On a semilinear elliptic eigenvalue problem

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Abstract. We obtain a description of the spectrum and estimates for generalized positive solutions of \(-\Delta u = \lambda (f(x) + h(u))\) in \(\Omega\), \(u|_{\partial \Omega} = 0\), where \(f(x)\) and \(h(u)\) satisfy minimal regularity assumptions.

Introduction. From various points of view there is still interest in the eigenvalue problem
\[(*) \quad -\Delta u = \lambda (f(x) + h(u)) \quad \text{in} \ \Omega, \quad u|_{\partial \Omega} = 0,\]
where \(\Omega \subset \mathbb{R}^N, 2 \leq N\), is bounded. Following the terminology of Krasnosel'skiǐ we define the spectrum of (*) to be the set of the values \(\lambda\) for which there exist positive solutions of (*). Various authors have obtained a description of the spectrum of the more general problem than (*), i.e.
\[-\Delta u = \lambda f(x, u) \quad \text{in} \ \Omega, \quad u|_{\partial \Omega} = 0,\]
where \(f(x, u)\) satisfies some regularity hypotheses and some increasing and/or convexity conditions with respect to \(u\) (see, for example, [7; 11; 13; 14]). When \(\lambda = 1\) in (*), the questions of multiplicity of solutions arise. As is well known this last problem has exhaustive answers if \(f(x) = 0\). When \(f(x) \neq 0\) the existence of solutions is in general an open question. Nevertheless if \(h(u)\) increases more slowly than \(u^p, p < 2^* - 1 = (n + 2)/(n - 2)\), as \(u \to \infty\) some multiplicity results have been obtained utilizing recent methods of the Calculus of Variations (see, for example, [1; 2; 6; 15]). Recently G. Bonanno and S. A. Marano in [3; 4] have demonstrated, together with an existence result for (*), also an estimate from below of the supremum of the spectrum of (*).

In this paper we obtain, under minimal assumptions on \(f(x)\) and \(h(u)\), a description of the spectrum and estimates of the generalized positive solu-


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tions of (⋆) near \( \partial \Omega \). Some results of the author (see [8; 9; 10]) are applied together with the method of sub-super solutions.

In the first section the main results are stated. Their proof and certain auxiliary results are contained in the second section.

1. Results. Let \( \Omega \subset \mathbb{R}^N \), \( 2 \leq N \), be a bounded domain with \( C^2 \) boundary. \( M^{r,p}(\Omega) \), \( N < r \), \( 2 < p \), denotes the space of all \( \gamma \in L^{r,\text{loc}}(\Omega) \) such that
\[
\lim_{x \to \partial \Omega} |\gamma(x)|d(x)|\ln d(x)|^p < \infty, \quad d(x) := \text{dist}(x, \partial \Omega).
\]
\( M^{r,p}(\Omega) \) is not empty and
\[
L^\infty(\Omega) \subset M^{r,p}(\Omega) \subset L^1(\Omega), \quad M^{r,p}(\Omega) \not\subset L^q(\Omega), \quad 1 < q < \infty
\]
(see [8, Lemma 1]).

Let \( |\cdot|_p \) be the norm of \( L^p(\Omega) \), \( 1 \leq p < \infty \), and \( |\cdot|_\infty \) denote the norm of \( L^\infty(\Omega) \) and \( C(\overline{\Omega}) \). As usual we put \( \mathbb{N} \setminus \{0\} = \mathbb{N}^* \) and given \( \alpha, \beta \in C(\overline{\Omega}) \) with \( \alpha \leq \beta \), \( [\alpha, \beta] \) denotes the set of \( v \in C(\overline{\Omega}) \) such that \( \alpha \leq v \leq \beta \). Let \( \varphi(x) \) be a positive eigenfunction of the Dirichlet problem for \(-\Delta\) in \( \Omega \).

The main result of this paper is the following:

**Theorem.** Let \( f \in M^{r,p}(\Omega) \), \( f \geq 0 \), \( f \neq 0 \), and \( h \in C(\mathbb{R}_+) \), \( h \geq 0 \). Define \( \Lambda \) to be the set of \( \lambda > 0 \) so that the problem
\[
(P_\lambda) \quad \begin{cases} 
-\Delta u = \lambda(f(x) + h(u)), & u > 0 \text{ in } \Omega; \quad u|_{\partial \Omega} = 0, \\
u \in W^{2,r}_{\text{loc}}(\Omega) \cap C^1(\overline{\Omega}),
\end{cases}
\]
has at least one solution. There exists \( \lambda^* \in ]0, \infty] \) such that
\[
[0, \lambda^*] \subset \Lambda \subset [0, \lambda^*].
\]
Moreover, for each solution \( u \) of \((P_\lambda)\) there exists \( c = c(\lambda) > 0 \) such that
\[
c^{-1}\varphi \leq u \leq c\varphi.
\]
Finally,
\[
\lim_{u \to \infty} h(u)/u = 0 \Rightarrow \lambda^* = \infty;
\]
\[
\lim_{u \to \infty} h(u)/u > 0 \Rightarrow \lambda^* < \infty.
\]

**Remark.** If \( f \in M^{r,p}(\Omega) \cap C^{0,\mu}(\Omega) \), \( h \in C^{0,\mu}(\mathbb{R}_+) \cap C(\mathbb{R}_+) \) and \( 0 < \mu < 1 \) then every solution of \((P_\lambda)\) is a classical solution, i.e. it belongs to \( C^2(\Omega) \cap C^1(\overline{\Omega}) \).

