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On a semilinear elliptic eigenvalue problem

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Abstract. We obtain a description of the spectrum and estimates for generalized positive solutions of $-\Delta u = \lambda(f(x) + h(u))$ in Ω, $u|_{\partial\Omega} = 0$, where $f(x)$ and $h(u)$ satisfy minimal regularity assumptions.

Introduction. From various points of view there is still interest in the eigenvalue problem

$$(*)\qquad -\Delta u = \lambda(f(x) + h(u)) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N$, $2 \le N$, is bounded. Following the terminology of Krasnosel'skiĭ we define the *spectrum* of $(*)$ to be the set of the values λ for which there exist positive solutions of $(*)$. Various authors have obtained a description of the spectrum of the more general problem than $(*)$, i.e.

$$-\Delta u = \lambda f(x,u) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0,$$

where $f(x,u)$ satisfies some regularity hypotheses and some increasing and/or convexity conditions with respect to u (see, for example, [7; 11; 13; 14]). When $\lambda = 1$ in $(*)$, the questions of multiplicity of solutions arise. As is well known this last problem has exhaustive answers if $f(x) = 0$. When $f(x) \neq 0$ the existence of solutions is in general an open question. Nevertheless if $h(u)$ increases more slowly than $u^p, p < 2^* - 1 = (n+2)/(n-2)$, as $u \to \infty$ some multiplicity results have been obtained utilizing recent methods of the Calculus of Variations (see, for example, [1; 2; 6; 15]). Recently G. Bonanno and S. A. Marano in [3; 4] have demonstrated, together with an existence result for $(*)$, also an estimate from below of the supremum of the spectrum of $(*)$.

In this paper we obtain, under minimal assumptions on $f(x)$ and $h(u)$, a description of the spectrum and estimates of the generalized positive solu-

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tions of $(*)$ near $\partial\Omega$. Some results of the author (see [8; 9; 10]) are applied together with the method of sub-super solutions.

In the first section the main results are stated. Their proof and certain auxiliary results are contained in the second section.

1. Results. Let $\Omega \subset \mathbb{R}^N$, $2 \le N$, be a bounded domain with C^2 boundary. $M^{r,p}(\Omega)$, $N < r$, $2 < p$, denotes the space of all $\gamma \in L^r_{\rm loc}(\Omega)$ such that

$$\overline{\lim_{x\to\partial\Omega}} |\gamma(x)| d(x) |\ln d(x)|^p < \infty, \qquad d(x) := \operatorname{dist}(x, \partial\Omega).$$

$M^{r,p}(\Omega)$ is not empty and

$$L^\infty(\Omega) \subset M^{r,p}(\Omega) \subset L^1(\Omega), \qquad M^{r,p}(\Omega) \not\subset L^q(\Omega), \qquad 1 < q < \infty$$

(see [8, Lemma 1]).

Let $|\cdot|_p$ be the norm of $L^p(\Omega)$, $1 \le p < \infty$, and $|\cdot|_\infty$ denote the norm of $L^\infty(\Omega)$ and $C(\overline{\Omega})$. As usual we put $\mathbb{N} \setminus \{0\} = \mathbb{N}^*$ and given $\alpha, \beta \in C(\overline{\Omega})$ with $\alpha \le \beta$, $[\alpha, \beta]$ denotes the set of $v \in C(\overline{\Omega})$ such that $\alpha \le v \le \beta$. Let $\varphi(x)$ be a positive eigenfunction of the Dirichlet problem for $-\Delta$ in Ω.

The main result of this paper is the following:

THEOREM. *Let $f \in M^{r,p}(\Omega)$, $f \ge 0$, $f \ne 0$, and $h \in C(\mathbb{R}_+)$, $h \ge 0$. Define Λ to be the set of $\lambda > 0$ so that the problem*

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda(f(x) + h(u)), \quad u > 0 \quad \text{in } \Omega; \quad u|_{\partial\Omega} = 0, \\ u \in W^{2,r}_{\rm loc}(\Omega) \cap C^1(\overline{\Omega}), \end{cases}$$

has at least one solution. There exists $\lambda^* \in]0, \infty]$ such that

$$]0, \lambda^*[\subset \Lambda \subset]0, \lambda^*].$$

Moreover, for each solution u of (P_λ) *there exists $c = c(\lambda) > 0$ such that*

$$c^{-1}\varphi \le u \le c\varphi.$$

Finally,

$$\lim_{u\to\infty} h(u)/u = 0 \Rightarrow \lambda^* = \infty;$$
$$\underline{\lim}_{u\to\infty} h(u)/u > 0 \Rightarrow \lambda^* < \infty.$$

Remark. If $f \in M^{r,p}(\Omega) \cap C^{0,\mu}(\Omega)$, $h \in C^{0,\mu}(\mathbb{R}^*_+) \cap C(\mathbb{R}_+)$ and $0 < \mu < 1$ then every solution of (P_λ) is a classical solution, i.e. it belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$.

