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Asymptotic behaviour of the iterates of nonnegative operators on a Banach lattice

by Jolanta Socała (Katowice)

Abstract. Asymptotic convergence theorems for nonnegative operators on Banach lattices, on L^∞, on $C(X)$ and on L^p $(1 \le p < \infty)$ are proved. The general results are applied to a class of integral operators on L^1.

Introduction. In ergodic theory some noncompact Markov operators play an important role. They can, for example, transform the unit ball of L^1 onto itself. A. Lasota and J. A. Yorke [LY] proved the convergence of the iterates of such operators under the assumption of the existence of a lower function. A. Zalewska [Z] considers the case where the operator is a nonnegative contractive operator on L^1. Positive operators on $C(X)$ have been investigated by R. Rudnicki [R], A. Lasota and R. Rudnicki [LR], A. Lasota and J. A. Yorke [LY1].

The purpose of this paper is to give a necessary and sufficient condition for the convergence of the iterates of nonnegative linear operators on Banach lattices. Our result is a straightforward extension of the results of Lasota–Yorke, Zalewska and Rudnicki and is based on the idea of the lower function. The main difference between our approach and the classical generalizations of the Krein–Rutman theorem [A], [ZKP] and [N] is that we do not assume any kind of compactness of the operators.

The organization of the paper is as follows. Section 1 contains the convergence theorems for nonnegative operators on Banach lattices. In Section 2 we prove the theorems. In Section 3 we discuss the case of $C(X)$ and L^∞. Section 4 contains a convergence theorem for L^p $(1 \le p < \infty)$. In Section 5 we show an application to integral operators on L^1. We follow the idea of

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A. Lasota and M. C. Mackey [LM] but our criterion is valid for nonnegative operators.

1. Convergence theorems. Let $(V, \leq, \|\cdot\|)$ be a Banach lattice. For $f \in V$ we define $f^+ = \max(0, f)$, $f^- = \max(0, -f)$. It is easy to verify that for $f, g, h \in V$ we have

(1) $(f+g)^- \leq f^- + g^-$;

(2) $f^- \leq g^-$ for $f \geq g$;

(3) $\|f^-\| \leq \|f\|$.

By V_+ we denote the set of all nontrivial nonnegative elements, that is,

$$V_+ = \{f \in V : f \geq 0,\ \|f\| \neq 0\}.$$

A linear continuous operator $P : V \to V$ is called *nonnegative* if $Pf \geq 0$ for $f \in V_+$. It is easy to see that a nonnegative operator P satisfies the condition

(4) $(P^n f)^- \leq P^n(f^-)$ and $(P^n f)^+ \leq P^n(f^+)$ for $f \in V$, $n \in \mathbb{N} = \{1, 2, \ldots\}$.

A nonnegative operator $P : V \to V$ is said to be *exponentially stationary* if there exist $\lambda > 0$, $f_0 \in V_+$ and a continuous linear functional $L : V \to \mathbb{R}$ such that

$$Pf_0 = \lambda f_0, \tag{5}$$

$$\lim_{n\to\infty} \|\lambda^{-n} P^n f - f_0 Lf\| = 0 \quad \text{for } f \in V. \tag{6}$$

THEOREM 1. *A nonnegative operator $P : V \to V$ is exponentially stationary if and only if there exist a dense subset D in V_+ and $h \in V_+$ such that $\|P^n f\| \neq 0$ for $f \in D \cup \{h\}$, $n \in \mathbb{N}$ and the following three conditions hold*:

(I) $\displaystyle\lim_{n\to\infty} \left\| \left(\frac{P^n f}{\|P^n f\|} - h \right)^- \right\| = 0$ *for* $f \in D$;

(II) $\displaystyle\limsup_{n\to\infty} \frac{\|P^n f\|}{\|P^n h\|} < \infty$ *for* $f \in V_+$;

(III) *The sequence* $\{P^n h / \|P^n h\|\}$ *has a convergent subsequence.*

REMARK 1. Example (a) (respectively (b), (c)) below shows that assumption I (respectively II, III) of Theorem 1 is essential:

(a) $Pf = f$ for $f \in V$, V a Banach space;

(b) (J. A. Yorke [Y])

$$Pf = 9 \cdot 1_{[0,1]} \int_0^1 f(x)\,dx + 9 \cdot 1_{[1,2]} \int_1^2 f(x)\,dx + 1_{[2,3]} \int_0^3 f(x)\,dx,$$

V the space of all integrable functions on $[0,3]$;

(c) (R. Rudnicki [R])

$$(Pf)(x) = xf(x) - 2^{-1}(1-x)f(1),$$

V the space of all continuous functions on $[0,1]$.

We shall use the following conditions:

III$'$. For some $g_0 \in D$ the sequence $\{P^n g_0/\|P^n g_0\|\}$ has a weakly convergent subsequence;

IV. There exists $\gamma > 0$ such that

$$\lim_{n\to\infty} \left\| \left(\gamma h - \frac{P^n h}{\|P^n h\|} \right)^- \right\| = 0;$$

IV$'$. There exists $\gamma > 0$ such that for $f \in D, m \in \mathbb{N}$,

$$\lim_{n\to\infty} \left\| \left(\frac{P^n f}{\|P^n f\|} - \frac{P^{n+m} h}{\gamma \|P^{n+m} h\|} \right)^- \right\| = 0.$$

THEOREM 2. *A nonnegative operator $P : V \to V$ is exponentially stationary if and only if there exist a dense subset D in V_+, $h \in V_+$ and $\gamma > 0$ such that $\|P^n f\| \neq 0$ for $f \in D \cup \{h\}$, $n \in \mathbb{N}$, and conditions* I, II, III$'$ *and* IV (*or* I, II, III$'$ *and* IV$'$) *are satisfied.*

REMARK 2. From the proof of Theorem 2 it follows that instead of $g_0 \in D$ (see condition III$'$), we can assume $g_0 \in V_+$ and $\liminf_{n\to\infty} \|P^n g_0\|/\|P^n h\| > 0$.

