ANNALES
POLONICI MATHEMATICI
LXVIII.1 (1998)

Parabolic differential-functional inequalities in viscosity sense

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Abstract. We consider viscosity solutions for second order differential-functional equations of parabolic type. Initial value and mixed problems are studied. Comparison theorems for subsolutions, supersolutions and solutions are considered.

1. Introduction. Let $\Omega \subseteq \mathbb{R}^n$ be any open domain and $T > 0$, $\tau_0, r \in \mathbb{R}_+ = [0, \infty)$ given constants. Define

$$\Omega_r = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) \le r\}, \quad \delta_0\Omega = \Omega_r \setminus \Omega, \quad \Theta = (0,T) \times \Omega,$$
$$\Theta_0 = [-\tau_0, 0] \times \Omega_r, \quad \delta_0\Theta = (0,T) \times \delta_0\Omega, \quad \Gamma = \Theta_0 \cup \delta_0\Theta, \quad E = \Gamma \cup \Theta.$$

(Note that if $\Omega = \mathbb{R}^n$ then $\Omega_r = \mathbb{R}^n$, $\delta_0\Theta = \delta_0\Omega = \emptyset$ and $\Gamma = \Theta_0$.) Let $D = [-\tau_0, 0] \times B(r)$, where $B(r) = \{x \in \mathbb{R}^n : |x| \le r\}$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. For every $z : E \to \mathbb{R}$ and $(t,x) \in \Theta$ we define a function $z_{(t,x)} : D \to \mathbb{R}$ by $z_{(t,x)}(s,y) = z(t+s, x+y)$ for $(s,y) \in D$.

For every metric space X we denote by $C(X)$ the class of all continuous functions from X into $\mathbb{R}$ and by $\mathrm{BUC}(X)$ the class of all uniformly continuous and bounded functions from X into $\mathbb{R}$. We will write $\|\cdot\|_X$ for the supremum norm. Let $\mathcal{M}(n)$ stand for the space of $n \times n$ real symmetric matrices. Recall that $A \ge B$ if for all $\xi \in \mathbb{R}^n$ we have $\langle A\xi, \xi\rangle \ge \langle B\xi, \xi\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the standard inner product. For $A \in \mathcal{M}(n)$ we denote by $\|A\|$ the norm of A. Let $F : \Theta \times \mathbb{R} \times C(D) \times \mathbb{R}^n \times \mathcal{M}(n) \to \mathbb{R}$ be a continuous function of the variables (t,x,u,w,p,A) and $g \in C(\Gamma)$ be a given function.

We write $C^{1,2}(\Theta)$ (resp. $C^{1,2}(E)$) for the set of all functions from Θ (resp. E) into $\mathbb{R}$ with continuous derivatives $D_t u, D_x u, D_x^2 u$.

We consider the initial-boundary value problem

$$D_t z + F(t,x,z(t,x),z_{(t,x)},D_x z(t,x),D_x^2 z(t,x)) = 0 \quad \text{in } \Theta, \tag{1}$$
$$z(t,x) = g(t,x) \quad \text{in } \Gamma. \tag{2}$$

1991 *Mathematics Subject Classification*: 35D99, 35B30.
Key words and phrases: viscosity solution, differential-functional equation.

Even though we say "initial-boundary value problem" it is an initial value problem for $\Theta = (0,T) \times \mathbb{R}^n$.

Problem (1), (2) contains as a particular case equations with retarded argument and a few kinds of differential-integral equations.

Definition 1. A function $u \in C(E)$ is called *F-subparabolic* (resp. *F-superparabolic*) provided for all $\psi \in C^{1,2}(\Theta)$, if $u - \psi$ attains a local maximum (resp. minimum) at $(t_0, x_0) \in \Theta$ then

$$F(t_0, x_0, u(t_0, x_0), u_{(t_0,x_0)}, D_x\psi(t_0, x_0), A) \geq F(t_0, x_0, u(t_0, x_0), u_{(t_0,x_0)}, D_x\psi(t_0, x_0), B)$$

whenever $A \leq B$.

A function $u \in C(E)$ is called *F-parabolic* if u is both F-subparabolic and F-superparabolic.