2. Preparatory results and proof of the Theorem. Let \( G(x, y) \) be the Green function of \(-\Delta\) with the Dirichlet condition on \( \partial \Omega \). From the properties of \( G(x, y) \) and \( \varphi(x) \) it follows that there exists a continuous
extension of $G(x,y)/\varphi(x)$ to $\Omega \times \Omega \setminus \{(x,x) \mid x \in \mathbb{R}^N\}$ (see [8; 12]), which we denote as $N(x,y)$. Let $G$ and $N$ be the operators
\[
G(v)(x) = \int_{\Omega} G(x,y)v(y) \, dy, \quad N(v)(x) = \int_{\Omega} N(x,y)v(y) \, dy.
\]
From Corollary 12 and Lemma 14 of [8] it follows that
\[
M^{r,p}(\Omega) \subset \text{Dom} \, G, \quad M^{r,p}(\Omega) \subset \text{Dom} \, N.
\]

**Theorem 1** ([8, Lemma 13; 9, Theorems 5 and 6]). (1) $G(v)$ and $N(v)$ belong to $C(\overline{\Omega})$ for all $v \in M^{r,p}(\Omega)$.

(2) For every $\mathcal{F} \subset M^{r,p}(\Omega)$ and $\beta \in M^{r,p}(\Omega)$, if $|v| \leq \beta$ a.e. in $\Omega$ for all $v \in \mathcal{F}$, then $G(\mathcal{F})$ and $N(\mathcal{F})$ are relatively compact in $C(\overline{\Omega})$.

(3) Let $v_n \in M^{r,p}(\Omega)$, $n \in \mathbb{N}$, and $\beta \in M^{r,p}(\Omega)$. If $v_n \to v$ in measure and $|v_n| \leq \beta$ a.e. in $\Omega$, then $v \in M^{r,p}(\Omega)$ and $G(v_n) \to G(v)$, $N(v_n) \to N(v)$ in $C(\overline{\Omega})$.

**Theorem 2** ([8, Theorem 16; 9, Theorem 8]). For all $f \in M^{r,p}(\Omega)$, the function $u = G(f)$ belongs to $W^{2,r}_{\text{loc}}(\Omega) \cap C^1(\overline{\Omega})$ and it is the unique solution of the problem
\[
-\Delta u = f \quad \text{in} \; \Omega, \quad u|_{\partial\Omega} = 0.
\]

**Theorem 3** ([8, Theorem 9; 10, Lemma 6]). Given $f \in M^{r,p}(\Omega)$, $f \geq 0$, $f \neq 0$ there exist $m = m(f) > 0$ and $M = M(f) > 0$ such that the solution $u$ of (4) satisfies the estimates
\[
m \varphi(x) \leq u(x) \leq M \varphi(x), \quad x \in \overline{\Omega}.
\]

To prove the Theorem we need some general results on semilinear problems
\[
-\Delta u = k(x,u) \quad \text{in} \; \Omega, \quad u|_{\partial\Omega} = 0,
\]
where $k(x,u)$ is a positive Carathéodory function defined in $\Omega \times \mathbb{R}^+$ ($k(\cdot, u)$ is measurable for every $u \geq 0$, and $k(x, \cdot)$ is continuous for a.e. $x \in \Omega$).

**Theorem 4.** Let $u, \varphi \in C(\overline{\Omega})$ and $\beta \in M^{r,p}(\Omega)$. If
\[
v \in [\varphi u, \varphi \overline{\varphi}] \Rightarrow |k(\cdot, v)| \leq \beta \quad \text{a.e. in} \; \Omega \quad \text{and} \quad N(k(\cdot, v)) \in [u, \varphi],
\]

then there exists a solution $u \in W^{2,r}_{\text{loc}}(\Omega) \cap C^1(\overline{\Omega}) \cap [\varphi u, \varphi \overline{\varphi}]$ of (5).

