A Schwarz lemma on complex ellipsoids

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Abstract. We give a Schwarz lemma on complex ellipsoids.

1. Introduction. Let \( \Delta = \{ z \in \mathbb{C} \mid |z| < 1 \} \) be the unit disc in \( \mathbb{C} \). The Schwarz lemma in one complex variable is as follows.

**Theorem 1.** (i) Let \( f : \Delta \to \Delta \) be a holomorphic map such that \( f(0) = 0 \). Then \( |f(z)| \leq |z| \) for all \( z \in \Delta \).

(ii) If, moreover, there exists \( z_0 \in \Delta \setminus \{0\} \) such that \( |f(z_0)| = |z_0| \), or if \( |f'(0)| = 1 \), then there exists a complex number \( \lambda \) of absolute value 1 such that \( f(z) = \lambda z \) and \( f \) is an automorphism of \( \Delta \).

Let \( D \) be the unit ball in \( \mathbb{C}^n \) for some norm \( \| \cdot \| \), and let \( f : D \to D \) be a holomorphic map such that \( f(0) = 0 \). By the Hahn–Banach theorem, we have \( \|f(z)\| \leq \|z\| \) for all \( z \in D \). As a generalization of part (ii) of the above theorem, Vigué [7] proved the following.

**Theorem 2.** Let \( D \) be the unit ball in \( \mathbb{C}^n \) for some norm \( \| \cdot \| \), and let \( f : D \to D \) be a holomorphic map such that \( f(0) = 0 \). Assume that every boundary point of \( D \) is a complex extreme point of \( D \). If one of the following conditions is satisfied, then \( f \) is a linear automorphism of \( \mathbb{C}^n \).

\( (H_1) \) There exists a nonempty open subset \( U \) of \( D \) such that \( \|f(x)\| = \|x\| \) on \( U \).

\( (H_2) \) There exists a nonempty open subset \( U \) of \( D \) such that \( c_D(f(0), f(x)) = c_D(0, x) \) on \( U \), where \( c_D \) denotes the Carathéodory distance on \( D \).

\( (H_3) \) There exists a nonempty open subset \( V \) of \( T_0(D) \) such that \( E_D(f(0), f'(0)v) = E_D(0, v) \) on \( V \), where \( E_D \) denotes the infinitesimal Carathéodory metric on \( D \).

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Moreover, he showed that if there exists a point \( a \in U \setminus \{0\} \) such that \( f(a) = a \), or if the boundary \( \partial D \) of \( D \) is a real-analytic submanifold of \( \mathbb{C}^n \), then \( f \) is a linear automorphism of \( D \). As a corollary, he proved that if \( D \) is the unit ball of \( \mathbb{C}^n \) for the Euclidean norm on \( \mathbb{C}^n \), then \( f \) is a linear automorphism of \( D \). But, in the above results, the conditions (H1) and (H2) are strong, because a point in \( \Delta \) is of codimension 1 and an open set in \( D \) is of codimension 0. The author [2] announced that Vigué’s results hold under the hypothesis that one of the conditions (H1), (H2) is satisfied for some local complex submanifold of codimension 1 instead of an open subset.

The aim of the present paper is to consider an analogous result on complex ellipsoids \( E(p) \). However, \( E(p) \) is not convex in general. For a bounded balanced convex domain \( D \), the Minkowski function \( h \) of \( D \) is a norm on \( \mathbb{C}^n \) and \( D \) is the unit ball in \( \mathbb{C}^n \) with respect to this norm. Also, \( c_D = \tilde{k}_D \) and \( E_D = \kappa_D \) in the convex case (Lempert [4], [5], Royden–Wong [6]), where \( \tilde{k}_D \) is the Lempert function and \( \kappa_D \) is the the Kobayashi–Royden pseudometric for \( D \). So we use \( h, \tilde{k}_D, \kappa_D \) instead of \( \| \cdot \|, c_D, E_D \). First we give a theorem on some bounded balanced pseudoconvex domains which corresponds to Theorem 2. Then we show that if \( D = E(p) \), then \( f \) is a linear automorphism of \( E(p) \). We also give an example showing that our hypothesis cannot be weakened.

Some ideas of this paper come from Dini–Primicerio [1] and Vigué [7], [8].