Definition 2. A function $u \in C(E)$ is a *viscosity subsolution* (resp. *supersolution*) of (1), (2) if u is F-subparabolic (resp. F-superparabolic) and provided for all $\varphi \in C^{1,2}(\Theta)$, if $u - \varphi$ attains a local maximum (resp. minimum) at $(t_0, x_0) \in \Theta$ then

$$D_t\varphi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), u_{(t_0,x_0)}, D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) \leq 0 \tag{3}$$

(resp. $D_t\varphi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), u_{(t_0,x_0)}, D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) \geq 0$) and

$$u(t,x) \leq g(t,x) \quad (\text{resp. } u(t,x) \geq g(t,x)) \quad \text{in } \Gamma \tag{4}$$

Definition 3. A function $u \in C(E)$ is a *viscosity solution* of (1), (2) if u is both a viscosity subsolution and supersolution of (1), (2).

We denote by $\mathrm{SUB}(F,g)$, $\mathrm{SUP}(F,g)$, $\mathrm{SOL}(F,g)$ the sets of all viscosity subsolutions, supersolutions and solutions of problem (1), (2).

The following is immediate:

Remark 1. If $u \in C(E) \cap C^{1,2}(\Theta)$ then $u \in \mathrm{SOL}(F,g)$ (resp. $u \in \mathrm{SUB}(F,g), \mathrm{SUP}(F,g)$) if and only if u is a classical solution (resp. subsolution, supersolution) of (1), (2).

This notion of solution was first introduced by M. G. Crandall and P. L. Lions in [4] and [6] for first order differential equations. The best general reference for second order equations is [3].

There are two ways of estimating solutions for parabolic inequalities. We can use one-variable or multi-variable comparison functions. The second method is presented in [5]. This work is devoted to the first. The main result for classical solutions were announced by J. Szarski in [7] and for functional-differential equations by the same author in [8, 9]. Sufficient conditions for the existence of classical solutions for functional-differential equations were given by Brzychczy in [1, 2].

2. Viscosity inequalities. A function ω is said to satisfy condition "P" if $\omega \in C([0,T]\times\mathbb{R}_+)$ is nondecreasing, positive and the right-hand maximum solution of the problem

$$y'(t) = \omega(t, y(t)), \quad y(0) = \sigma, \tag{5}$$

exists in $[0,T]$. We will denote this solution by $\mu(t,\sigma)$.

Write $a^+ = \max(0,a)$, $a^- = \max(0,-a)$ for $a \in \mathbb{R}$. For $G \subseteq \mathbb{R}^{n+1}$ set $G_t = \{(s,x) \in G : -\tau_0 \le s \le t\}$.

PROPOSITION 1. *Let $a > 0$ and $h, H \in C([0,a])$. Assume that h is a viscosity solution of $h' \le H$ (i.e. h is a viscosity subsolution of $h' = H$) in $(0,a)$. Then*

$$h(t) \le h(s) + \int_s^t H(\tau)\,d\tau \quad \textit{for } 0 \le s \le t \le a.$$

The proof can be found in [4], p. 12.

We will need the following

ASSUMPTION 1. 1) *There exists a function ω satisfying condition "P" such that for all $(t,x,u,w) \in \Theta \times \mathbb{R} \times C(D)$, if $u \ge 0$ then*

$$F(t,x,u,w,0,0) \ge -\omega(t, \max(u, \|w^+\|_D)).$$

2) *For every $R > 0$ and $|u|, \|w\|_D \le R$,*

$$[F(t,x,u,w,0,0) - F(t,x,u,w,p,A)]^+ \to 0 \quad \textit{as } p, A \to 0$$

uniformly with respect to $(t,x,u,w) \in \Theta \times \mathbb{R} \times C(D)$.

THEOREM 1. *Suppose that F satisfies Assumption* 1 *and $z \in \mathrm{BUC}(E) \cap \mathrm{SUB}(F,g)$. Then*

$$\|z^+\|_{E_t} \le \mu(t, \|g^+\|_{\Gamma_t}) \quad \textit{for } t \in [0,T]. \tag{6}$$

Proof. Put

$$M(t) = \|z^+\|_{\overline{\Theta}_t}, \quad \overline{M}(t) = \|z^+\|_{\overline{E}_t}, \quad M_0(t) = \|z^+\|_{\overline{\Gamma}_t} \quad \text{for } t \in [0,T]. \tag{7}$$

Since z is uniformly continuous it follows that $M, \overline{M}, M_0$ are continuous. (Note that if $\Omega = \mathbb{R}^n$ then $M_0(t) \equiv M_0(0)$.) It is evident that it suffices to show (6) for $t = T$.

