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On some radius results for normalized analytic functions

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Abstract. We investigate some radius results for various geometric properties concerning some subclasses of the class $\mathcal{S}$ of univalent functions.

1. Introduction. Let $\mathcal{A}$ denote the class of all normalized functions $f(z)$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are *univalent* in $\mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{S}$ consisting of all functions which are, respectively, *starlike* and *convex of order* α in $\mathcal{U}$ $(0 \le \alpha < 1)$, that is,

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathcal{U} \right\} \tag{1.2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathcal{U} \right\}. \tag{1.3}$$

Further, we introduce the sets

$$\mathrm{UST} := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left(\frac{f(z) - f(\zeta)}{(z-\zeta)f'(z)}\right) \ge 0,\ (z,\zeta) \in \mathcal{U} \times \mathcal{U} \right\} \tag{1.4}$$

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and

$$\text{(1.5)}\quad \mathrm{UCV} := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0,\ (z,\zeta) \in \mathcal{U}\times\mathcal{U} \right\}$$

which were defined by Goodman [3, 4].

Each of the classes UST and UCV has a natural geometric interpretation: $f \in$ UST if and only if the image of every circular arc in $\mathcal{U}$ with center ζ also in $\mathcal{U}$ is *starlike* with respect to $f(\zeta)$, and $f \in$ UCV if and only if the image of every circular arc is *convex*.

Note that if we take $\zeta = 0$ in (1.4) and (1.5) we have the usual classes of *starlike* and *convex* functions, and if we let $\zeta \to z$, then the conditions are trivially fulfilled.

Let $S_p(\alpha)$ be the class defined by

$$\text{(1.6)}\qquad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)} - \alpha.$$

We see that for all $\alpha \in [-1, 1)$ we have $S_p(\alpha) \subset \mathcal{S}^*(0)$. Introducing the class UCV(α) (uniformly convex functions of order α) by $g \in UCV(\alpha) \Leftrightarrow zg' \in S_p(\alpha)$, we observe that UCV$(\alpha) \subset \mathcal{K}(0)$ for $\alpha \in [-1, 1)$ (see [7, 8]).

Then $f \in UCV(\alpha)$ if and only if

$$\text{(1.7)}\qquad \operatorname{Re}\left\{ 1 + (z-\zeta)\frac{f''(z)}{f'(z)} \right\} \geq \alpha, \qquad (z,\zeta) \in \mathcal{U}\times\mathcal{U}.$$

Clearly we have UCV$(0) =$ UCV. We easily find that [6]

$$g \in UCV \Leftrightarrow zg' \in S_p(0) \equiv S_p.$$

Let α_j $(j = 1, \ldots, p)$ and β_j $(j = 1, \ldots, q)$ be complex numbers with

$$\beta_j \neq 0, -1, -2, \ldots, \qquad j = 1, \ldots, q.$$

Then the *generalized hypergeometric function* ${}_pF_q(z)$ is defined by

$$\text{(1.8)}\qquad \begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \cdot \frac{z^n}{n!}, \qquad p \leq q+1, \end{aligned}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the gamma function, by

$$\text{(1.9)}\qquad \begin{aligned} (\lambda)_n &:= \Gamma(\lambda+n)/\Gamma(\lambda) \\ &= \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \end{cases} \end{aligned}$$

The ${}_pF_q(z)$ series in (1.8) converges absolutely for $|z| < \infty$ if $p < q + 1$, and for $z \in \mathcal{U}$ if $p = q + 1$. Furthermore, if we set

$$w = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.10}$$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is absolutely convergent for

$$|z| = 1 \text{ if } \operatorname{Re}(w) > 0,$$

and conditionally convergent for

$$|z| = 1 \ (z \neq 1) \text{ if } -1 < \operatorname{Re}(w) \leq 0.$$

Let $\sigma_\alpha(f)$ denote the largest number r such that $f(z)$ is univalent on $\mathcal{U}_r := \{z \in \mathbb{C} : |z| < r \leq 1\}$ and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad \text{on } \mathcal{U}_r \tag{1.11}$$

and let $k_\alpha(f)$ denote the largest number r such that $f(z)$ is univalent on $\mathcal{U}_r$ and