2. Proofs of the convergence theorems. Let P be a nonnegative operator defined on a Banach space V. In Lemmas 1–4 we shall assume that there exists a set $D \subseteq V_+$ dense in V_+ and $h \in V_+$ such that $\|P^n f\| \neq 0$ for $f \in D \cup \{h\}$, $n \in \mathbb{N}$ and conditions I and II hold.

LEMMA 1. *There exists $\alpha < \infty$ such that*:

(a) *The following condition holds*:

$$\|P^n f\| \le \alpha \|P^n h\| \cdot \|f\| \quad \text{for } f \in V,\ n \in \mathbb{N}; \tag{7}$$

(b) *If $f, g \in V$ and $\|P^n f\| \neq 0 \neq \|P^n g\|$ for $n \in \mathbb{N}$ then*

$$\left\| \frac{P^n g}{\|P^n g\|} - \frac{P^n f}{\|P^n f\|} \right\| \le 2\alpha \|g - f\| \frac{\|P^n h\|}{\|P^n g\|} \quad \text{for } n \in \mathbb{N}; \tag{8}$$

(c) *The following condition holds*:

$$\lim_{n\to\infty} \left\| \left(\frac{P^n h}{\|P^n h\|} - h \right)^- \right\| = 0. \tag{9}$$

Proof. (a) Every $f \in V$ can be written in the form $f = f^+ - f^-$, so it is easy to see that condition II holds for $f \in V$ (not only for $f \in V_+$).

Consider a sequence $\{P^n/\|P^n h\|\}$ of continuous operators. From Corollary 21 of [Dun] (Part I, II.3) it follows immediately that there exists $\alpha < \infty$ such that condition (7) holds.

(b) Let $f, g \in V$. From (7) we have

$$\begin{aligned}\left\|\frac{P^n g}{\|P^n g\|} - \frac{P^n f}{\|P^n f\|}\right\| &\le \left\|\frac{P^n g - P^n f}{\|P^n g\|}\right\| + \left\|\frac{P^n f}{\|P^n g\|} - \frac{P^n f}{\|P^n f\|}\right\| \\ &\le \left\|\frac{P^n g - P^n f}{\|P^n g\|}\right\| + \left|\frac{\|P^n f\| - \|P^n g\|}{\|P^n g\|}\right| \\ &\le 2\frac{\|P^n(g-f)\|}{\|P^n h\|}\,\frac{\|P^n h\|}{\|P^n g\|} \le 2\alpha\|g-f\|\frac{\|P^n h\|}{\|P^n g\|}.\end{aligned}$$

(c) From (1), (3) and (8) it follows that for $f \in D$, $n \in \mathbb{N}$ we have

$$\begin{aligned}\left\|\left(\frac{P^n h}{\|P^n h\|} - h\right)^-\right\| &\le \left\|\frac{P^n h}{\|P^n h\|} - \frac{P^n f}{\|P^n f\|}\right\| + \left\|\left(\frac{P^n f}{\|P^n f\|} - h\right)^-\right\| \\ &\le 2\alpha\|h-f\| + \left\|\left(\frac{P^n f}{\|P^n f\|} - h\right)^-\right\|.\end{aligned}$$

Since the set D is dense in V_+, by condition I this finishes the proof. ∎

Lemma 2. *Assume condition* IV′ *holds. Then for $g \in V_+$ such that $\|P^n g\| \ne 0$ for $n \in \mathbb{N}$ the following conditions are equivalent*:

(a) $\displaystyle\lim_{n\to\infty}\left\|\left(\frac{P^n g}{\|P^n g\|} - \frac{P^{n+m}h}{\gamma\|P^{n+m}h\|}\right)^-\right\| = 0$ *for* $m \in \mathbb{N}$;

(b) $\displaystyle\lim_{n\to\infty}\left\|\left(\frac{P^n g}{\|P^n g\|} - \frac{h}{\gamma}\right)^-\right\| = 0$;

(c) *There exists $m_0 \in \mathbb{N}$ such that*

$$\|P^n h\|\cdot\|P^m g\| \le 2\gamma\|P^{n+m}g\| \quad \text{for } m \ge m_0,\ n \in \mathbb{N};$$

(d) *There exist $\delta_g < \infty$ and $n_0 = n_0(g) \in \mathbb{N}$ such that*

$$\|P^n h\| \le \delta_g\|P^n g\| \quad \text{for } n \ge n_0.$$

Proof. (a)⇒(b). From (1) it follows that for $m \in \mathbb{N}$,

$$\begin{aligned}&\left\|\left(\frac{P^n g}{\|P^n g\|} - \frac{h}{\gamma}\right)^-\right\| \\ &\quad\le \left\|\left(\frac{P^n g}{\|P^n g\|} - \frac{P^{n+m}h}{\gamma\|P^{n+m}h\|}\right)^-\right\| + \left\|\left(\frac{P^{n+m}h}{\gamma\|P^{n+m}h\|} - \frac{h}{\gamma}\right)^-\right\|.\end{aligned}$$

By (a) and (9) this finishes the proof.