If $M(T) \le M_0(T)$ there is nothing to prove. Suppose that $M(T) > M_0(T)$. Since $M(0) \le M_0(0)$ there exists $t^* \in [0,T)$ such that

$$\overline{M}(t^*) = M(t^*) = M_0(t^*) \quad \text{and} \quad M(t) > M_0(t) \quad \text{for } t \in (t^*, T]. \tag{8}$$

We will show that

$$M'(t) \le \omega(t, M(t)) \quad \text{in viscosity sense for } t \in (t^*, T) \tag{9}$$

(i.e. M is a viscosity subsolution of $y' = \omega(t,y)$). Let $\eta \in C^1((t^*,T))$ and suppose $M-\eta$ attains a local maximum at $t_0 \in (t^*,T)$. Since M is nondecreasing it is clear that $\eta'(t_0) \geq 0$. We claim that

$$\eta'(t_0) \leq \omega(t_0, M(t_0)). \tag{10}$$

Indeed, if $\eta'(t_0) = 0$ then (10) is obvious. Let $\eta'(t_0) > 0$. It follows from Lemma 1.4 of [4] that we can find a nondecreasing function $\overline{\eta} \in C^1([t^*,T])$ such that $\overline{\eta}'(t_0) = \eta'(t_0)$ and $(M-\overline{\eta})(t_0) > (M-\overline{\eta})(t)$ for $t \neq t_0$. To simplify notation we continue to write η for $\overline{\eta}$.

Put $I = [t^*,T]$. Define $\Phi : I \times \overline{\Omega} \to \mathbb{R}$ by

$$\Phi(t,x) = z(t,x)^+ - \eta(t). \tag{11}$$

Let $\delta > 0$ and let $(t',x') \in I \times \overline{\Omega}$ be such that $\Phi(t',x') > \sup \Phi - \delta$. Put

$$\Psi(t,x) = \Phi(t,x) + 2\delta\xi(x) \quad \text{for } (t,x) \in I \times \overline{\Omega} \tag{12}$$

where $\xi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \xi \leq 1$, $\xi(x') = 1$, $|D\xi|, \|D^2\xi\| \leq 1$ and $D\xi, D^2\xi$ are the derivatives of ξ. Since $\Psi = \Phi$ outside the support of ξ and $\Psi(t',x') > \sup \Phi + \delta$ there exists $(t_\delta, x_\delta) \in I \times \overline{\Omega}$ such that $\Psi(t_\delta, x_\delta) = \sup \Psi$. By the compactness of I we can assume, taking a subsequence if necessary, that $t_\delta \to \overline{t}$ as $\delta \to 0$.

We claim that $\overline{t} = t_0$. Indeed, since

$$z(t_\delta, x_\delta)^+ - \eta(t_\delta) + 2\delta \geq z(s,x)^+ - \eta(s) \quad \text{for } t^* \leq s \leq t \in I \tag{13}$$

and $\eta(s) \leq \eta(t)$ we obtain, by (8),

$$M(t_\delta) - \eta(t_\delta) + 2\delta \geq M(t) - \eta(t) \quad \text{for } t \in I. \tag{14}$$

Note that in view of (8), $M(t) = \sup\{z^+(s,x) : (s,x) \in \Theta_t \setminus \Theta_{t^*}\}$ for $t \in I$. Letting $\delta \to 0$ in (14) we get

$$M(\overline{t}) - \eta(\overline{t}) \geq M(t) - \eta(t) \quad \text{for } t \in I,$$

which means by the definition of t_0 that $\overline{t} = t_0$.

