Frölicher–Nijenhuis bracket of two classical projectable tangent valued forms on E_2 in the sense of the second author [9]. Here, we know that (17) well defines the Frölicher–Nijenhuis bracket as a projectable tangent valued $(k+l)$-form. But the combinatorics is the same in both situations, which proves our claim.

Applying Lemma 3.1, we obtain directly

Proposition 4.2. *For every differentiable projectable tangent valued k-form A and l-form B on $\mathcal{F}(E_1,E_2)$, $[A,B]$ is a projectable tangent valued $(k+l)$-form on $\mathcal{F}(E_1,E_2)$, which will be called the Frölicher–Nijenhuis* (*F-N for short*) *bracket of A and B. The following diagram commutes*:

$$\begin{array}{ccc}
\mathcal{F}(E_1,E_2) \times_M \bigwedge^{k+l} TM & \xrightarrow{[A,B]} & T\mathcal{F}(E_1,E_2) \\
\Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle Tp} \\
\bigwedge^{k+l} TM & \xrightarrow[{[A^0,B^0]}]{} & TM
\end{array}$$

where $[A^0,B^0]$ is the classical F-N bracket.

In [1] Cabras and the first author introduced the canonical involution κ of the iterated tangent bundle $TT\mathcal{F}(E_1,E_2)$. If A is a differentiable vector field on $\mathcal{F}(E_1,E_2)$, then $\kappa \circ TA$ is a vector field on TA.

Definition 4.1. A vector field A on $\mathcal{F}(E_1,E_2)$ is said to be *twice differentiable* if $\kappa\circ TA$ is a differentiable vector field on $T\mathcal{F}(E_1,E_2)$. A projectable tangent valued k-form A is said to be *twice differentiable* if $A(X_1,\ldots,X_k)$ is a twice differentiable vector field on $\mathcal{F}(E_1,E_2)$, for all $X_1,\ldots,X_k \in C^\infty TM$.

If A and B are twice differentiable vector fields on $\mathcal{F}(E_1,E_2)$, then the bracket $[A,B]$ is differentiable. Clearly, every finite order vector field is twice differentiable.

There is a proof of the Jacobi identity for classical vector fields on a manifold M, which is based on the concept of strong difference and on the geometry of the third tangent bundle $TTTM$ (see [11]). That approach implies directly that every triple of twice differentiable vector fields on $\mathcal{F}(E_1,E_2)$ satisfies the Jacobi identity. Then we obtain analogously to the classical case

Proposition 4.3. *The F-N bracket of twice differentiable projectable tangent valued forms on $\mathcal{F}(E_1, E_2)$ has the algebraic properties of a graded Lie algebra.*

Let A and B be projectable tangent valued forms on $\mathcal{F}(E_1, E_2)$ of order r and s, respectively. By (12), the order of $[A, B]$ is $r + s$.

Proposition 4.4. *Let* (16) *be the associated map of A and $B^i_{j_1\ldots j_l}(x)$, $B^a_{j_1\ldots j_l}(x^i, y^p, z^a_\beta)$, with $0 \le |\beta| \le s$, be the associated map of B. Then the associated map of $[A, B]$ is given by $[A^0, B^0]$ and by the antisymmetrization in $i_1, \ldots, i_{k+l}$ of the following expression*:

$$
\begin{aligned}
(18)\quad & A^i_{i_1\ldots i_k}\frac{\partial B^a_{i_{k+1}\ldots i_{k+l}}}{\partial x^i} + A^b_{i_1\ldots i_k}\frac{\partial B^a_{i_{k+1}\ldots i_{k+l}}}{\partial z^b} + \ldots + D_\beta A^b_{i_1\ldots i_k}\frac{\partial B^a_{i_{k+1}\ldots i_{k+l}}}{\partial z^b_\beta} \\
& - (-1)^{kl}\bigg(B^i_{i_1\ldots i_l}\frac{\partial A^a_{i_{l+1}\ldots i_{k+l}}}{\partial x^i} + B^b_{i_1\ldots i_l}\frac{\partial A^a_{i_{l+1}\ldots i_{k+l}}}{\partial z^b} + \ldots \\
& \qquad \ldots + D_\alpha B^b_{i_1\ldots i_l}\frac{\partial A^a_{i_{l+1}\ldots i_{k+l}}}{\partial z^b_\alpha}\bigg) \\
& - kA^a_{i_1\ldots i_{k-1}i}\frac{\partial B^a_{i_{k+1}\ldots i_{k+l}}}{\partial x^{i_k}} + (-1)^{kl} l B^i_{i_1\ldots i_{l-1}i}\frac{\partial A^i_{i_{l+1}\ldots i_{k+l}}}{\partial x^{i_l}}.
\end{aligned}
$$

Proof. This follows directly from (12) and (17).