2. Preparatory results and proof of the Theorem. Let $G(x, y)$ be the Green function of $-\Delta$ with the Dirichlet condition on $\partial\Omega$. From the properties of $G(x, y)$ and $\varphi(x)$ it follows that there exists a continuous extension of $G(x,y)/\varphi(x)$ to $\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)\mid x\in\mathbb{R}^N\}$ (see [8; 12]), which we denote as $N(x,y)$. Let G and N be the operators

$$G(v)(x)=\int_\Omega G(x,y)v(y)\,dy,\qquad N(v)(x)=\int_\Omega N(x,y)v(y)\,dy.$$

From Corollary 12 and Lemma 14 of [8] it follows that

$$M^{r,p}(\Omega)\subset \operatorname{Dom} G,\qquad M^{r,p}(\Omega)\subset \operatorname{Dom} N.$$

THEOREM 1 ([8, Lemma 13; 9, Theorems 5 and 6]). (1) *$G(v)$ and $N(v)$ belong to $C(\overline{\Omega})$ for all $v\in M^{r,p}(\Omega)$.*

(2) *For every $\mathcal{F}\subset M^{r,p}(\Omega)$ and $\beta\in M^{r,p}(\Omega)$, if $|v|\le\beta$ a.e. in Ω for all $v\in\mathcal{F}$, then $G(\mathcal{F})$ and $N(\mathcal{F})$ are relatively compact in $C(\overline{\Omega})$.*

(3) *Let $v_n\in M^{r,p}(\Omega)$, $n\in\mathbb{N}$, and $\beta\in M^{r,p}(\Omega)$. If $v_n\to v$ in measure and $|v_n|\le\beta$ a.e. in Ω, then $v\in M^{r,p}(\Omega)$ and $G(v_n)\to G(v)$, $N(v_n)\to N(v)$ in $C(\overline{\Omega})$.*

THEOREM 2 ([8, Theorem 16; 9, Theorem 8]). *For all $f\in M^{r,p}(\Omega)$, the function $u=G(f)$ belongs to $W^{2,r}_{\rm loc}(\Omega)\cap C^1(\overline{\Omega})$ and it is the unique solution of the problem*

$$-\Delta u=f\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0. \tag{4}$$

THEOREM 3 ([8, Theorem 9; 10, Lemma 6]). *Given $f\in M^{r,p}(\Omega)$, $f\ge0$, $f\ne0$ there exist $m=m(f)>0$ and $M=M(f)>0$ such that the solution u of* (4) *satisfies the estimates*

$$m\varphi(x)\le u(x)\le M\varphi(x),\qquad x\in\overline{\Omega}.$$

To prove the Theorem we need some general results on semilinear problems

$$-\Delta u=k(x,u)\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0, \tag{5}$$

where $k(x,u)$ is a positive Carathéodory function defined in $\Omega\times\mathbb{R}_+$ ($k(\cdot,u)$ is measurable for every $u\ge0$, and $k(x,\cdot)$ is continuous for a.e. $x\in\Omega$).

THEOREM 4. *Let $\underline{u},\overline{u}\in C(\overline{\Omega})$ and $\beta\in M^{r,p}(\Omega)$. If*

$$v\in[\varphi\underline{u},\varphi\overline{u}]\Rightarrow|k(\cdot,v)|\le\beta\ \text{a.e. in }\Omega\text{ and }N(k(\cdot,v))\in[\underline{u},\overline{u}],$$

then there exists a solution $u\in W^{2,r}_{\rm loc}\cap C^1(\overline{\Omega})\cap[\varphi\underline{u},\varphi\overline{u}]$ of (5).