**Proof.** Since $k(\cdot, v) \in M^{r,p}(\Omega)$ and $v \in [\varphi u, \varphi \overline{\varphi}]$, by Theorem 2 there exists a solution $U(v) \in W^{2,r}_{\text{loc}}(\Omega) \cap C^1(\overline{\Omega})$ of (5) and $U(v) = G(k(\cdot, v))$. The hypothesis implies that $U(v) \in [\varphi u, \varphi \overline{\varphi}]$. By Theorem 1 and the Schauder Theorem, $U$ has at least one fixed point. From Theorem 2, this fixed point is a solution of (5). \[\square\]
$k(x, u)$ is called \textit{sublinear} as $u \to \infty$ if there exists $b \in M^{r, p}(\Omega)$ with $0 < b(x)$ for a.e. $x \in \Omega$ such that
\begin{equation}
\lim_{u \to \infty} \frac{k(x, u)}{b(x)u} = 0,
\end{equation}
uniformly with respect to a.e. $x \in \Omega$. The hypotheses of the preceding theorem are satisfied if $k(x, u)$ is sublinear as $u \to \infty$. Therefore we obtain:
\textbf{Theorem 5.} If $k(x, u)$ is sublinear as $u \to \infty$ and $\sup_{0 \leq s \leq s} k(\cdot, t) \in M^{r, p}(\Omega)$ for all $s \geq 0$, then there exist $R > 0$ and a solution $u \in W^{2, r}_{loc} \cap C^{1} (\overline{\Omega}) \cap [0, R\varphi]$ of (5).

\textbf{Proof.} Since for all $v \in C(\overline{\Omega})$ with $0 \leq v$ we have
\[ k(x, v(x)) \leq \max_{0 \leq u \leq |v|_{\infty}} k(x, u), \]
it follows that $k(\cdot, v) \in M^{r, p}(\Omega)$. Let $U(v) = G(k(\cdot, v))$, a positive solution of (5).

Now we observe that
\begin{equation}
\lim_{R \to 0} \frac{1}{R} N(k(\cdot, v)) = 0,
\end{equation}
uniformly with respect to $v$ in $[0, R\varphi]$ and $x \in \overline{\Omega}$. For $\varepsilon > 0$, there exists $s_0 > 0$ such that $s_0 \leq u \Rightarrow k(x, u) \leq \varepsilon b(x)u$ for a.e. $x \in \Omega$.

Then it follows that
\begin{align*}
N(k(\cdot, v))(x)_{0 \leq v \leq R\varphi} &= \left( \int_{v \leq s_0} + \int_{s_0 \leq v} \right) N(x, y)k(y, v(y)) \, dy \\
&\leq |N( \sup_{0 \leq v \leq s_0} k(\cdot, v))|_{\infty} + \varepsilon |N(bv)(x)|_{0 \leq v \leq R\varphi} \\
&\leq \varepsilon R |N(b\varphi)|_{\infty}.
\end{align*}
From this (7) follows.

Let $R > 0$ (independent of $x$) be such that
\[ 0 \leq v \leq R\varphi \Rightarrow 0 \leq N(k(\cdot, v)) \leq R \Leftrightarrow 0 \leq G(k(\cdot, v)) \leq R\varphi. \]
By virtue of the previous theorem the assertion follows. \hfill \blacksquare

\textbf{Proof of Theorem.} Firstly we observe that for all $v \in C(\overline{\Omega})$ and $\lambda > 0$,
\[ \lambda(f + h(v)) \in M^{r, p}(\Omega), \quad \lambda(f + \sup_{0 \leq u \leq |v|_{\infty}} h(u)) \in M^{r, p}(\Omega). \]
Therefore, putting $h_0 := \sup\{h(s) \mid 0 \leq s \leq |\varphi|_{\infty}\}$, from Corollary 12 of [8] we have $|N(f + h_0)|_{\infty} < \infty$. 

Now the proof is divided into five steps.

**Step 1.** Since for every $v \in [0, \varphi]$ we have

$$0 \leq N[\lambda(f + h(v))](x) \leq \lambda|N(f + h_0)|_{\infty} \leq 1,$$

from Theorem 4 we conclude that $(P_\lambda)$ has at least one solution. Then

$$[0, 1/|N(f + h_0)|_{\infty}] \subset \Lambda.$$

**Step 2.** To prove that $\Lambda$ is an interval we show that

$$\lambda \in \Lambda, 0 < \mu < \lambda \Rightarrow \mu \in \Lambda.$$

Let $u_\lambda$ be a solution of $(P_\lambda)$, and consider the function

$$k(x, u) = \mu(f(x) + h(\min\{u, u_\lambda(x)\})).$$

The following properties are valid:

$$0 \leq k(x, u), \quad k(x, u) \neq 0;$$

$$0 \leq k(\cdot, u) \in M^{r,p}(\Omega);$$

$$0 \leq k(x, u) \text{ sublinear as } u \to \infty.$$

From Theorem 5 we know that there exists

$$u_{\mu} \in W^{2,r}_{\text{loc}}(\Omega) \cap C^1(\Omega)$$

such that

$$-\Delta u_{\mu} = k(x, u_{\mu}), \quad 0 < u_{\mu} \quad \text{in } \Omega, \quad u_{\mu}|_{\partial \Omega} = 0.$$