(b)⇒(c). Let

$$r_m = \frac{P^m g}{\|P^m g\|} - \frac{h}{\gamma}. \tag{10}$$

Condition (b) implies that there exists $m_0 \in \mathbb{N}$ such that

$$\|r_m^-\| \le \frac{1}{2\alpha\gamma} \quad \text{for } m \ge m_0.$$

By (7) for $m \ge m_0$ we obtain

$$\|P^n(r_m^-)\| \le \alpha\|P^n h\| \cdot \|r_m^-\| \le \frac{\|P^n h\|}{2\gamma}.$$

Since P is monotonic, according to (10) for $m \ge m_0$ we have

$$\frac{\|P^{n+m}g\|}{\|P^m g\|} \ge \left\| \frac{P^n h}{\gamma} + P^n(r_m^+) \right\| - \|P^n(r_m^-)\| \ge \frac{\|P^n h\|}{2\gamma}.$$

(c)$\Rightarrow$(d). From assumption (c) we obtain

$$\|P^{n+m_0}h\| \le \|P\|^{m_0}\|P^n h\| \le \delta_g\|P^{n+m_0}g\|$$

where

$$\delta_g = \frac{2\gamma\|P\|^{m_0}}{\|P^{m_0}g\|}.$$

(d)$\Rightarrow$(a). By (1), (3), (8) and (d) for $f \in D$, $n \ge n_0$ we have

$$\begin{aligned}
&\left\| \left(\frac{P^n g}{\|P^n g\|} - \frac{P^{n+m}h}{\gamma\|P^{n+m}h\|} \right)^- \right\| \\
&\qquad \le \left\| \frac{P^n g}{\|P^n g\|} - \frac{P^n f}{\|P^n f\|} \right\| + \left\| \left(\frac{P^n f}{\|P^n f\|} - \frac{P^{n+m}h}{\gamma\|P^{n+m}h\|} \right)^- \right\| \\
&\qquad \le 2\alpha\|g - f\|\delta_g + \left\| \left(\frac{P^n f}{\|P^n f\|} - \frac{P^{n+m}h}{\gamma\|P^{n+m}h\|} \right)^- \right\|.
\end{aligned}$$

Since D is dense in V_+, by assumption IV$'$ this finishes the proof. ■

LEMMA 3. *Assume condition* IV$'$ *holds. Then there exists* $\beta < \infty$ *such that*

$$\|P^m h\| \cdot \|P^n h\| \le \beta\|P^{n+m}h\| \quad \textit{for } n, m \in \mathbb{N}. \tag{11}$$

P r o o f. Condition (d) from Lemma 2 holds for $g = h$. By (d)$\Rightarrow$(c) we have (11), where $\beta = \max\{2\gamma, \beta_1\}$ and

$$\beta_1 = \max\left\{ \frac{\|P^m h\| \cdot \|P^n h\|}{\|P^{n+m}h\|} : 0 \le n \le m_0,\ 0 \le m \le m_0 \right\}. \ ■$$

LEMMA 4. *Assume condition* IV$'$ *holds. Then for every* $f \in V$ *there exists a number* μ *such that*

$$\lim_{n\to\infty} \frac{\|P^n f - \mu P^n h\|}{\|P^n h\|} = 0. \tag{12}$$

Moreover, if $f \in D$ (*or* $f \in V_+$ *and* $\liminf_{n\to\infty} \|P^n f\|/\|P^n h\| > 0$) *then*

$$\lim_{n\to\infty} \left\| \frac{P^n f}{\|P^n f\|} - \frac{P^n h}{\|P^n h\|} \right\| = 0. \tag{13}$$

Proof. It suffices to prove (12) for $f \in V_+$. Choose $f \in V_+$ and define

$$\Delta_f = \left\{ \eta \geq 0 : \lim_{n\to\infty} \frac{\|(P^n f - \eta P^n h)^-\|}{\|P^n h\|} = 0 \right\}.$$

Since $0 \in \Delta_f$, the set Δ_f is not empty. Now we prove that it is bounded. Let $\eta \in \Delta_f$ and $s_n = P^n f - \eta P^n h$. There is $n_0 \in \mathbb{N}$ such that

$$\|s_{n_0}^-\| \leq \eta \|P^{n_0} h\|/2.$$

Consequently,

$$\|P^{n_0} f\| \geq \|\eta P^{n_0} h + s_{n_0}^+\| - \|s_{n_0}^-\| \geq \|\eta P^{n_0} h\| - \|s_{n_0}^-\| \geq \eta \|P^{n_0} h\|/2$$

and from (7) it follows that $\eta \leq 2\alpha\|f\|$. We define $\mu = \sup \Delta_f$. There exists $\{\eta_m\}$ such that $\eta_m \in \Delta_f$ and $\lim_{m\to\infty} \eta_m = \mu$. From (1) and (3) we have

$$\frac{\|(P^n f - \mu P^n h)^-\|}{\|P^n h\|} \leq \frac{\|(P^n f - \eta_m P^n h)^-\|}{\|P^n h\|} + |\eta_m - \mu| \quad \text{for } n, m \in \mathbb{N}$$

and consequently $\mu \in \Delta_f$. Let $q_n = P^n f - \mu P^n h$ for $n \in \mathbb{N}$. Now we prove that there is a sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} \|q_{n_k}^+\|/\|P^{n_k} h\| = 0. \tag{14}$$