Proof. Since $k(\cdot,v)\in M^{r,p}(\Omega)$ and $v\in[\varphi\underline{u},\varphi\overline{u}]$, by Theorem 2 there exists a solution $U(v)\in W^{2,r}_{\rm loc}(\Omega)\cap C^1(\overline{\Omega})$ of (5) and $U(v)=G(k(\cdot,v))$. The hypothesis implies that $U(v)\in[\varphi\underline{u},\varphi\overline{u}]$. By Theorem 1 and the Schauder Theorem, U has at least one fixed point. From Theorem 2, this fixed point is a solution of (5). ∎

$k(x,u)$ is called *sublinear as* $u \to \infty$ if there exists $b \in M^{r,p}(\Omega)$ with $0 < b(x)$ for a.e. $x \in \Omega$ such that

$$\lim_{u\to\infty} \frac{k(x,u)}{b(x)u} = 0, \tag{6}$$

uniformly with respect to a.e. $x \in \Omega$. The hypotheses of the preceding theorem are satisfied if $k(x,u)$ is sublinear as $u \to \infty$. Therefore we obtain:

THEOREM 5. *If $k(x,u)$ is sublinear as $u \to \infty$ and $\sup_{0\le t\le s} k(\cdot,t) \in M^{r,p}(\Omega)$ for all $s \ge 0$, then there exist $R > 0$ and a solution $u \in W^{2,r}_{\mathrm{loc}} \cap C^1(\overline{\Omega}) \cap [0, R\varphi]$ of* (5).

Proof. Since for all $v \in C(\overline{\Omega})$ with $0 \le v$ we have

$$k(x, v(x)) \le \max_{0\le u\le |v|_\infty} k(x,u),$$

it follows that $k(\cdot,v) \in M^{r,p}(\Omega)$. Let $U(v) = G(k(\cdot,v))$, a positive solution of (5).

Now we observe that

$$\lim_{R\to 0} \frac{1}{R} N(k(\cdot,v)) = 0, \tag{7}$$

uniformly with respect to v in $[0, R\varphi]$ and $x \in \overline{\Omega}$. For $\varepsilon > 0$, there exists $s_0 > 0$ such that

$$s_0 \le u \Rightarrow k(x,u) \le \varepsilon b(x) u \text{ for a.e. } x \in \Omega.$$

Then it follows that

$$\begin{aligned} N(k(\cdot,v))(x)|_{0\le v\le R\varphi} &= \Big(\int\limits_{v\le s_0} + \int\limits_{s_0\le v}\Big) N(x,y) k(y, v(y))\, dy \\ &\le |N(\sup_{0\le v\le s_0} k(\cdot,v))|_\infty + \varepsilon N(bv)(x)|_{0\le v\le R\varphi} \\ &\le |N(\sup_{0\le v\le s_0} k(\cdot,v))|_\infty + \varepsilon R |N(b\varphi)|_\infty. \end{aligned}$$

From this (7) follows.

Let $R > 0$ (independent of x) be such that

$$0 \le v \le R\varphi \Rightarrow 0 \le N(k(\cdot,v)) \le R \Leftrightarrow 0 \le G(k(\cdot,v)) \le R\varphi.$$

By virtue of the previous theorem the assertion follows. ∎

Proof of Theorem. Firstly we observe that for all $v \in C(\overline{\Omega})$ and $\lambda > 0$,

$$\lambda(f + h(v)) \in M^{r,p}(\Omega), \qquad \lambda(f + \sup_{0\le u\le |v|_\infty} h(u)) \in M^{r,p}(\Omega).$$

Therefore, putting $h_0 := \sup\{h(s) \mid 0 \le s \le |\varphi|_\infty\}$, from Corollary 12 of [8] we have $|N(f + h_0)|_\infty < \infty$.

Now the proof is divided into five steps.

STEP 1. Since for every $v \in [0, \varphi]$ we have

$$0 \le N[\lambda(f + h(v))](x) \le \lambda |N(f + h_0)|_\infty \le 1,$$

from Theorem 4 we conclude that (P_λ) has at least one solution. Then

$$]0, 1/|N(f + h_0)|_\infty] \subset \Lambda.$$

STEP 2. To prove that Λ is an interval we show that

$$\lambda \in \Lambda,\ 0 < \mu < \lambda \Rightarrow \mu \in \Lambda.$$

Let u_λ be a solution of (P_λ), and consider the function

$$k(x, u) = \mu(f(x) + h(\min\{u, u_\lambda(x)\})).$$

The following properties are valid:

$$\begin{aligned} &0 \le k(x, u), \qquad k(x, u) \ne 0;\\ &0 \le k(\cdot, u) \in M^{r,p}(\Omega);\\ &0 \le k(x, u) \ \text{ sublinear as } u \to \infty. \end{aligned}$$