Now we prove that $u_{\mu} \leq u_\lambda$. Otherwise $A = \{x \in \Omega \mid u_{\mu}(x) > u_\lambda(x)\} \neq \emptyset$. Since

$$x \in A \Rightarrow -\Delta u_{\mu} = \mu(f(x) + h(\min\{u_{\mu}(x), u_\lambda(x)\})) \leq \lambda(f(x) + h(u_\lambda(x))) = -\Delta u_\lambda,$$

we obtain

$$-\Delta(u_{\mu} - u_\lambda) \leq 0 \quad \text{in } A \quad \text{and} \quad (u_{\mu} - u_\lambda)|_{\partial A} = 0.$$

By the Maximum Principle (see [5]), $u_{\mu} \leq u_\lambda$ in $A$. But this is not true since $A \neq \emptyset$. Therefore $u_{\mu} \leq u_\lambda$.

We conclude that $u_{\mu}$ is a solution of $(P_\lambda)$, and so $\mu \in \Lambda$.

**Step 3.** The estimate for positive solutions of $(P_\lambda)$ follows by Theorem 3.

**Step 4.** Let $\lim_{u \to \infty} h(u)/u = 0$; the Carathéodory function

$$k(x, u) := \lambda(f(x) + h(u))$$

is positive and sublinear. In fact, the function $b(x) := 1 + f(x)$ belongs to $M^{r,p}(\Omega)$ and (6) is satisfied. From the previous theorem, $(P_\lambda)$ has at least one solution $u$. Moreover, if $u \in W^{2,r}_{\text{loc}}(\Omega) \cap C^1(\Omega)$ is a solution of

$$-\Delta u = f(x), \quad u > 0 \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0,$$

(see Theorem 2), from the Maximum Principle we deduce $\lambda u \leq u$. Since by virtue of Theorem 3, $u > 0$, we conclude that $u > 0$. 
Step 5. Let \( \lim_{u \to \infty} h(u)/u > 0 \). There exist \( s_0 \geq 0 \) and \( m > 0 \) such that \( h(u) \geq mu \) for \( u \geq s_0 \). Arguing by contradiction, suppose that \( \lambda^* = \infty \). From the Maximum Principle (see [5]) it follows that \( \lambda u \leq u_\lambda \). Let \( \lambda_0 > 0 \) be such that the open set \( T = \{ x \in \Omega \mid s_0 < \lambda_0 u_\lambda(x) \} \) is not empty. Hence, putting \( \Omega_\lambda = \{ x \in \Omega \mid s_0 < u_\lambda(x) \} \), we obtain
\[
\lambda_0 \leq \lambda \Rightarrow T \subset \Omega_\lambda \Rightarrow 0 < |T| \leq |\Omega_\lambda|.
\]
Then
\[
\int_{\Omega_\lambda} u_\lambda \varphi \, dx \geq \lambda \int_T u \varphi \, dx \geq \lambda \frac{s_0}{\lambda_0} \int_T \varphi \, dx
\]
and \( \int_T \varphi \, dx > 0 \) (see [8, Theorem 9]) imply
\begin{equation}
\lim_{\lambda \to \infty} \int_{\Omega_\lambda} u_\lambda \varphi \, dx = \infty.
\end{equation}

Therefore since \( u_\lambda \) is a solution of (P\( \lambda \)) it follows that
\[
\lambda_1 \int_{\Omega_\lambda} u_\lambda \varphi \, dx = \lambda \int_{\Omega} f \varphi \, dx + \lambda \int_{\Omega} h(u_\lambda) \varphi \, dx \geq \lambda \int_{\Omega} f \varphi \, dx + \lambda m \int_{\Omega_\lambda} u_\lambda \varphi \, dx.
\]
Then
\[
\lambda_1 \int_{\Omega_\lambda} u_\lambda \varphi \, dx + \lambda_1 \int_{\Omega \setminus \Omega_\lambda} u_\lambda \varphi \, dx \geq \lambda \int_{\Omega} f \varphi \, dx + \lambda m \int_{\Omega_\lambda} u_\lambda \varphi \, dx
\]
\[
\Rightarrow (\lambda_1 - \lambda m) \int_{\Omega_\lambda} u_\lambda \varphi \, dx + \lambda_1 s_0 \int_{\Omega \setminus \Omega_\lambda} \varphi \, dx \geq \lambda \int_{\Omega} f \varphi \, dx.
\]
This inequality is impossible, because, from (8), the first term goes to \( -\infty \) as \( \lambda \to \infty \). Therefore the original assumption is false. Thus \( \lambda^* < \infty \). □

References


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