2. Main results. The Lempert function \( \tilde{k}_D \) and the Kobayashi–Royden pseudometric \( \kappa_D \) for a domain \( D \) in \( \mathbb{C}^n \) are defined as follows:

\[
\tilde{k}_D(x,y) = \inf \{ \rho(\xi,\eta) \mid \xi,\eta \in \Delta, \exists \varphi \in H(\Delta,D) \text{ such that } \varphi(\xi) = x, \varphi(\eta) = y \},
\]

\[
\kappa_D(z;X) = \inf \{ \gamma(\lambda) \mid \exists \varphi \in H(\Delta,D), \exists \lambda \in \Delta \text{ such that } \varphi(\lambda) = z, \alpha \varphi'(\lambda) = X \},
\]

where \( \rho \) is the Poincaré distance on the unit disc \( \Delta \) and \( \gamma(\lambda) = 1/(1 - |\lambda|^2) \).

Let \( D \) be a balanced pseudoconvex domain with Minkowski function \( h \) in \( \mathbb{C}^n \). Then we have (Propositions 3.1.10 and 3.5.3 of Jarnicki and Pflug [3])

(1) \( \tilde{k}_D(0,x) = \rho(0,h(x)) \) for any \( x \in D \),
(2) \( \kappa_D(0,X) = h(X) \) for any \( X \) in \( \mathbb{C}^n \).

Let \( f \) be a holomorphic map from \( D \) to \( D \) such that \( f(0) = 0 \). By (1) and the distance decreasing property of the Lempert functions, we have

\[
\rho(0,h(z)) = \tilde{k}_D(0,z) \geq \tilde{k}_D(0,f(z)) = \rho(0,h(f(z))).
\]

Since \( \rho(0,r) \) is increasing for \( 0 \leq r < 1 \), we obtain \( h(f(z)) \leq h(z) \). This is a generalization of part (i) of the Schwarz lemma to balanced pseudoconvex domains.
A boundary point \( x \) of \( D \) is said to be an extreme point of \( \overline{D} \) if there is no non-constant holomorphic mapping \( g : \Delta \to \overline{D} \) with \( x = g(0) \). For example, \( C^\infty \)-smooth strictly pseudoconvex boundary points are extreme points (p. 257 of Jarnicki and Pflug [3]).

A mapping \( \varphi \in H(\Delta, D) \) is said to be a complex \( \tilde{k}_D \)-geodesic for \((x, y)\) if there exist points \( \xi, \eta \in \Delta \) such that \( \varphi(\xi) = x, \varphi(\eta) = y \), and \( \tilde{k}_D(x, y) = g(\xi, \eta) \).

A mapping \( \varphi \in H(\Delta, D) \) is said to be a complex \( \kappa_D \)-geodesic for \((z, X)\) if there exist \( \lambda \in \Delta \) and \( \alpha \in \mathbb{C} \) such that \( \varphi(\lambda) = z, \alpha\varphi'(\lambda) = X \), and \( \kappa_D(z, X) = \gamma(\lambda)\alpha \).

Using (1), (2) and complex \( \tilde{k}_D \)-geodesics or \( \kappa_D \)-geodesics, we have the following proposition (cf. Vigué [7], [8], Hamada [2]).

**Proposition 1.** Let \( D_j \) be bounded balanced pseudoconvex domains with Minkowski functions \( h_j \) in \( \mathbb{C}^n_j \) for \( j = 1, 2 \), and let \( f : D_1 \to D_2 \) be a holomorphic map such that \( f(0) = 0 \). Let \( f(z) = \sum_{m=1}^{\infty} P_m(z) \) be the development of \( f \) in vector-valued homogeneous polynomials \( P_m \) in a neighborhood of \( 0 \), where \( \deg P_m = m \) for each \( m \). Let \( x \in D_1 \setminus \{0\} \). If one of the following conditions is satisfied, then we have \( P_m(x) = 0 \) for \( m \geq 2 \).

- \((H'_1)\) \( h_2(f(x)) = h_1(x) \) and \( f(x)/h_2(f(x)) \) is an extreme point of \( \overline{D}_2 \).
- \((H'_2)\) \( \tilde{k}_{D_2}(f(0), f(x)) = \tilde{k}_{D_1}(0, x) \) and \( f(x)/h_2(f(x)) \) is an extreme point of \( \overline{D}_2 \).
- \((H'_3)\) \( \kappa_{D_2}(f(0), f'(0)(x)) = \kappa_{D_1}(0, x) \) and \( f'(0)(x)/h_2(f'(0)(x)) \) is an extreme point of \( \overline{D}_2 \).

