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Aspects of unconditionality of bases in spaces of compact operators

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Abstract. E. Tutaj has introduced classes of Schauder bases termed "unconditional-like" (UL) and "unconditional-like*" (UL*) whose intersection is the class of unconditional bases. In view of this association with unconditional bases, it is interesting to note that there exist Banach spaces which have no unconditional basis and yet have a basis of one of these two types (e.g., the space $\mathcal{O}[0,1]$). In the same spirit, we show in this paper that the space of all compact operators on a reflexive Banach space with an unconditional basis has a basis of type UL*, even though it is well-known that this space has no unconditional basis.

1. Introduction. In the papers [6] and [7] E. Tutaj has introduced and studied the fundamental properties of two classes of Schauder bases in Banach spaces related closely to unconditional bases and consequently termed "unconditional-like" (UL) and "unconditional-like*" (UL*). In particular, a Schauder basis $\{x_n\}_{n=1}^{\infty}$ for a real Banach space E is said to be *of type* UL if convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies convergence of $\sum_{n=1}^{\infty} |a_n| x_n$, and *of type* UL* if whenever $\sum_{n=1}^{\infty} |a_n| x_n$ converges, so does $\sum_{n=1}^{\infty} a_n x_n$. Since an unconditional basis is one for which convergence of the series $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of every rearrangement as well, one can show that a basis is unconditional if and only if it is of type UL and of type UL* (see [6]), and in view of this relationship it is natural to designate bases of either of these two types as "unconditional-like" and to think of each as having some essential aspect of unconditionality even if they are, in fact, conditional. What is interesting is that though a Banach space may not have an unconditional basis, it may still have a basis of type UL or UL*. For example, Tutaj [6] has demonstrated the existence of a basis of type UL in the space D of Lindenstrauss, which is known to have no unconditional basis

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[3], and a basis of type UL* in $\mathcal{C}[0,1]$ (see [7]), another space which has no unconditional basis [2].

In the same spirit we will show in this paper that if E is a reflexive Banach space having an unconditional basis $\{x_n, x_n^*\}_{n=1}^\infty$, then the standard "tensor product" basis $\{x_n^* \otimes x_m\}$ of one-dimensional operators for the space $K(E)$ of compact operators on E is of type UL*, even though it is known from results of Pełczyński and Kwapień [4] that $K(E)$ has no unconditional basis.

2. Recall that if E is a reflexive Banach space having a Schauder basis $\{x_n\}_{n=1}^\infty$ with coefficient functionals $\{x_n^*\}_{n=1}^\infty$ in E^*, then $\{x_n^*\}_{n=1}^\infty$ is a basis for E^* and the sequence of one-dimensional operators $\{x_n^* \otimes x_m\}$ ordered in "blocks" $\{B_k\}_{k=1}^\infty$ of $2k-1$ operators, $k = 1, 2, \ldots$, as $\{B_1, \ldots, B_k, \ldots\} = \{x_1^* \otimes x_1, x_1^* \otimes x_2, x_2^* \otimes x_2, x_2^* \otimes x_1, \ldots, x_1^* \otimes x_k, x_k^* \otimes x_{k-1}, \ldots, x_k^* \otimes x_1, \ldots\}$ is a basis for $E^* \otimes_\lambda E$, the completion of the linear space $E^* \otimes E$ of all finite-dimensional operators on E in the operator norm, and hence for $K(E)$, which is identified with $E^* \otimes_\lambda E$ in this case. Gelbaum and Gil de Lamadrid [1] showed that if $\{x_n, x_n^*\}_{n=1}^\infty$ is an unconditional basis in E, the tensor product basis $\{x_n^* \otimes x_m\}$ for $K(E)$ need not be unconditional, even for the case of an orthonormal basis in Hilbert space, a result extended by Pełczyński and Kwapień in the paper referred to above [4]. Our purpose is to prove the following result which shows that, in spite of these negative results concerning the existence of unconditional bases in $K(E)$, any tensor product basis $\{x_n^* \otimes x_m\}$ is always of type UL*.

THEOREM. *If E is a reflexive Banach space and $\{x_n, x_n^*\}_{n=1}^\infty$ an unconditional basis for E, the basis $\{x_n^* \otimes x_m\}$ for $K(E)$ is of type* UL*.