Assume the contrary. Then there exist $\delta > 0$ and $l_0 \in \mathbb{N}$ such that $\|q_m^+\| \geq \delta \|P^m h\|$ for $m \geq l_0$. Choose $n \in \mathbb{N}$. Since $q_{n+m} = P^m q_n$, from (4) and (11) it follows that

$$\frac{\|P^m(q_n^+)\|}{\|P^m h\|} \geq \|P^n h\| \frac{\|q_{n+m}^+\|}{\|P^{n+m} h\|} \cdot \frac{\|P^{n+m} h\|}{\|P^n h\| \cdot \|P^m h\|} \geq \frac{\delta \|P^n h\|}{\beta} > 0$$

for $m \geq l_0$. By Lemma 2(d)⇒(a) we have

$$\lim_{m\to\infty} \left\| \left(\frac{P^m(q_n^+)}{\|P^m(q_n^+)\|} - \frac{P^{n+m} h}{\gamma \|P^{n+m} h\|} \right)^- \right\| = 0. \tag{15}$$

Choose $m \geq l_0$. The inequalities (11) and (7) imply that

$$\frac{\|P^m(q_n^-)\|}{\|P^{n+m} h\|} = \frac{\|P^m h\| \cdot \|P^n h\|}{\|P^{n+m} h\|} \cdot \frac{\|P^m(q_n^-)\|}{\|P^m h\|} \cdot \frac{1}{\|P^n h\|} \leq \frac{\beta\alpha \|q_n^-\|}{\|P^n h\|}.$$

It follows easily from the definition of δ and (4) that $\delta \|P^{n+m} h\| \leq \|P^m(q_n^+)\|$ and according to (2) we have

$$\frac{\|(P^m(q_n^+) - (\delta/\gamma) P^{n+m} h)^-\|}{\|P^{n+m} h\|} \leq \delta \left\| \left(\frac{P^m(q_n^+)}{\|P^m(q_n^+)\|} - \frac{P^{n+m} h}{\gamma \|P^{n+m} h\|} \right)^- \right\|.$$

By (1) and (4) we obtain

$$\begin{aligned}\frac{\|(P^{n+m}f-(\mu+(\delta/\gamma))P^{n+m}h)^-\|}{\|P^{n+m}h\|}\\
&\le \frac{\|P^m(q_n^-)\|}{\|P^{n+m}h\|}+\frac{\|(P^m(q_n^+)-(\delta/\gamma)P^{n+m}h)^-\|}{\|P^{n+m}h\|}\\
&\le \frac{\beta\alpha\|q_n^-\|}{\|P^nh\|}+\delta\bigg\|\bigg(\frac{P^m(q_n^+)}{\|P^m(q_n^+)\|}-\frac{P^{n+m}h}{\gamma\|P^{n+m}h\|}\bigg)^-\bigg\|.\end{aligned}$$

Since $\mu\in\varDelta_f$, from (15) it follows that $\mu+(\delta/\gamma)\in\varDelta_f$, which contradicts the condition $\mu=\sup\varDelta_f$. Consequently, there is a sequence $\{n_k\}$ such that (14) holds. Conditions (4), (11) and (7) imply that

$$\begin{aligned}\frac{\|q^+_{m+n_k}\|}{\|P^{m+n_k}h\|}&\le\frac{\|P^mh\|\cdot\|P^{n_k}h\|}{\|P^{m+n_k}h\|}\cdot\frac{\|P^m(q^+_{n_k})\|}{\|P^mh\|}\cdot\frac{1}{\|P^{n_k}h\|}\\
&\le\frac{\beta\alpha\|q^+_{n_k}\|}{\|P^{n_k}h\|}\quad\text{ for } m,k\in\mathbb{N}.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|q_n^+\|/\|P^nh\|=0.$$

Since $\mu\in\varDelta_f$, condition (12) holds. If $f\in D$, then from IV′ it follows that f satisfies condition (a) of Lemma 2. By Lemma 2(a)⇒(d) there is $\delta_f<\infty$ such that (d) holds. Thus, finally

$$\begin{aligned}\bigg\|\frac{P^nf}{\|P^nf\|}-\frac{P^nh}{\|P^nh\|}\bigg\|&\le\frac{\|r_n\|}{\|P^nf\|}+\bigg|\frac{\mu\|P^nh\|}{\|P^nf\|}-1\bigg|\\
&\le\frac{2\|r_n\|}{\|P^nf\|}\le\frac{2\delta_f\|r_n\|}{\|P^nh\|},\end{aligned}$$

where $r_n=P^nf-\mu\le P^nh$. By (12), f satifies condition (13). ■

Proof of Theorem 2. Let P be exponentially stationary. Since L is linear and $L\ne0$, the set $D=\{f\in V_+: Lf\ne0\}$ is dense in V_+. Moreover, from (5) and (6) it follows that $h=g_0=f_0/\|f_0\|$ and $\gamma=1$ satisfy conditions I, II, III′, IV, IV′. Now assume conditions I, II, III′, IV′ hold. We are going to prove that P is exponentially stationary. By III′ we can assume that some sequence $\{P^{n_k}g_0/\|P^{n_k}g_0\|\}$ is weakly convergent to some f_0V. Since P is continuous, from (13) it follows that $\{P^{n_k}h/\|P^{n_k}h\|\}$ and $\{P^{n_k+1}h/\|P^{n_k}h\|\}$ are weakly convergent to f_0 and Pf_0. Lemma 4 implies that there exists a number λ such that

$$\lim_{n\to\infty}\|P^{n+1}h-\lambda P^nh\|/\|P^nh\|=0.$$

Hence $\{P^{n_k+1}h/\|P^{n_k}h\|\}$ is weakly convergent to λf_0. Consequently, $Pf_0=$

λf_0. By Lemma 4 there is a number μ such that

$$\lim_{n\to\infty}\frac{\|P^n f_0-\mu P^n h\|}{\|P^n h\|}=0,\qquad \lim_{n\to\infty}\frac{\|P^n f_0\|}{\|P^n h\|}=\mu. \tag{16}$$