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad \text{on } \mathcal{U}_r. \tag{1.12}$$

Similarly, $\sigma_{\mathrm{UST}}(f)$ denotes the largest number r such that $f(z)$ is univalent on $\mathcal{U}_r$ and

$$\operatorname{Re}\left\{\frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)}\right\} \geq 0, \quad (z, \zeta) \in \mathcal{U}_r \times \mathcal{U}_r, \tag{1.13}$$

$\sigma_{S_p(\alpha)}(f)$ denotes the largest number r such that $f(z)$ is univalent on $\mathcal{U}_r$ and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\frac{zf'(z)}{f(z)} - \alpha, \quad z \in \mathcal{U}_r, \tag{1.14}$$

and $k_{\mathrm{UCV}(\alpha)}(f)$ denotes the largest number r such that $f(z)$ is univalent on $\mathcal{U}_r$ and

$$\operatorname{Re}\left\{1 + (z - \zeta)\frac{f''(z)}{f'(z)}\right\} \geq \alpha, \quad (z, \zeta) \in \mathcal{U}_r \times \mathcal{U}_r. \tag{1.15}$$

For $0 < p \leq \infty$ and a function $f(z)$ in $\mathcal{U}$, define the integral means $M_p(r, f)$ by

$$M_p(r, f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & \text{if } 0 < p < \infty, \\ \max_{|z| \leq r} |f(z)| & \text{if } p = \infty. \end{cases} \tag{1.16}$$

Then, by definition, an analytic function $f(z)$ in $\mathcal{U}$ belongs to the *Hardy space* $\mathcal{H}^p$ $(0 < p \leq \infty)$ if

$$\|f\|_p := \lim_{r \to 1-} M_p(r, f) < \infty. \tag{1.17}$$

For $f \in \mathcal{A}$ we set

$$\Phi_p(r,f) = \frac{r}{\{1 + M_p(r, f' - 1)^p\}^{1/p}} \quad (0 \le r < 1), \tag{1.18}$$

and

$$\Phi_p(f) = \sup_{0\le r<1} \Phi_p(r,f) \quad (0 < p < \infty). \tag{1.19}$$

For the functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \quad (a_{j,1} := 1;\ j = 1,2), \tag{1.20}$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) := \sum_{n=0}^{\infty} a_{1,n+1} a_{2,n+1} z^{n+1} \quad (a_{j,1} := 1;\ j = 1,2). \tag{1.21}$$

Let λ denote normalized Lebesgue area measure on $\mathcal{U}$; and, for $\beta > -1$, λ_β denote the finite measure defined on $\mathcal{U}$ by

$$d\lambda_\beta(z) = (1 - |z|^2)^\beta d\lambda(z). \tag{1.22}$$

For $\beta > -1$ and $0 < p < \infty$ the *weighted Bergman space* A^p_β is the collection of all functions f holomorphic in $\mathcal{U}$ for which

$$\|f\|^p_{p,\beta} = \int_{\mathcal{U}} |f|^p \, d\lambda_\beta < \infty. \tag{1.23}$$

The *weighted Dirichlet space* D_β $(\beta > -1)$ is the collection of all functions f holomorphic in $\mathcal{U}$ for which the derivative f' belongs to A^2_β. It is well known that A^p_β is a complete linear metric space for $p > 0$, a Banach space if $p \ge 1$, and a Hilbert space if $p = 2$.

The space D_β is a Hilbert space with the norm $\|\cdot\|_{D_\beta}$ defined by

$$\|f\|^2_{D_\beta} = |f(0)|^2 + \int_{\mathcal{U}} |f'|^2 \, d\lambda_\beta. \tag{1.24}$$

In this paper, we investigate some radii problems for various geometric properties concerning the subclasses of the class $\mathcal{S}$ of univalent functions.

2. A set of lemmas.

The following lemmas will be required in our investigation.