Proof. For convenience of notation, we will denote the basis $\{x_n^* \otimes x_m\}$ for $K(E)$ (in the order described above) as a sequence $\{T_j\}_{j=1}^\infty$ of one-dimensional operators of the form $x_n^* \otimes x_m$; i.e., $\{T_j\}_{j=1}^\infty = \{x_1^* \otimes x_1, x_2^* \otimes x_1, x_2^* \otimes x_2, x_1^* \otimes x_2, \ldots\} = \{B_1, \ldots, B_k \ldots\}$, as we described earlier. Correspondingly, the series $\sum_{n,m} c_{nm} x_n^* \otimes x_m$ will be more simply denoted by $\sum_{j=1}^\infty b_j T_j$, where $b_j = c_{nm}$ in the appropriate ordering of the bases.

Let us also recall the following characterization of unconditional bases [5, p. 500] which is a quantitative version of Tutaj's observation [6] concerning the equivalence of unconditionality and the properties UL and UL*:

(∗) *A basis $\{x_n\}_{n=1}^\infty$ for E is unconditional $\Leftrightarrow$ there are constants α and $\beta, 0 < \alpha \le 1 \le \beta$, so that $\alpha \|\sum_{i=p}^q |a_i| x_i\| \le \|\sum_{i=p}^q a_i x_i\| \le \beta \|\sum_{i=p}^q |a_i| x_i\|$ for all $1 \le p \le q < \infty$ and for all scalars $a_p, a_{p+1}, \ldots, a_q$.*

Now, suppose $\{x_n, x_n^*\}_{n=1}^{\infty}$ is an unconditional basis for a reflexive Banach space E and the series $\sum_{n,m} |a_{nm}| x_n^* \otimes x_m = \sum_{j=1}^{\infty} |b_j| T_j$ converges in the space $K(E)$ of compact operators on E. Then the sequence of partial sums of this series is Cauchy, so given any $\varepsilon > 0$ there exists some p_0 for which $\| \sum_{j=p_0}^{q} |b_j| T_j \| < \varepsilon$ for all $q \geq p_0$. An inspection of the ordering of the basis $\{x_n^* \otimes x_m\}$ shows that any such partial sum is of the form $\sum_{k \in A_q} x_k^* \otimes |v_k|$, where A_q is a subset of $\{1, \ldots, N_q\}$ for some N_q, $v_k = \sum_{i \in \sigma_k} b_{i_k} x_i$ for σ_k some subset of $\{1, \ldots, N_k\}$, and $|v_k| = \sum_{i \in \sigma_k} |b_{i_k}| x_i$ for each $k \in A_q$.

If $\| \sum_{j=p_0}^{q} |b_j| T_j \| < \varepsilon$ for some p_0 and some $q \geq p_0$, then by the above (and the definition of the norm in $K(E)$) we have

$$\sup_{\|x\| \leq 1} \Big\| \sum_{k \in A_q} \langle x_k^*, x \rangle |v_k| \Big\| < \varepsilon.$$

From the characterization $(*)$ of unconditional bases it follows that there are positive constants α and β so that, for each $x = \sum_{n=1}^{\infty} \langle x_n^*, x \rangle x_n$ with $\|x\| \leq 1$, we have $\big\| |x| \big\| = \big\| \sum_{n=1}^{\infty} |\langle x_n^*, x \rangle| x_n \big\| \leq \alpha$, so

$$\begin{aligned}
\varepsilon > \Big\| \sum_{j=p_0}^{q} |b_j| T_j \Big\| &= \Big\| \sum_{k \in A_q} x_k^* \otimes |v_k| \Big\| \\
&\geq \alpha \sup_{\|x\| \leq 1} \Big\| \sum_{k \in A_q} \langle x_k^*, x \rangle |v_k| \Big\| \\
&= \alpha \sup_{\|x\| \leq 1} \Big\| \sum_{k \in A_q} \langle x_k^*, x \rangle \sum_{i \in \sigma_k} |b_{i_k}| x_i \Big\| \\
&\geq \frac{\alpha}{\beta} \sup_{\|x\| \leq 1} \Big\| \sum_{k \in A_q} \langle x_k^*, x \rangle \sum_{i \in \sigma_k} b_{i_k} x_i \Big\| \qquad \text{(again, by } (*)) \\
&= \frac{\alpha}{\beta} \Big\| \sum_{k \in A_q} x_k^* \otimes v_k \Big\| = \frac{\alpha}{\beta} \Big\| \sum_{j=p_0}^{q} b_j T_j \Big\|,
\end{aligned}$$

and so $\| \sum_{j=p_0}^{q} b_j T_j \| < (\beta/\alpha) \cdot \varepsilon$. That is, if $\sum_{j=1}^{\infty} |b_j| T_j$ converges then the sequence of partial sums of the series $\sum_{j=1}^{\infty} b_j T_j$ is Cauchy, hence convergent in $K(E)$, and the theorem is proved.

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