Since $g_0\in D$, by condition IV′ and Lemma 2(a)⇒(b) we have $\lim_{k\to\infty}\|r_k^-\|=0$, where

$$r_k=\frac{P^{n_k}g_0}{\|P^{n_k}g_0\|}-\frac{h}{\gamma}\quad \text{for } k=1,2,\ldots$$

From III′ it follows that $\{r_k^+\}$ is weakly convergent to $f_0-\gamma^{-1}h$. Since $r_k^+\ge 0$, we have $f_0-\gamma^{-1}h\ge 0$ and

$$\|P^n f_0\|\ge \|P^n(\gamma^{-1}h)\|>0. \tag{17}$$

By (16) we obtain $\mu\ne 0$. Choose $f\in V$. From Lemma 4 and (16) there is a number η_f such that

$$\lim_{n\to\infty}\frac{\|P^n f-\eta_f P^n f_0\|}{\|P^n h\|}=0,\qquad \lim_{n\to\infty}\frac{\|P^n f-\eta_f P^n f_0\|}{\|P^n f_0\|}=0.$$

Since $Pf_0=\lambda f_0$, we have

$$\lim_{n\to\infty}\|\lambda^{-n}P^n f-\eta_f f_0\|=0.$$

Define $L:V\to\mathbb{R}$ by $Lf=\eta_f$. It is obvious that L is linear. By (7) and (17) we have

$$\eta_f=\lim_{n\to\infty}\frac{\|P^n f\|}{\|P^n h\|}\cdot\frac{\|P^n h\|}{\|P^n f_0\|}\le \alpha\gamma\|f\|.$$

Thus L is continuous and P is exponentially stationary. Now assume conditions I, II, III′, IV hold. From I, IV and (1) it follows that condition IV′ holds and this finishes the proof. ■

Proof of Theorem 1. It is easy to show that if P is exponentially stationary, then $D=\{f\in V_+ : Lf\ne 0\}$ and $h=f_0/\|f_0\|$ satisfy conditions I–III. Now we assume conditions I–III hold. By Lemma 1(c) we can assume that $h\in D$ and condition III′ holds. We now show that condition IV′ holds. Choose $f\in D$ and $m\in\mathbb{N}$. By (2), (4), (7), (11) we obtain

$$\begin{aligned}&\left\|\left(\frac{P^{n+i}f}{\|P^{n+i}f\|}-\frac{\delta P^{n+j}h}{\|P^{n+j}h\|}\right)^-\right\|\\ &\quad\le \alpha\|P^n h\|\cdot\left\|\left(\frac{P^i f}{\|P^{i+n}f\|}-\frac{\delta P^j h}{\|P^{n+j}h\|}\right)^-\right\|\\ &\quad\le \left\|\left(\frac{P^i f}{\|P^i f\|}-\frac{\alpha\beta\delta P^j h}{\|P^j h\|}\right)^-\right\|\quad \text{for } i,n\in\mathbb{N},\ j\in\mathbb{N}\cup\{0\},\ \delta\in\mathbb{R}.\end{aligned} \tag{18}$$

From III it follows that there exists a sequence $\{P^{n_k}h/\|P^{n_k}h\|\}$ convergent to some $f_0\in V$. Since the sequence $\{n_k\}$ is increasing, we have $n_{2k}-n_k\ge k$

and $l_k := n_{2k} - m > n_k$ for $k > m$. Then, according to (1), (3) and (18) for $k > m$ we have

$$\begin{aligned}
&\left\| \left(\frac{P^{l_k} f}{\|P^{l_k} f\|} - \frac{\|h\| P^{m+l_k} h}{\alpha\beta \|P^{m+l_k} h\|} \right)^- \right\| \\
&\quad \le \left\| \left(\frac{P^{l_k} f}{\|P^{l_k} f\|} - \frac{\|h\| P^{n_k} h}{\alpha\beta \|P^{n_k} h\|} \right)^- \right\| + \alpha^{-1}\beta^{-1}\|h\| \cdot \left\| \frac{P^{n_k} h}{\|P^{n_k} h\|} - \frac{P^{n_{2k}} h}{\|P^{n_{2k}} h\|} \right\| \\
&\quad \le \left\| \left(\frac{P^{l_k - n_k} f}{\|P^{l_k - n_k} f\|} - h \right)^- \right\| + \alpha^{-1}\beta^{-1}\|h\| \cdot \left\| \frac{P^{n_k} h}{\|P^{n_k} h\|} - f_0 \right\| \\
&\qquad + \alpha^{-1}\beta^{-1}\|h\| \cdot \left\| f_0 - \frac{P^{n_{2k}} h}{\|P^{n_{2k}} h\|} \right\|.
\end{aligned}$$

Since $\lim_{k\to\infty}(l_k - n_k) = \infty$, by condition I we obtain

$$\lim_{k\to\infty} \left\| \left(\frac{P^{l_k} f}{\|P^{l_k} f\|} - \frac{\|h\| P^{m+l_k} h}{\alpha\beta \|P^{m+l_k} h\|} \right)^- \right\| = 0.$$

The inequality (18) implies that

$$\begin{aligned}
&\left\| \left(\frac{P^{l_k+n} f}{\|P^{l_k+n} f\|} - \frac{\|h\| P^{m+l_k+n} h}{\alpha^2\beta^2 \|P^{m+l_k+n} h\|} \right)^- \right\| \\
&\qquad\qquad \le \left\| \left(\frac{P^{l_k} f}{\|P^{l_k} f\|} - \frac{\|h\| P^{m+l_k} h}{\alpha\beta \|P^{m+l_k} h\|} \right)^- \right\| \quad \text{for } n, m \in \mathbb{N}.
\end{aligned}$$

Hence condition IV$'$ holds and $\gamma = \alpha^2\beta^2\|h\|^{-1}$. By Theorem 2 this finishes the proof. ■

3. Convergence theorem for $C(X)$ and L^∞. Let V be a Banach lattice. We shall assume that the unit ball of V contains a largest element 1_X, that is, $\|1_X\| = 1$ and $f \le 1_X$ for $f \in V$, $\|f\| \le 1$. Define a set $V_{++} \subseteq V$ as follows: if $f \in V_+$ and there exists $\alpha > 0$ such that $f \ge \alpha \cdot 1_X$, then $f \in V_{++}$.