Lemma 1 (Hausdorff–Young [1, Theorem 6.1, p. 94]). *Let $f \in \mathcal{H}^p$, $1 \le p \le 2$. Then*

$$\Big(\sum_{n=0}^{\infty} |a_n|^q\Big)^{1/q} \le \|f\|_p, \quad 1/p + 1/q = 1,$$

where the left-hand side is $\sup_{n\ge0} |a_n|$ *if* $p = 1$.

Lemma 2 (H. Silverman [9, Theorem 1, Corollary, p. 110]). *Let $f(z)$ be defined by* (1.1) *and* $0 \le \alpha < 1$. *Then*

(i) $$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha \Rightarrow \sigma_\alpha(f) = 1,$$

(ii) $$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le 1-\alpha \Rightarrow k_\alpha(f) = 1.$$

Lemma 3. *Let $f(z)$ be defined by* (1.1) *and* $0 \le \alpha < 1$. *Then*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha \Rightarrow k_\alpha(f) \ge 1/2.$$

Further, the constant $1/2$ *is best possible.*

Proof. Let $f(z) \in \mathcal{A}$ be such that $\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha$. Put $g(z) = 2f(z/2) = z + \sum_{n=2}^{\infty} a_n (1/2)^{n-1} z^n \equiv \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$. Then

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|c_n| \le \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| \le 1.$$

By Lemma 2, we obtain $k_\alpha(g) = 1$ and $k_\alpha(f) \ge 1/2$. ∎

Lemma 4 (A. W. Goodman [4, Theorem 6, p. 369; 3, Theorem 6, p. 91]). *Let $f(z)$ be defined by* (1.1). *Then*

(i) $$\sum_{n=2}^{\infty} n|a_n| \le \sqrt{2}/2 \Rightarrow \sigma_{\mathrm{UST}}(f) = 1,$$

(ii) $$\sum_{n=2}^{\infty} n(n-1)|a_n| \le 1/3 \Rightarrow k_{\mathrm{UCV}}(f) = 1.$$

Further, the number $1/3$ *above is the largest possible.*

Lemma 5. *Let $f(z)$ be defined by* (1.1) *and* $-1 \le \alpha < 1$. *Then*

(i) $$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \frac{1-\alpha}{3-\alpha} \Rightarrow k_{\mathrm{UCV}(\alpha)}(f) = 1.$$

Further, the constant $\frac{1-\alpha}{3-\alpha}$ *above cannot be replaced by a larger number.*

(ii) $$\sum_{n=2}^{\infty}(n-1)|a_n| \le \frac{1-\alpha}{3-\alpha} \Rightarrow k_{S_p(\alpha)}(f) = 1.$$

(iii) $$\sum_{n=2}^{\infty}(n-1)|a_n| \le \frac{1-\alpha}{3-\alpha} \Rightarrow k_{\mathrm{UCV}(\alpha)}(f) \ge 1/2.$$

P r o o f. (i) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \frac{1-\alpha}{3-\alpha}.$$

Then

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{1-\alpha}{3-\alpha}.$$

Further,

$$\begin{aligned} 1 + \mathrm{Re}\left\{\frac{f''(z)(z-\zeta)}{(1-\alpha)f'(z)}\right\} &\ge 1 - \frac{1}{1-\alpha} \cdot \frac{\sum_{n=2}^{\infty} n(n-1)|a_n| \cdot |z^{n-2}|}{1 - \sum_{n=2}^{\infty} n|a_n| \cdot |z^{n-1}|}|z-\zeta| \\ &\ge 1 - \frac{\frac{2(1-\alpha)}{3-\alpha}}{(1-\alpha)\big(1 - \frac{1-\alpha}{3-\alpha}\big)} = 0. \end{aligned}$$

Thus $k_{\mathrm{UCV}(\alpha)}(f) = 1$. But equality is attained for the function $f(z) = z - \frac{1-\alpha}{6-2\alpha} z^2$ with $z = 1$ and $\zeta = -1$.

