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Hedgehogs of constant width and equichordal points

by Yves Martinez-Maure (Rueil-Malmaison)

Abstract. We give a characterization of convex hypersurfaces with an equichordal point in terms of hedgehogs of constant width.

I. Introduction and statement of results. Let K be a convex body in $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ and let S be its boundary. An interior point o of K is called an *equichordal point* of S if all chords of S passing through o have the same length.

A famous unsolved problem is whether there exist plane convex curves with two equichordal points. A discussion of this problem, first raised by Fujiwara [2] and independently by Blaschke, Rothe and Weitzenböck [1], is given by Klee ([4] and [5]). Wirsing [10] proved (assuming their existence) that such curves are analytic. Petty and Crotty [8] have proved the existence of Minkowski spaces of arbitrary dimension in which there are convex hypersurfaces with exactly two equichordal points.

Let S be a smooth convex hypersurface in $\mathbb{E}^{n+1}$. The *pedal hypersurface* $P(S)$ with respect to an interior point o is defined as follows: for each $m \in S$ the point $P(m)$ is the foot of the perpendicular from the point o to the tangent hyperplane of S at m. If S is of constant width then $P(S)$ has o as an equichordal point, but $P(S)$ is not necessarily convex.

Conversely, Kelly [3] has shown that if a plane convex curve C has o as an equichordal point, then C is the pedal curve with respect to o of a curve $P^{-1}(C)$ with a kind of constant width. This curve $P^{-1}(C)$, called the *negative pedal* of the curve C, is not necessarily convex.

In this paper, we prove the following generalization to hypersurfaces.

Theorem 1. *If S is a smooth convex hypersurface with an equichordal point o, then S is the pedal hypersurface with respect to o of a hypersurface*

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$P^{-1}(S)$ with a kind of constant width. More precisely, $P^{-1}(S)$ is a hedgehog of constant width as defined below.

DEFINITION 1. For any $f \in C^\infty(\mathbb{S}^n;\mathbb{R})$, we define the *hedgehog* with supporting function f as the envelope $\mathcal{H}_f$ of the family of hyperplanes defined by the equations

$$\langle x, p\rangle = f(p)$$

(these hyperplanes are called the *supporting hyperplanes* of $\mathcal{H}_f$). In other words, $\mathcal{H}_f$ is the hypersurface (with possible singularities) parametrized by

$$x_f : \mathbb{S}^n \to \mathcal{H}_f, \quad p \mapsto x_f(p),$$

where $x_f(p) = f(p)p + (\operatorname{grad} f)(p)$ is the unique solution of the system

$$\langle x, p\rangle = f(p), \quad \langle x, \cdot\rangle = df_p(\cdot).$$

Remark. When $\mathcal{H}_f$ has a well defined tangent hyperplane at $x_f(p)$, say T, then T is defined by the equation $\langle x, p\rangle = f(p)$: the unit vector p is normal to T and $f(p)$ may be interpreted as the signed distance from the origin to T. Thus, any smooth part of $\mathcal{H}_f$ inherits a natural transverse orientation for which x_f is the reverse Gauss map. A singularity-free hedgehog is simply a convex hypersurface. For a general study of hedgehogs see R. Langevin, G. Levitt and H. Rosenberg [6].

DEFINITION 2. The hedgehog with supporting function f is said to be *of constant width* if the distance between two parallel supporting hyperplanes is constant, that is, if $f(p) + f(-p)$ is constant on $\mathbb{S}^n$.

We next prove the following results.

THEOREM 2. *Let S be a smooth convex hypersurface with the origin as an equichordal point. The negative pedal hypersurface of S with respect to the origin is convex if and only if the hypersurface obtained from S by inversion with respect to $\mathbb{S}^n$ is convex.*

THEOREM 3. *Let $\mathcal{H}_f$ be a hedgehog of constant width such that f is never zero. Then the pedal hypersurface of $\mathcal{H}_f$ with respect to the origin is a smooth hypersurface with the origin as an equichordal point. Furthermore, $P(\mathcal{H}_f)$ is convex if and only if $1/f$ is the supporting function of a convex hedgehog.*

Note that these problems are related to equireciprocal points of convex bodies (see [4]).

II. Proof of results

Proof of Theorem 1. Assume without loss of generality that o is the origin. Since S is starlike relative to o, S has a parametrization of the form

$$X : \mathbb{S}^n \to S, \quad p \mapsto g(p)p,$$

where the function g is > 0. The condition that S have the origin as an equichordal point is simply that g be of the form $g = h + r$, where r is a constant and h is a function such that

$$\forall p \in \mathbb{S}^n, \quad h(-p) = -h(p).$$

Note that the hedgehog $\mathcal{H}_h$ can be considered as a hedgehog of zero width: such a hedgehog is said to be *projective*. For a study of projective hedgehogs, see [7].

Since $(\operatorname{grad} g)(p) \in T_p\mathbb{S}^n$, it follows from the parametrization $x_g(p) = g(p)p + (\operatorname{grad} g)(p)$ of the hedgehog $\mathcal{H}_g$ that S is the pedal hypersurface of $\mathcal{H}_g$ with respect to o. Furthermore, this hedgehog $\mathcal{H}_g$ is of constant width since the distance $d(p)$ between the two supporting hyperplanes of $\mathcal{H}_g$ which are orthogonal to $p \in \mathbb{S}^n$ is given by

$$d(p) = g(p) + g(-p) = 2r = \text{const.} \ \blacksquare$$

Theorems 2 and 3 are based on the following result (see for example the book by R. Schneider [9], Sections 1.6 and 1.7).

LEMMA. *Let h_L (resp. ϱ_L) denote the supporting (resp. radial) function of a convex body L with the origin as an interior point. If a convex body K has the origin as an interior point, then its polar body K^* also has the origin as an interior point, and we have*

$$h_{K^*} = 1/\varrho_K \quad \textit{and} \quad \varrho_{K^*} = 1/h_K.$$

Proof of Theorem 2. Let Σ and Σ^* denote respectively the negative pedal hypersurface $P^{-1}(S)$ and the hypersurface obtained from S by inversion with respect to $\mathbb{S}^n$. We can deduce from the Lemma that if Σ or Σ^* is convex, then Σ and Σ^* are the boundaries of polar bodies K and K^*. Theorem 2 follows immediately. $\blacksquare$

Proof of Theorem 3. The pedal hypersurface of $\mathcal{H}_f$ is the smooth hypersurface parametrized by

$$X : \mathbb{S}^n \to S, \quad p \mapsto f(p)p,$$

which has the origin as an equichordal point since $f(p) + f(-p)$ is constant. Furthermore, we can deduce from the Lemma that if $P(\mathcal{H}_f)$ or $\mathcal{H}_{1/f}$ is convex, then $P(\mathcal{H}_f)$ and $\mathcal{H}_{1/f}$ are the boundaries of polar bodies. Theorem 3 follows immediately. $\blacksquare$

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1, rue Auguste Perret
92500 Rueil-Malmaison, France

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