THEOREM 3 (see R. Rudnicki [R], A. Lasota and R. Rudnicki [LR], A. Lasota and J. A. Yorke [LY1]). *Let P be a nonnegative operator on V, D be a dense subset of V_+ and $\alpha > 0$. Assume that for every $f \in D$ there is an integer $n_0(f)$ such that*

$$\|P^n f\| \ne 0, \qquad P^n f / \|P^n f\| \ge \alpha \cdot 1_X \qquad \textit{for } n \ge n_0(f)$$

and that, for some $g \in V_{++}$, the sequence $\{P^n g/\|P^n g\|\}$ has a weakly convergent subsequence. Then P is exponentially stationary.

P r o o f. The element $h = \alpha \cdot 1_X$ satisfies condition I. Since $f \le \alpha^{-1}\|f\| h$ for $f \in V_+$ and P is monotonic, it follows that $P^n f \le \alpha^{-1}\|f\| P^n h$ for $n \in \mathbb{N}$ and condition II holds. Moreover,

$$P^n h \le \alpha^{-1}\|P^n h\| h \quad \text{for } n \in \mathbb{N}$$

and $\gamma = \alpha^{-1}$ satisfies condition IV. Since $g \in V_{++}$, there exists $\beta > 0$ such that $g \geq \beta h$. Consequently, $P^n g \geq \beta P^n h$ for $n \in \mathbb{N}$ and

$$\liminf_{n\to\infty} \|P^n g\|/\|P^n h\| > 0.$$

By Theorem 2 and Remark 2 this finishes the proof. ■

4. Convergence theorem for L^p. Let (X, Σ, μ) be a σ-finite measure space. We deal with the space $V = L^p = L^p(X, \Sigma, \mu)$ $(1 \leq p < \infty)$ with the norm $\| * \| = \| * \|_{L^p}$. It is easy to verify that for $f, g, h \in L^p$ we have

$$(f - \max(g,h))^- \leq (f-g)^- + (f-h)^-. \tag{19}$$

THEOREM 4. *A nonnegative operator $P : L^p \to L^p$ is exponentially stationary if and only if there exist a dense subset $D \subset L^p_+ = V_+$ and $h \in L^p_+$ such that*

$$\|P^n f\| \neq 0 \quad \text{for } f \in D \cup \{h\},\ n \in \mathbb{N}$$

and the following two conditions hold:

(I) $$\lim_{n\to\infty} \left\| \left(\frac{P^n f}{\|P^n f\|} - h \right)^- \right\| = 0 \quad \text{for } f \in D;$$

(II) $$\limsup_{n\to\infty} \frac{\|P^n f\|}{\|P^n h\|} < \infty \quad \text{for } f \in L^p_+.$$

REMARK 3. Every assumption of Theorem 4 is essential (see Remark 1).

REMARK 4. The above theorem implies the Lasota–Yorke theorem [LY] and Zalewska's theorem [Z].

Proof of Theorem 4. If P is exponentially stationary, then $D = \{f \in L^p_+ : Lf \neq 0\}$ and $h = f_0/\|f_0\|$ satisfy conditions I and II (see the proof of Theorem 2).

Now assume conditions I and II hold. We are going to prove that P is exponentially stationary. It is easy to show (see Lemma 1) that

$$\lim_{n\to\infty} \left\| \left(\frac{P^n h}{\|P^n h\|} - h \right)^- \right\| = 0$$

and there exists $\alpha < \infty$ such that

$$\|P^n f\| \leq \alpha \|P^n h\| \cdot \|f\| \quad \text{for } f \in L^p,\ n \in \mathbb{N}. \tag{20}$$

So we can assume $h \in D$. Define

$$G = \left\{ g \in L^p_+ : \lim_{n\to\infty} \left\| \left(\frac{P^n f}{\|P^n f\|} - g \right)^- \right\| = 0 \text{ for } f \in D \right\}.$$

From (20), (2) and next (4), (2) it follows that for $f \in D$ we have

$$\begin{aligned}(21)\quad &\left\| \left(\frac{P^{n+m} f}{\|P^{n+m} f\|} - \frac{P^m h}{\alpha \|P^m h\|} \right)^- \right\| \\ &\le \alpha^{-1} \|P^m h\|^{-1} \left\| \left(\frac{P^{n+m} f}{\|P^n f\|} - P^m h \right)^- \right\| \\ &\le \alpha^{-1} \|P^m h\|^{-1} \left\| P^m \left(\left(\frac{P^n f}{\|P^n f\|} - h \right)^- \right) \right\| \le \left\| \left(\frac{P^n f}{\|P^n f\|} - h \right)^- \right\|.\end{aligned}$$

Then by condition I we have

$$\frac{P^m h}{\alpha \|P^m h\|} \in G \quad \text{for } m \in \mathbb{N}.$$

Define $h_0 = h$ and

$$h_{m+1} = \sup \left(h_m, \frac{P^{m+1} h}{\alpha \|P^{m+1} h\|} \right) \quad \text{for } m \ge 0.$$