**Proof.** By (1), the conditions \((H'_1)\) and \((H'_2)\) are equivalent. Let

\[
\varphi(\zeta) = \zeta \frac{x}{h_1(x)}.
\]

Then \( \varphi \) is a complex \( \tilde{k}_{D_1} \)-geodesic and \( \kappa_{D_1} \)-geodesic for \((0, x)\). Suppose that \((H'_1)\) or \((H'_2)\) is satisfied. Since

\[
\tilde{k}_{D_2}(f \circ \varphi(0), f \circ \varphi(h_1(x))) = \tilde{k}_{D_2}(0, f(x)) = \tilde{k}_{D_1}(0, x) = g(0, h_1(x)),
\]

\( f \circ \varphi \) is a complex \( \tilde{k}_{D_2} \)-geodesic for \((0, f(x))\). By Proposition 8.3.5(a) of Jarnicki and Pflug [3],

\[
f \circ \varphi(\zeta) = \zeta \frac{f(x)}{h_2(f(x))}.
\]

Since

\[
f \circ \varphi(\zeta) = \sum P_m \left( \zeta \frac{x}{h_1(x)} \right) = \sum \left( \zeta \frac{x}{h_1(x)} \right)^m P_m(x)
\]

in a neighborhood of \( 0 \), \( P_m(x) = 0 \) for \( m \geq 2 \).
Suppose that $(H'_3)$ is satisfied. Since 
\[ \kappa_{D_2}(0, f'(0)x) = \kappa_{D_1}(0, x) = h_1(x) \quad \text{and} \quad h_1(x)(f \circ \varphi)'(0) = f'(0)x, \]
$f \circ \varphi$ is a complex $\kappa_{D_2}$-geodesic for $(0, f'(0)x)$. By Proposition 8.3.5(a) of Jarnicki and Pflug [3],
\[ f \circ \varphi(\zeta) = \zeta e^{i\theta} \frac{f'(0)x}{h_2(f'(0)x)} \]
for some $\theta \in \mathbb{R}$. The rest of the argument is the same as above. This completes the proof.

The following proposition is a key for proving our theorem (Hamada [2]).

**Proposition 2.** Let $U$ be an open subset of $\mathbb{C}^n$. Let $M$ be a complex submanifold of $U$ of dimension $n - 1$. Assume that there exists a point $a$ in $M$ such that $a + T_a(M)$ does not contain the origin. Then there exists a neighborhood $U_1$ of $a$ in $\mathbb{C}^n$ such that $U_1 \subset CM = \{ tx \mid t \in \mathbb{C}, \; x \in M \}$.

**Proof.** To prove this proposition, it is enough to prove the following claim.

**Claim.** For any $x$ in $M$, let $g(x)$ be the intersection point of $a + T_a(M)$ and the complex line through $x$ and the origin $O$. Then $g$ is a biholomorphic map from a neighborhood $W_M$ of $a$ in $M$ onto a neighborhood $W_T$ of $a$ in $a + T_a(M)$.

Assume the claim is proved. Since there exists an open neighborhood $U_1$ of $a$ in $\mathbb{C}^n$ such that $U_1 \subset CW_T$, we obtain $U_1 \subset CM$.

Now we will prove the claim. By an affine coordinate change, we may assume that $a = 0$, $M = \{ z_n = \psi(z') \}$ with $\psi(0) = 0$, $d\psi(0) = 0$, where $(z', z_n) \in \mathbb{C}^n$. Then $(z', \psi(z'))$ gives a local parametrization of $M$ at $a$, $a + T_a(M) = \{ z_n = 0 \}$ and $O = (b_1, \ldots, b_n)$ with $b_n \neq 0$. Let $g(z', \psi(z')) = (g_1(z'), \ldots, g_{n-1}(z'), 0)$. Since
\[ g_i(z', \psi(z')) = b_i + \frac{b_n}{b_n - \psi(z')}(z_i - b_i) \]
for sufficiently small $z'$, we have
\[ \frac{\partial g_i}{\partial z_j}(0) = \delta_{ij} \quad (1 \leq i, j \leq n - 1). \]

Therefore $g$ is biholomorphic in a neighborhood $W_M$ of $a$. This completes the proof.

From now on, we assume that $D$ is a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ which satisfies the following condition:

(*) For any $1 \leq j_1 < \ldots < j_k \leq n$ ($0 \leq k \leq n - 1$), let
\[ \tilde{D} = D \cap \{ z_{j_1} = \ldots = z_{j_k} = 0 \} \]
be a domain in \( \mathbb{C}^{n-k} \). Then every point of \( \partial \bar{D} \cap (\mathbb{C}^*)^{n-k} \) is an extreme point of \( \bar{D} \).

By the above two propositions, we have the following theorem.