(ii) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with

$$\sum_{n=2}^{\infty} (n-1)|a_n| \le \frac{1-\alpha}{3-\alpha}.$$

Then there exists

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n, \quad \text{i.e.} \quad zg'(z) = f(z),$$

such that

$$\sum_{n=2}^{\infty} n(n-1)|b_n| = \sum_{n=2}^{\infty} (n-1)|a_n| \le \frac{1-\alpha}{3-\alpha}.$$

Thus, by (i), $k_{\mathrm{UCV}(\alpha)}(g) = 1$, i.e. $g(z) \in \mathrm{UCV}(\alpha)$. Therefore, by the relation between $\mathrm{UCV}(\alpha)$ and $S_p(\alpha)$, $f \in S_p(\alpha)$, i.e. $\sigma_{S_p(\alpha)}(f) = 1$.

(iii) The proof is much akin to that of Lemma 3, with (i) above used in place of Lemma 2. ■

3. Results. By using Lemmas 1 and 2, we obtain

THEOREM 1. *Let $f(z)$ be defined by* (1.1). *Then*

$$\sigma_\alpha(f) \ge \Phi_p(g_\alpha) \quad (0 \le \alpha < 1;\ 1 \le p \le 2), \tag{3.1}$$

where

$$g_\alpha(z) = \left[\frac{1}{1-\alpha}\left\{\frac{z}{1-z} + \alpha \log(1-z)\right\}\right] * f(z). \tag{3.2}$$

Moreover,

$$g_\alpha(z) = [z\ {}_3F_2(2-\alpha,1,1;1-\alpha,2;z)] * f(z) = \sum_{n=1}^{\infty}\frac{n-\alpha}{1-\alpha}\cdot\frac{1}{n}a_n z^n. \tag{3.3}$$

P r o o f. We may put $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \neq z$.

For fixed r, $0<r<1$, define

$$R = \Phi_p(r, g_\alpha) \quad (0 \le \alpha < 1). \tag{3.4}$$

Then we easily find that $0<R<r$.

Set $h(z) = g'_\alpha(rz) - 1$. Then Lemma 1 gives

$$\left\{\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n|r^{n-1}\right)^q\right\}^{1/q} \le \|h\|_p = M_p(r, g'_\alpha - 1), \tag{3.5}$$

where $1/p+1/q=1$, and the left-hand side of (3.5) attains its supremum when $p=1$. Thus

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n|R^{n-1} &= \sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n|r^{n-1}(R/r)^{n-1}\\ &\le \left\{\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n|r^{n-1}\right)^q\right\}^{1/q}\left\{\sum_{n=2}^{\infty}(R/r)^{pn-p}\right\}^{1/p}\\ &\le M_p(r, g'_\alpha - 1)\left\{\sum_{n=2}^{\infty}(R/r)^{pn-p}\right\}^{1/p} = 1,\end{aligned}$$

by the Hölder inequality.

Lemma 2 shows that $\sigma_\alpha(u) = 1$ for $u(z) = R^{-1}f(Rz)$ and $\sigma_\alpha(f) \ge R$, since r is arbitrary. Hence we get the inequality $\sigma_\alpha(f) \ge \Phi_p(g_\alpha)$. ■

THEOREM 2. *Let $f(z)$ be defined by* (1.1) *and let $g_\alpha(z)$, $0 \le \alpha < 1$, be defined by* (3.2). *Then*

$$\sigma_\alpha(f) \ge \left\{\frac{1}{(\beta+2)\|g_\alpha\|^2_{D_\beta}}\right\}^{1/2} \quad (\beta > -1). \tag{3.6}$$

P r o o f. By Theorem 1 and (1.18), we have

$$\sigma_\alpha(f) \ge \frac{r}{\{1 + M_2(r, g'_\alpha - 1)^2\}^{1/2}}. \tag{3.7}$$