It also follows from (13), (14) (for $t = t_0$) that

$$\begin{aligned} M(t_0) - \eta(t_0) &\geq \limsup_{\delta \to 0} z(t_\delta, x_\delta)^+ - \eta(t_0) \\ &\geq \liminf_{\delta \to 0} z(t_\delta, x_\delta)^+ - \eta(t_0) \geq M(t_0) - \eta(t_0), \end{aligned}$$

which yields

$$\lim_{\delta \to 0} z(t_\delta, x_\delta)^+ = M(t_0). \tag{15}$$

Observe now that we may assume that $x_\delta \in \Omega$. Indeed, if $x_\delta \to x_0 \in \delta_0\Omega$ then $z(t_0,x_0)^+ \leq M_0(t_0)$ and by (15) we have $M(t_0) \leq M_0(t_0)$, which contradicts (8). Moreover, by (8), (15) we can also assume that $z^+(t_\delta, x_\delta) = z(t_\delta, x_\delta) > 0$. Put

$$\lambda(t,x) = \eta(t) - 2\delta\xi(x).$$

Notice that $z-\lambda$ attains a local maximum at $(t_\delta, x_\delta)\in(t^*,T)\times\Omega$. Since

$$D_t\lambda(t_\delta,x_\delta)=\eta'(t_\delta),\quad D_x\lambda(t_\delta,x_\delta)=-2\delta D\xi(x_\delta),$$
$$D_x^2\lambda(t_\delta,x_\delta)=-2\delta D^2\xi(x_\delta)$$

and $z\in \mathrm{SUB}(F,g)$ in $\Theta\setminus\Theta_{t^*}$ we obtain

$$\eta'(t_\delta)+F(t_\delta,x_\delta,z(t_\delta,x_\delta),z_{(t_\delta,x_\delta)},-2\delta D\xi(x_\delta),-2\delta D^2\xi(x_\delta))\le 0$$

and

$$\eta'(t_\delta)+F(t_\delta,x_\delta,z(t_\delta,x_\delta),z_{(t_\delta,x_\delta)},-2\delta D\xi(x_\delta),-2\delta D^2\xi(x_\delta))$$
$$-F(t_\delta,x_\delta,z(t_\delta,x_\delta),z_{(t_\delta,x_\delta)},0,0)+F(t_\delta,x_\delta,z(t_\delta,x_\delta),z_{(t_\delta,x_\delta)},0,0)\le 0.$$

It follows from Assumption 1 that

$$\eta'(t_\delta)-\omega(t_\delta,\max(z(t_\delta,x_\delta),\|z^+_{(t_\delta,x_\delta)}\|_D))-A_\delta\le 0$$

where $A_\delta\to 0$ as $\delta\to 0$. Hence,

$$\eta'(t_\delta)-\omega(t_\delta,\|z^+_{(t_\delta,x_\delta)}\|_D))-A_\delta\le 0. \tag{16}$$

Notice that

$$\lim_{\delta\to 0}\|z^+_{(t_\delta,x_\delta)}\|_D=M(t_0).$$

This fact follows from (15) and from the inequality

$$z(t_\delta,x_\delta)^+\le\|z^+_{(t_\delta,x_\delta)}\|_D\le z(t_\delta,x_\delta)^++2\delta$$

where the right-hand estimate is a consequence of (13) (for $t=t_\delta$). Letting $\delta\to 0$ in (16) we get (10). It now follows from Proposition 1 (if we put $H(t)=\omega(t,M(t)))$ that

$$M(t)\le M(t^*)+\int_{t^*}^{t}\omega(s,M(s))\,ds,\qquad t\in[t^*,T], \tag{17}$$

which in view of (8) implies

$$M(t)\le M_0(T)+\int_{0}^{t}\omega(s,M(s))\,ds\qquad \text{for } t\in[t^*,T]. \tag{18}$$

Since

$$M(t)\le M(t^*)=M_0(t^*)\le M_0(T)\qquad\text{for } t<t^*$$

inequality (18) holds for $t\in[0,T]$. It follows from standard theorems that

$$M(t)\le\mu(t,M_0(T))\qquad\text{for } t\in[0,T].$$

Putting $t=T$ we complete the proof.

REMARK 2. If we assume that $\|g^+\|_{\Gamma_t}\le\mu(t,\|g^+\|_{\Gamma_0})$ for $t\in[0,T]$ then

$$\|z^+\|_{E_t}\le\mu(t,\|g^+\|_{\Gamma_0}). \tag{19}$$

P r o o f. It follows from (17) and (8) that

$$M(t) \le \mu(t^*, M_0(0)) + \int_{t_*}^{t} \omega(s, M(s))\, ds \quad \text{for } t \in [t^*, T]$$

and as a result

$$M(t) \le \mu(t; t^*, \mu(t^*, M_0(0))) = \mu(t, M_0(0)) \quad \text{for } t \in [t^*, T]$$

where $\mu(t; t^*, \mu(t^*, M_0(0)))$ denotes the right-hand maximum solution of (16) through $(t^*, \mu(t^*, M_0(0)))$.