**Theorem 3.** Let \( D \) be a bounded balanced pseudoconvex domain with Minkowski function \( h \) in \( \mathbb{C}^n \) which satisfies the condition \((\ast)\), and let \( f : D \to D \) be a holomorphic map such that \( f(0) = 0 \). Let \( M \) be a connected complex submanifold of dimension \( n-1 \) of an open subset \( U \) of \( D \) such that \( a + T_a(M) \) does not contain the origin for some \( a \) in \( M \). Let \( V \) be a connected open subset of \( T_0(D) \). If one of the following conditions is satisfied, then \( f \) is a linear automorphism of \( \mathbb{C}^n \).

\[
\begin{align*}
(H''_1) & \ h(f(x)) = h(x) \text{ on } M. \\
(H''_2) & \ k_D(f(0), f(x)) = k_D(0, x) \text{ on } M. \\
(H''_3) & \ k_D(f(0), f'(0)v) = k_D(0, v) \text{ on } V.
\end{align*}
\]

**Proof.** Suppose that \((H''_1)\) or \((H''_2)\) is satisfied. We may assume that for any \( a \in M \), \( a + T_a(M) \) does not contain the origin, the functions \( f_1, \ldots, f_k \) do not vanish on \( M \) and the functions \( f_{k+1}, \ldots, f_n \) are identically 0 on \( M \) for some \( k \), \( 1 \leq k \leq n \). Let

\[ \bar{D} = D \cap \{ z_{k+1} = \ldots = z_n = 0 \} \quad \text{and} \quad \bar{f} = (f_1, \ldots, f_k). \]

Then \( \bar{D} \) is a bounded balanced pseudoconvex domain in \( \mathbb{C}^k \) with Minkowski function \( \bar{h} = h| \bar{D} \), and \( \bar{f} \) is a holomorphic map from \( D \) to \( \bar{D} \) with \( \bar{f}(0) = 0 \). Since the functions \( f_1, \ldots, f_k \) do not vanish on \( M \), \( \bar{f}(x)/\bar{h}(\bar{f}(x)) \) is an extreme point of \( \bar{D} \) for any \( x \in M \). Let

\[ \bar{f}(z) = \sum_{m=1}^{\infty} P_m(z) \]

be the development of \( \bar{f} \) in vector-valued homogeneous polynomials \( P_m \) in a neighborhood of 0, where \( \deg P_m = m \) for each \( m \). Since \( \bar{h}(\bar{f}(x)) = \bar{h}(f(x)) = h(x) \) on \( M \), we have \( P_m(x) = 0 \) on a nonempty open subset \( U_1 \) of \( D \) for \( m \geq 2 \) by Propositions 1 and 2. By the analytic continuation theorem, \( P_m \) is identically 0 for \( m \geq 2 \). Therefore \( \bar{f} \) is linear. By Proposition 2, we have \( \bar{h}(\bar{f}(x)) = h(x) \) on \( U_1 \). We can show that \( \text{Ker}(\bar{f}) = 0 \) as in Vigué [7]. Then \( k \) must be \( n \) and \( f \) is a linear automorphism of \( \mathbb{C}^n \).

Suppose that \((H''_3)\) is satisfied. We may assume that \( \partial f_1(0), \ldots, \partial f_k(0) \) are not 0 and \( \partial f_{k+1}(0), \ldots, \partial f_n(0) \) are 0 for some \( k \), \( 1 \leq k \leq n \). Let

\[ \bar{D} = D \cap \{ z_{k+1} = \ldots = z_n = 0 \} \quad \text{and} \quad \bar{f} = (f_1, \ldots, f_k). \]

Then \( \bar{D} \) is a bounded balanced pseudoconvex domain in \( \mathbb{C}^k \) with Minkowski function \( h = h| \bar{D} \), and \( \bar{f} \) is a holomorphic map from \( D \) to \( \bar{D} \) with \( \bar{f}(0) = 0 \).
We may assume that $\partial f_1(0) \cdot (\sum v_j \partial /\partial z_j), \ldots, \partial f_k(0) \cdot (\sum v_j \partial /\partial z_j)$ do not vanish for any $v \in V$. Then $\hat{f}'(0) v = h(\hat{f}'(0) v)$ is an extreme point of $\overline{D}$ for any $v \in V$. The rest of the argument is the same as above. This completes the proof.