From Theorem 5 we know that there exists $u_\mu \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^1(\overline{\Omega})$ such that

$$-\Delta u_\mu = k(x, u_\mu), \quad 0 < u_\mu \quad \text{in } \Omega, \quad u_\mu|_{\partial\Omega} = 0.$$

Now we prove that $u_\mu \le u_\lambda$. Otherwise $A = \{x \in \Omega \mid u_\mu(x) > u_\lambda(x)\} \ne \emptyset$. Since

$$\begin{aligned} x \in A \Rightarrow -\Delta u_\mu &= \mu(f(x) + h(\min\{u_\mu(x), u_\lambda(x)\}))\\ &\le \lambda(f(x) + h(u_\lambda(x))) = -\Delta u_\lambda, \end{aligned}$$

we obtain

$$-\Delta(u_\mu - u_\lambda) \le 0 \quad \text{in } A \quad \text{and} \quad (u_\mu - u_\lambda)|_{\partial A} = 0.$$

By the Maximum Principle (see [5]), $u_\mu \le u_\lambda$ in A. But this is not true since $A \ne \emptyset$. Therefore $u_\mu \le u_\lambda$.

We conclude that u_μ is a solution of (P_λ), and so $\mu \in \Lambda$.

STEP 3. The estimate for positive solutions of (P_λ) follows by Theorem 3.

STEP 4. Let $\lim_{u\to\infty} h(u)/u = 0$; the Carathéodory function

$$k(x, u) := \lambda(f(x) + h(u))$$

is positive and sublinear. In fact, the function $b(x) := 1 + f(x)$ belongs to $M^{r,p}(\Omega)$ and (6) is satisfied. From the previous theorem, (P_λ) has at least one solution u. Moreover, if $\underline{u} \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^1(\overline{\Omega})$ is a solution of

$$-\Delta u = f(x), \quad u > 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

(see Theorem 2), from the Maximum Principle we deduce $\lambda\underline{u} \le u$. Since by virtue of Theorem 3, $\underline{u} > 0$, we conclude that $u > 0$.

Step 5. Let $\underline{\lim}_{u\to\infty} h(u)/u > 0$. There exist $s_0 \geq 0$ and $m > 0$ such that $h(u) \geq mu$ for $u \geq s_0$. Arguing by contradiction, suppose that $\lambda^* = \infty$. From the Maximum Principle (see [5]) it follows that $\lambda\underline{u} \leq u_\lambda$. Let $\lambda_0 > 0$ be such that the open set $T = \{x \in \Omega \mid s_0 < \lambda_0\underline{u}(x)\}$ is not empty. Hence, putting $\Omega_\lambda = \{x \in \Omega \mid s_0 < u_\lambda(x)\}$, we obtain

$$\lambda_0 \leq \lambda \Rightarrow T \subset \Omega_\lambda \Rightarrow 0 < |T| \leq |\Omega_\lambda|.$$

Then

$$\int_{\Omega_\lambda} u_\lambda\varphi\,dx \geq \lambda\int_T \underline{u}\varphi\,dx \geq \lambda\frac{s_0}{\lambda_0}\int_T \varphi\,dx$$

and $\int_T \varphi\,dx > 0$ (see [8, Theorem 9]) imply

$$\lim_{\lambda\to\infty}\int_{\Omega_\lambda} u_\lambda\varphi\,dx = \infty. \tag{8}$$

Therefore since u_λ is a solution of (P_λ) it follows that

$$\lambda_1\int_\Omega u_\lambda\varphi\,dx = \lambda\int_\Omega f\varphi\,dx + \lambda\int_\Omega h(u_\lambda)\varphi\,dx \geq \lambda\int_\Omega f\varphi\,dx + \lambda m\int_{\Omega_\lambda} u_\lambda\varphi\,dx.$$

Then

$$\lambda_1\int_{\Omega_\lambda} u_\lambda\varphi\,dx + \lambda_1\int_{\Omega\setminus\Omega_\lambda} u_\lambda\varphi\,dx \geq \lambda\int_\Omega f\varphi\,dx + \lambda m\int_{\Omega_\lambda} u_\lambda\varphi\,dx$$
$$\Rightarrow (\lambda_1 - \lambda m)\int_{\Omega_\lambda} u_\lambda\varphi\,dx + \lambda_1 s_0\int_{\Omega\setminus\Omega_\lambda}\varphi\,dx \geq \lambda\int_\Omega f\varphi\,dx.$$

This inequality is impossible, because, from (8), the first term goes to $-\infty$ as $\lambda \to \infty$. Therefore the original assumption is false. Thus $\lambda^* < \infty$. ∎

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