ASSUMPTION 2. 1) *There exists a function* ω *satisfying condition "P" such that for all* $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(D)$, *if* $u \le 0$ *then*

$$F(t, x, u, w, 0, 0) \le \omega(t, \max(u, \|w^-\|_D)).$$

2) *For every* $R > 0$ *and* $|u|, \|w\|_D \le R$,

$$[F(t, x, u, w, 0, 0) - F(t, x, u, w, p, A)]^- \to 0 \quad \text{as } p, A \to 0$$

uniformly with respect to $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(D)$.

COROLLARY 1. *Suppose that* F *satisfies Assumption* 2 *and* $z \in \mathrm{BUC}(E) \cap \mathrm{SUP}(F, g)$ *then*

$$\|z^-\|_{E_t} \le \mu(t, \|g^-\|_{\Gamma_t}) \quad \text{for } t \in [0, T]. \tag{20}$$

Moreover, if we assume that $\|g^-\|_{\Gamma_t} \le \mu(t, \|g^-\|_{\Gamma_0})$ *for* $t \in [0, T]$ *then*

$$\|z^-\|_{E_t} \le \mu(t, \|g^-\|_{\Gamma_0}). \tag{21}$$

P r o o f. Notice that if $z \in \mathrm{SUP}(F, g)$ that $-z \in \mathrm{SUB}(\widetilde{F}, -g)$ where

$$\widetilde{F}(t, x, u, w, p, A) = -F(t, x, -u, -w, -p, -A). \tag{22}$$

It is easy to check that F satisfies Assumption 2 if and only if $\widetilde{F}$ satisfies Assumption 1. Therefore Theorem 1 and Remark 2 imply (20) and (21).

Let us now introduce:

ASSUMPTION 3. 1) *There exists a function* ω *satisfying condition "P" such that for all* $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(D)$, *if* $u \ge 0$ *then*

$$F(t, x, u, w, 0, 0) \ge -\omega(t, \max(|u|, \|w\|_D)),$$

and if $u \le 0$ *then*

$$F(t, x, u, w, 0, 0) \le \omega(t, \max(|u|, \|w\|_D)).$$

2) *For every* $R > 0$ *and* $|u|, \|w\|_D \le R$,

$$F(t, x, u, w, p, A) \to F(t, x, u, w, 0, 0) \quad \text{as } p, A \to 0$$

uniformly with respect to $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(D)$.

THEOREM 2. *Suppose that F satisfies Assumption* 3 *and* $z \in \mathrm{BUC}(E) \cap \mathrm{SOL}(F, g)$. *Then*

$$\|z\|_{E_t} \le \mu(t, \|g\|_{\Gamma_t}) \quad \text{for } t \in [0, T]. \tag{23}$$

Moreover, *if* $\|g\|_{\Gamma_t} \le \mu(t, \|g\|_{\Gamma_0})$ *for* $t \in [0, T]$ *then*

$$\|z\|_{E_t} \le \mu(t, \|g\|_{\Gamma_0}). \tag{24}$$

Proof. The proof follows by the same method as for Theorem 1. The only difference is that we put $|z|$ in place of z^+. Now, we have

$$M(t) = \|z\|_{\overline{\Theta}_t}, \quad \overline{M}(t) = \|z\|_{\overline{E}_t}, \quad M_0(t) = \|z\|_{\overline{\Gamma}_t} \quad \text{for } t \in [0, T],$$
$$\Phi(t, x) = |z(t, x)| - \eta(t),$$

and since $|z(t_\delta, x_\delta)| > 0$ we consider two cases $z(t_\delta, x_\delta) > 0$ and $z(t_\delta, x_\delta) < 0$. Both lead in view of Assumption 3 to (23) and (24).

3. Comparison results. Let $F, \overline{F} : \Theta \times \mathbb{R} \times C(D) \times \mathbb{R}^n \times \mathcal{M}(n) \to \mathbb{R}$ and $g, \overline{g} : \Gamma \to \mathbb{R}$ be continuous functions.