For $p = (p_1, \ldots, p_n)$ with $p_1, \ldots, p_n > 0$, let

$$\mathcal{E}(p) = \{ (z_1, \ldots, z_n) \mid \sum_{j=1}^n |z_j|^{2p_j} < 1 \}.$$ 

Then $\mathcal{E}(p)$ is a bounded balanced pseudoconvex domain which satisfies the condition $(\ast)$ (cf. p. 264 of Jarnicki and Pflug [3]). Let $f$ be a holomorphic map from $\mathcal{E}(p)$ to itself which satisfies the condition of Theorem 3. Then $f$ is a linear automorphism of $\mathbb{C}^n$ by Theorem 3. Moreover, we can show that $f$ is a linear automorphism of $\mathcal{E}(p)$ using the idea of Dini and Primicerio [1].

**Theorem 4.** Let $f$ be a holomorphic map from $\mathcal{E}(p)$ to itself such that $f(0) = 0$. Let $\mathcal{M}$ be a connected complex submanifold of dimension $n - 1$ of an open subset $U$ of $\mathcal{E}(p)$ such that $a + T_0(M)$ does not contain the origin for some $a$ in $\mathcal{M}$. Let $V$ be a connected open subset of $T_0(D)$. If one of the following conditions is satisfied, then $f$ is a linear automorphism of $\mathcal{E}(p)$.

1. $h(f(x)) = h(x)$ on $\mathcal{M}$, where $h$ is the Minkowski function of $\mathcal{E}(p)$.
2. $k_{\mathcal{E}(p)}(f(0), f(x)) = k_{\mathcal{E}(p)}(0, x)$ on $\mathcal{M}$.
3. $\kappa_{\mathcal{E}(p)}(f(0), f'(0)v) = \kappa_{\mathcal{E}(p)}(0, v)$ on $V$.

**Proof.** By Theorem 3 and its proof, we may assume that $f$ is a linear automorphism of $\mathbb{C}^n$ and there exists an open set $U_1$ in $\mathcal{E}(p)$ such that on $U_1$, the functions $z_1, \ldots, z_n, f_1, \ldots, f_n$ do not vanish and $h(f(x)) = h(x)$. Then there exists an open connected set $U_2$ in $\mathbb{C}^n$ such that:

1) $U_2 \cap \partial \mathcal{E}(p) \neq \emptyset$.
2) the mapping $g = (z_1^{p_1}, \ldots, z_n^{p_n})$ is well-defined and 1-1 on $U_2$ and $f(U_2)$.
3) $f(U_2 \cap \partial \mathcal{E}(p)) \subset \partial \mathcal{E}(p)$.

Then the map $F = g \circ f \circ g^{-1}$ is holomorphic and 1-1 on $g(U_2)$ and $F(g(U_2) \cap \partial \mathbb{B}^n) \subset \partial \mathbb{B}^n$. By the proof of Theorem 1.1 and Corollary 1.2 of Dini and Primicerio [1], $f$ is a linear automorphism of $\mathcal{E}(p)$.

**Corollary 1.** Let $f$ be a holomorphic map from $D = \{(z_1, \ldots, z_n) \mid \|z\|_q = \sum_{j=1}^n |z_j|^q < 1 \} \ (q \geq 1)$ to itself such that $f(0) = 0$. Let $\mathcal{M}$ be a connected complex submanifold of dimension $n - 1$ of an open subset $U$ of $D$ such that $a + T_0(M)$ does not contain the origin for some $a$ in $\mathcal{M}$. Let $V$ be a connected open subset of $T_0(D)$. If one of the following conditions is satisfied, then $f$ is a linear automorphism of $D$. 

\[ (H''_1) \|f(x)\|_q = \|x\|_q \text{ on } M. \]
\[ (H''_2) \tilde{k}_D(f(0), f(x)) = \tilde{k}_D(0, x) \text{ on } M. \]
\[ (H''_3) \kappa_D(f(0), f'(0)v) = \kappa_D(0, v) \text{ on } V. \]

**Example 1.** Let \( f(z) = (z_1, \ldots, z_{n-1}, z_n^2). \) Then \( f \) maps \( E(p) \) into itself and \( f(0) = 0. \)

(i) Let \( M = \{ z_n = 0 \}. \) We have \( h(f(z)) = h(z) \) on \( M. \) Since \( f \) is not linear, the condition that \( a + T_a(M) \) does not contain the origin cannot be omitted.

(ii) For \( k \geq 2, \) let \( M_{n-k} = \{ z_{n-k+1} = b, \ldots, z_n = 0 \}, \) where \( b \neq 0. \) The complex dimension of \( M_{n-k} \) is \( n - k, \) and for any \( a \in M_{n-k}, \) \( a + T_a(M) \) does not contain the origin. Since \( h(f(z)) = h(z) \) on \( M_{n-k} \) and \( f \) is not linear, the condition that the complex dimension of \( M \) is \( n - 1 \) cannot be omitted.

**References**


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