Since

$$d\lambda_\beta(z) \ = \ \frac{\beta+1}{\pi}(1-r^2)^\beta r\,dr\,d\theta \quad (|z| = r),$$

we obtain

$$\begin{aligned}(3.8)\qquad \|g_\alpha\|_{D_\beta}^2 &= \frac{\beta+1}{\pi}\int_0^1\int_0^{2\pi} |g_\alpha'(re^{i\theta})|^2(1-r^2)^\beta r\,d\theta\,dr\\ &= 2(\beta+1)\int_0^1 M_2(r,g_\alpha')^2(1-r^2)^\beta r\,dr.\end{aligned}$$

From (3.3) we observe that

$$(3.9)\qquad M_2(r,g_\alpha')^2 = 1 + M_2(r,g_\alpha'-1)^2 = \sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\right)^2 |a_n|^2.$$

Hence

$$\begin{aligned}(3.10)\qquad &\{\sigma_\alpha(f)\}^2\|g_\alpha\|_{D_\beta}^2\\ &= 2(\beta+1)\int_0^1 \{\sigma_\alpha(f)\}^2\{1+M_2(r,g_\alpha'-1)\}(1-r^2)^\beta r\,dr\\ &\ge 2(\beta+1)\int_0^1 r^3(1-r^2)^\beta\,dr = (\beta+1)\int_0^1 r(1-r)^\beta\,dr\\ &= (\beta+1)B(2,\beta+1) = \frac{1}{\beta+2},\end{aligned}$$

where $B(\alpha,\beta)$ denotes the beta function. Hence the proof is complete. ∎

Remark. Letting $\beta\to -1$, we easily find that

$$(3.11)\qquad \sigma_\alpha(f) \ge 1/\|g_\alpha'\|_2.$$

Furthermore, for $\alpha = 0$, we obtain the result of Goluzin [2, Theorem 23, p. 187].

Theorem 3. *Let $f(z)$ be defined by* (1.1) *and let $g_\alpha(z)$, $0\le\alpha<1$, be defined by* (3.2). *Then*

$$(3.12)\qquad k_\alpha(f) \ge \Phi_p(g_\alpha)/2 \quad (1\le p\le 2).$$

Proof. The proof is much akin to that of Theorem 1 which we have detailed above. Indeed, in place of Lemma 2, we make use of Lemma 3. ∎

Remark. If we put $\alpha = 0$ in Theorems 1 and 3, then we easily find that

$$(3.13)\qquad \sigma_0(f) \ge \Phi_p(f)$$

and

$$(3.14)\qquad k_0(f) \ge \Phi_p(f)/2,$$

which are the results of Yamashita [11, Theorem 2, Theorem 2C, pp. 1095–1096].

From Lemmas 1 and 4 we have

THEOREM 4. *Let* $f(z)$ *be defined by* (1.1) . *Then*

$$\sigma_{\mathrm{UST}}(f) \geq \Phi_p(v) \quad (1 \leq p \leq 2), \tag{3.15}$$

where

$$v(z) = z + \sqrt{2}\,(f(z) - z). \tag{3.16}$$

Define

$$h_\alpha(z) = z + \frac{3-\alpha}{1-\alpha}\left\{\frac{z}{1-z} + \log(1-z)\right\} \quad (-1 \leq \alpha < 1;\ z \in \mathcal{U}). \tag{3.17}$$

Put $u_\alpha(z) = h_\alpha * f(z)$. Then

$$u_\alpha(z) = z + \sum_{n=2}^{\infty} \frac{3-\alpha}{1-\alpha}\left(1 - \frac{1}{n}\right) a_n z^n. \tag{3.18}$$

Hence, by using Lemmas 1 and 5, we have

THEOREM 5. *Let* $f(z)$ *be defined by* (1.1) *and* $-1 \leq \alpha < 1$. *Then for* $1 \leq q \leq 2$,

$$\sigma_{S_p(\alpha)}(f) \geq \Phi_q(u_\alpha) \tag{3.19}$$

and

$$k_{\mathrm{UCV}(\alpha)}(f) \geq \Phi_q(u_\alpha)/2, \tag{3.20}$$

where u_α *is defined by* (3.18).

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