By (19), an induction argument shows that $h_m \in G$ for $m \ge 1$. Then from the inequality

$$\begin{aligned}1 &\ge \left\| h_m + \left(\frac{P^n f}{\|P^n f\|} - h_m \right)^+ \right\| - \left\| \left(\frac{P^n f}{\|P^n f\|} - h_m \right)^- \right\| \\ &\ge \|h_m\| - \left\| \left(\frac{P^n f}{\|P^n f\|} - h_m \right)^- \right\| \quad \text{for } f \in D,\end{aligned}$$

it follows that $\|h_m\| \le 1$. Since the sequence $\{h_m\}$ is increasing, the strong limit $h_* = \lim_{m\to\infty} h_m$ exists. We show that h_* satisfies the assumptions of Theorem 2. From (1) we obtain

$$\left\| \left(\frac{P^n f}{\|P^n f\|} - h_* \right)^- \right\| \le \left\| \left(\frac{P^n f}{\|P^n f\|} - h_m \right)^- \right\| + \|h_m - h_*\|$$

for $f \in D$, $m \in \mathbb{N}$. Since $h_m \in G$, we have $h_* \in G$ (assumption I of Theorem 2). As $h \le h_*$, we obtain

$$\|P^n f\| / \|P^n h_*\| \le \|P^n f\| / \|P^n h\| \quad \text{for } n \in \mathbb{N}$$

(assumption II). We have $P^m h / \alpha \|P^m h\| \le h_*$ for $m \in \mathbb{N}$ and (see [LM], Remark 5.1.3) the sequence $\{P^m h / (\alpha \|P^m h\|)\}$ has a weakly convergent subsequence (assumption III′). It is easy to show that there exists $\beta < \infty$ such that

$$\|P^n f\| \le \beta \|P^n h_*\| \cdot \|f\| \quad \text{for } f \in L^p,\ n \in \mathbb{N}$$

(see Lemma 1) and $P^m h_* / (\beta \|P^m h_*\|) \in G$ (see (21)). Moreover, from (2) and $h \in D$ we obtain

$$\limsup_{n\to\infty} \left\| \left(\alpha h_* - \frac{P^m h_*}{\beta \|P^m h_*\|} \right)^- \right\| \le \limsup_{n\to\infty} \left\| \left(\frac{P^{n+m} h}{\|P^{n+m} h\|} - \frac{P^m h_*}{\beta \|P^m h_*\|} \right)^- \right\| = 0 \quad \text{for } m \in \mathbb{N}$$

(assumption IV). According to Theorem 2 the operator P is exponentially stationary. ■

5. The integral operators. Let (X, Σ, μ) be a σ-finite measure space. We deal with the space $V = L^1(X, \Sigma, \mu)$ with the norm $\|\cdot\| = \|\cdot\|_{L^1}$. Let $K : X \times X \to [0, \infty)$ be a measurable function. Assume that there exist numbers α, β $(0 < \beta \le \alpha)$ such that

$$\beta \le \int_X K(x,y)\,dx \le \alpha \quad \text{for } y \in X. \tag{22}$$

Further, we define an integral operator P by

$$Pf(x) = \int_X K(x,y) f(y)\,dy \quad \text{for } f \in V. \tag{23}$$

The operator P is clearly linear and nonnegative. It is easy to verify (see [LM]) that the operator P^n can be written in the form

$$P^n f(x) = \int_X K_n(x,y) f(y)\,dy$$

where $K_1 = K$ and

$$K_{n+m}(x,y) = \int_X K_n(x,z) K_m(z,y)\,dz \quad \text{for } n, m \in \mathbb{N}.$$

By (22) an induction argument shows that

$$\beta^n \le \int_X K_n(x,y)\,dx \le \alpha^n \quad \text{for } y \in X \tag{24}$$

and

$$0 < \beta^n \|f\| \le \|P^n f\| \le \alpha^n \|f\| \quad \text{for } f \in V_+. \tag{25}$$

THEOREM 5. *Assume that there exist integers m_1, m_2 and positive numbers α, β, δ such that condition* (22) *holds and*

$$\sup_y K_{m_1}(x,y) \le \delta \inf_y K_{m_2}(x,y) \quad \textit{for } x \in X. \tag{26}$$

Then the operator P defined by (23) *is exponentially stationary.*

Proof. Define $D = V_+$ and

$$h(x) = \alpha^{-m_2} \inf_y K_{m_2}(x,y).$$

Then, according to (26) and (24),

$$\|h\| > \delta^{-1}\alpha^{-m_2} \int_X \sup_y K_{m_1}(x,y)\,dx \ge \beta^{m_1}\delta^{-1}\alpha^{-m_2} > 0.$$

From (25) it follows that for $f \in D, n \in \mathbb{N}$ we have

$$\begin{aligned} P^{n+m_2} f(x) &= \int_X \int_X K_{m_2}(x,z) K_n(z,y) f(y)\,dy\,dz \\ &\ge \alpha^{m_2} h(x) \|P^n f\| \ge h(x)\|P^{n+m_2} f\| \end{aligned} \tag{27}$$

and $\|P^{n+m_2} f\| \neq 0$. By (26) we obtain

$$P^{n+m_1} f(x) = \int_X \int_X K_n(x,z) K_{m_1}(z,y) f(y)\,dy\,dz \le \delta\alpha^{m_2} P^n h(x)\|f\|.$$

Then, according to (25) we have $\beta^{m_1}\|P^n h\| \le \|P^{n+m_1} h\|$ and

$$\|P^{n+m_1} f\| \le \eta \|P^{n+m_1} h\| \cdot \|f\| \qquad \text{where } \eta = \delta\alpha^{m_2}\beta^{-m_1}. \tag{28}$$

By Theorem 4 this finishes the proof. ∎

Let X be an unbounded measurable subset of a d-dimensional Euclidean space $\mathbb{R}^d$. We call a continuous nonnegative function $V : X \to \mathbb{R}$ satisfying $\lim_{|x|\to\infty} V(x) = \infty$ a *Lyapunov function.*

Consider a measurable function $K : X^2 \to [0,\infty)$ which satisfies condition (22) for some numbers α, β $(0 < \beta \le \alpha)$.