ASSUMPTION 4. 1) *There exists a function ω satisfying condition "P" such that for all* (t, x, u, w, p, A), $(t, x, v, z, p, A) \in \Theta \times \mathbb{R} \times C(D) \times \mathbb{R}^n \times \mathcal{M}(n)$, *if* $u \ge v$ *then*

$$F(t, x, u, w, p, A) - \overline{F}(t, x, v, z, p, A) \ge -\omega(t, \max(|u - v|, \|(w - z)^+\|_D)).$$

2) *For every $R > 0$ and* $|u|, \|w\|_D \le R$, $F(t, x, u, w, \cdot, \cdot)$ *is continuous uniformly with respect to* $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(D)$.

ASSUMPTION 5. 1) *There exists a function ω satisfying condition "P" such that for all* (t, x, u, w, p, A), $(t, x, v, z, p, A) \in \Theta \times \mathbb{R} \times C(D) \times \mathbb{R}^n \times \mathcal{M}(n)$, *if* $u \ge v$ *then*

$$F(t, x, u, w, p, A) - \overline{F}(t, x, v, z, p, A) \ge -\omega(t, \max(|u - v|, \|w - z\|_D)),$$

and if $u \le v$ *then*

$$F(t, x, u, w, p, A) - \overline{F}(t, x, v, z, p, A) \le \omega(t, \max(|u - v|, \|w - z\|_D)).$$

2) $F(t, x, u, w, \cdot, \cdot)$ *and* $\overline{F}(t, x, u, w, \cdot, \cdot)$ *are continuous uniformly with respect to* $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(D)$ *for every* $R > 0$ *and* $|u|, \|w\|_D \le R$.

THEOREM 3. *Suppose that F and $\overline{F}$ satisfy Assumption* 4 *and* $u \in \mathrm{BUC}(E) \cap \mathrm{SUB}(F, g)$, $v \in C^{1,2}(\overline{\Theta}) \cap \mathrm{BUC}(E) \cap \mathrm{SUP}(\overline{F}, \overline{g})$. *Then*

$$\|(u - v)^+\|_{E_t} \le \mu(t, \|(g - \overline{g})^+\|_{\Gamma_t}) \quad \text{for } t \in [0, T]. \tag{25}$$

Moreover, *if* $\|(g - \overline{g})^+\|_{\Gamma_t} \le \mu(t, \|(g - \overline{g})^+\|_{\Gamma_0})$ *for* $t \in [0, T]$ *then*

$$\|(u - v)^+\|_{E_t} \le \mu(t, \|(g - \overline{g})^+\|_{\Gamma_0}). \tag{26}$$

Proof. It is easily seen that $w^* = u - v \in \mathrm{SUB}(F[v], g - \overline{g})$ where

$$F[v](t,x,z,w,p,A) = F(t,x,z+v(t,x), w + v_{(t,x)}, p + D_x v(t,x), A + D_x^2 v(t,x)) - \overline{F}(t,x,v(t,x), v_{(t,x)}, D_x v(t,x), D_x^2 v(t,x))$$

satisfies Assumption 1. Theorem 1 and Remark 2 imply the desired assertions.

Similar reasoning yields

THEOREM 4. *Suppose that* F *and* $\overline{F}$ *satisfy Assumption* 5 *and* $u \in \mathrm{BUC}(E) \cap \mathrm{SOL}(F,g)$, $v \in C^{1,2}(\overline{\Theta}) \cap \mathrm{BUC}(E) \cap \mathrm{SOL}(\overline{F}, \overline{g})$. *Then*

$$\|u - v\|_{E_t} \le \mu(t, \|g - \overline{g}\|_{\Gamma_t}) \quad \text{for } t \in [0,T]. \tag{27}$$

Moreover, *if* $\|g - \overline{g}\|_{\Gamma_t} \le \mu(t, \|g - \overline{g}\|_{\Gamma_0})$ *for* $t \in [0,T]$ *then*

$$\|u - v\|_{E_t} \le \mu(t, \|g - \overline{g}\|_{\Gamma_0}). \tag{28}$$

REMARK 3. For first order equations (with $F, \overline{F}$ not depending on A) some results, which are not consequences of the above, are presented in [10].

REMARK 4. The above results may by extended to weakly coupled systems of differential-functional equations.

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Reçu par la Rédaction le 10.4.1995