5. Connections on $\mathcal{F}(E_1, E_2)$.

In accordance with [1], a *connection Γ* on $\mathcal{F}(E_1, E_2)$ can be defined as a differentiable projectable tangent valued 1-form on $\mathcal{F}(E_1, E_2)$ over id_{TM}. On the one hand, we have the curvature $C\Gamma$ of Γ introduced in [1]. On the other hand, we can construct the F-N bracket $[\Gamma, \Gamma]$.

Proposition 5.1. *We have $[\Gamma, \Gamma] = 2C\Gamma$.*

Proof. For 1-forms A and B, formula (17) reads

$$
\begin{aligned}
(19)\quad [A, B](X, Y) = {} & [A(X), B(Y)] + [B(X), A(Y)] + A(B^0([X, Y])) \\
& + B(A^0([X, Y])) - A([X, B^0(Y)]) \\
& - A([B^0(X), Y]) - B([X, A^0(Y)]) - B([A^0(X), Y]).
\end{aligned}
$$

Setting $A = B = \Gamma$, $A^0 = B^0 = \mathrm{id}$, we obtain $2[\Gamma(X), \Gamma(Y)] - 2\Gamma([X, Y])$. By Proposition 9 of [1], this is $2C\Gamma(X, Y)$. ■

Let Γ be a connection and A a differentiable projectable tangent valued k-form on $\mathcal{F}(E_1, E_2)$. The following definition generalizes [9].

Definition 5.1. The F-N bracket $[\Gamma, A]$ is said to be the *covariant exterior differential* of A with respect to Γ.

PROPOSITION 5.2 (Bianchi identity). *Let Γ be a twice differentiable connection on $\mathcal{F}(E_1, E_2)$. Then $[\Gamma, C\Gamma] = 0$.*

Proof. The relation $[\Gamma, [\Gamma, \Gamma]] = 0$ follows directly from the graded Jacobi identity. ∎

Consider another connection Δ on $\mathcal{F}(E_1, E_2)$.

DEFINITION 5.2. The F-N bracket $[\Gamma, \Delta]$ is said to be the *mixed curvature* of Γ and Δ.

PROPOSITION 5.3. *We have*

$$[\Gamma, \Delta](X, Y) = [\Gamma(X), \Delta(Y)] + [\Delta(X), \Gamma(Y)] - \Gamma([X, Y]) - \Delta([X, Y])$$

for every $X, Y \in C^\infty TM$.

Proof. This follows from (19). ∎

References

[1] A. Cabras and I. Kolář, *Connections on some functional bundles*, to appear.

[2] A. Frölicher, *Smooth structures*, in: Category Theory 1981, Lecture Notes in Math. 962, Springer, 1982, 69–81.

[3] A. Frölicher and A. Nijenhuis, *Theory of vector valued differential forms, I*, Indag. Math. 18 (1956), 338–359.

[4] A. Jadczyk and M. Modugno, *An outline of a new geometrical approach to Galilei general relativistic quantum mechanics*, to appear.

[5] —, —, *Galilei general relativistic quantum mechanics*, preprint 1993, 1–220.

[6] I. Kolář, *On the second tangent bundle and generalized Lie derivatives*, Tensor (N.S.) 38 (1982), 98–102.

[7] I. Kolář, P. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

[8] Y. Kosmann-Schwarzbach, *Vector fields and generalized vector fields on fibred manifolds*, in: Lecture Notes in Math. 792, Springer, 1982, 307–355.

[9] L. Mangiarotti and M. Modugno, *Graded Lie algebras and connections on fibred spaces*, J. Math. Pures Appl. 83 (1984), 111–120.

[10] P. Michor, *Gauge Theory for Fiber Bundles*, Bibliopolis, Napoli, 1991.

[11] A. Vanžurová, *On geometry of the third order tangent bundle*, Acta Univ. Palack. Olomuc. Math. 24 (1985), 81–96.

Department of Mathematics
Masaryk University
Janáčkovo nám. 2a
662 95 Brno, Czech Republic
E-mail: kolar@math.muni.cz

Department of Applied Mathematics "G. Sansone"
Via S. Marta 3
50139 Florence, Italy
E-mail: modugno@dma.unifi.it

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