Theorem 6. *Assume that there exists a Lyapunov function $V : X \to \mathbb{R}$ such that*

$$\int_X K(x,y) V(x)\,dx \le \gamma V(y) + \eta, \tag{29}$$

where $\gamma \ge 0$, $\eta \ge 0$ and $\gamma/\beta < 1$. Moreover, assume that there exist $l, m \in \mathbb{N}$ such that for every positive ε we can choose δ_ε which satisfies

$$\sup_y K_l(x,y) \le \delta_\varepsilon \inf\{K_m(x,y) : |y| \le \varepsilon\} \qquad \text{for } x \in X. \tag{30}$$

Then the operator $P : L^1 \to L^1$ defined by equation (23) *is exponentially stationary.*

Proof. First we define

$$E_n(V \mid f) = \int_X V(x) P^n f(x)\,dx, \qquad D = \{f \in L^1_+ : E_0(V \mid f) < \infty\}.$$

Choose $f \in$D. From the inequality (29), we have

$$\begin{aligned} E_{n+1}(V \mid f) &= \int_X \int_X V(x) K(x,y) P^n f(y)\,dy\,dx \\ &\le \eta \|P^n f\| + \gamma E_n(V \mid f). \end{aligned}$$

Since $\gamma/\beta < 1$, by (25) an induction argument shows that

$$\frac{E_{n+1}(V \mid f)}{\|P^{n+1}f\|} \le \eta\beta^{-1} + \gamma\beta^{-1}\frac{E_n(V \mid f)}{\|P^n f\|}$$

and

$$\frac{E_n(V \mid f)}{\|P^n f\|} \le \frac{\eta}{\beta-\gamma} + \left(\frac{\gamma}{\beta}\right)^n \frac{E_0(V \mid f)}{\|f\|}.$$

There exists an integer $n_0 = n_0(f)$ such that

$$E_n(V \mid f) \le \alpha_1 \|P^n f\| \quad \text{for } n \ge n_0, \tag{31}$$

where $\alpha_1 = 2\eta/(\beta-\gamma)$. Now let

$$U = \{x \in X : |x| \le \varepsilon\}.$$

Since $V(x) \to \infty$ as $|x| \to \infty$, we can choose $\varepsilon > 0$ such that $V(x) \ge 2\alpha_1$ for $x \in X - U$. From (31) it follows that

$$2\alpha_1 \int\limits_{X-U} P^n f(x)\,dx \le \int\limits_{X-U} V(x)P^n(x)\,dx \le \alpha_1\|P^n f\|$$

for $n \ge n_0$. Then by (25) we obtain

$$\begin{aligned} P^{n+m}f(x) &\ge \int\limits_U K_m(x,y)P^n f(y)\,dy \\ &\ge 2\alpha^m h(x)\int\limits_U P^n f(y)\,dy \ge \alpha^m h(x)\|P^n f\| \ge h(x)\|P^{n+m}f\| \end{aligned}$$

for $x \in X,\ n \ge n_0$, where

$$h(x) = (2\alpha^m)^{-1}\inf\{K_m(x,y) : y \in U\} \quad \text{for } x \in X.$$

From (30), (24) and (25) we have $\|h\| > 0$ and $\|P^i f\| \ne 0 \ne \|P^i h\|$ for $i \in \mathbb{N}$. Hence condition I of Theorem 4 holds. From (30) it follows that

$$P^{n+l}f(x) = \int\limits_X \int\limits_X K_n(x,z)K_l(z,y)f(y)\,dy\,dz \le 2\delta_\varepsilon \alpha^m P^n h(x)\|f\|$$

and by (25),

$$\|P^{n+l}f\| \le 2\delta_\varepsilon\alpha^m\beta^{-l}\|P^{n+l}h\| \cdot \|f\| \quad \text{for } n \in \mathbb{N}.$$

Hence h satisfies condition II. By Theorem 4 this finishes the proof. ∎

Remark 5. Instead of (30) we can assume the following two conditions:

(32) There exists $m \in \mathbb{N}$ such that

$$\int\limits_X \inf\{K_m(x,y) : |y| \le \varepsilon\}\,dx > 0 \quad \text{for } \varepsilon > 0;$$

(33) There exist $\delta \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$\sup_y \int_X K_n(x,y)\,dx \le \delta \inf_y \int_X K_n(x,y)\,dx \quad \text{for } n \ge n_0.$$

In this case for $n \ge n_0$ we have

$$\begin{aligned}\|P^n f\| \cdot \|h\| &= \int_X \int_X K_n(x,y) f(y)\,dy\,dx\,\|h\| \\ &\le \delta \inf_y \int_X K_n(x,y)\,dx\,\|f\| \cdot \|h\| \\ &\le \delta \int_X \int_X K_n(x,y) h(y)\,dy\,dx\,\|f\| = \delta \|P^n h\| \cdot \|f\|\end{aligned}$$

and h satisfies condition II.

REMARK 6. Theorems 5 and 6 imply the theorems of A. Lasota and M. C. Mackey [LM].

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Institute of Mathematics
Silesian University
40-007 Katowice, Poland
E-mail: jsocala@ux2.math.us.edu.pl

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