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A Schwarz lemma on complex ellipsoids

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Abstract. We give a Schwarz lemma on complex ellipsoids.

1. Introduction. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc in $\mathbb{C}$. The Schwarz lemma in one complex variable is as follows.

Theorem 1. (i) *Let $f : \Delta \to \Delta$ be a holomorphic map such that $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in \Delta$.*

(ii) *If, moreover, there exists $z_0 \in \Delta \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, or if $|f'(0)| = 1$, then there exists a complex number λ of absolute value 1 such that $f(z) = \lambda z$ and f is an automorphism of Δ.*

Let D be the unit ball in $\mathbb{C}^n$ for some norm $\|\cdot\|$, and let $f : D \to D$ be a holomorphic map such that $f(0) = 0$. By the Hahn–Banach theorem, we have $\|f(z)\| \le \|z\|$ for all $z \in D$. As a generalization of part (ii) of the above theorem, Vigué [7] proved the following.

Theorem 2. *Let D be the unit ball in $\mathbb{C}^n$ for some norm $\|\cdot\|$, and let $f : D \to D$ be a holomorphic map such that $f(0) = 0$. Assume that every boundary point of D is a complex extreme point of $\overline{D}$. If one of the following conditions is satisfied, then f is a linear automorphism of $\mathbb{C}^n$.*

- (H_1) *There exists a nonempty open subset U of D such that $\|f(x)\| = \|x\|$ on U.*
- (H_2) *There exists a nonempty open subset U of D such that $c_D(f(0), f(x)) = c_D(0, x)$ on U, where c_D denotes the Carathéodory distance on D.*
- (H_3) *There exists a nonempty open subset V of $T_0(D)$ such that $E_D(f(0), f'(0)v) = E_D(0, v)$ on V, where E_D denotes the infinitesimal Carathéodory metric on D.*

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Moreover, he showed that if there exists a point $a \in U \setminus \{0\}$ such that $f(a) = a$, or if the boundary ∂D of D is a real-analytic submanifold of $\mathbb{C}^n$, then f is a linear automorphism of D. As a corollary, he proved that if D is the unit ball of $\mathbb{C}^n$ for the Euclidean norm on $\mathbb{C}^n$, then f is a linear automorphism of D. But, in the above results, the conditions (H_1) and (H_2) are strong, because a point in Δ is of codimension 1 and an open set in D is of codimension 0. The author [2] announced that Vigué's results hold under the hypothesis that one of the conditions (H_1), (H_2) is satisfied for some local complex submanifold of codimension 1 instead of an open subset.

The aim of the present paper is to consider an analogous result on complex ellipsoids $\mathcal{E}(p)$. However, $\mathcal{E}(p)$ is not convex in general. For a bounded balanced convex domain D, the Minkowski function h of D is a norm on $\mathbb{C}^n$ and D is the unit ball in $\mathbb{C}^n$ with respect to this norm. Also, $c_D = \widetilde{k}_D$ and $E_D = \kappa_D$ in the convex case (Lempert [4], [5], Royden–Wong [6]), where $\widetilde{k}_D$ is the Lempert function and κ_D is the the Kobayashi–Royden pseudometric for D. So we use $h, \widetilde{k}_D$ and κ_D instead of $\|\cdot\|, c_D$ and E_D. First we give a theorem on some bounded balanced pseudoconvex domains which corresponds to Theorem 2. Then we show that if $D = \mathcal{E}(p)$, then f is a linear automorphism of $\mathcal{E}(p)$. We also give an example showing that our hypothesis cannot be weakened.

Some ideas of this paper come from Dini–Primicerio [1] and Vigué [7], [8].

2. Main results. The Lempert function $\widetilde{k}_D$ and the Kobayashi–Royden pseudometric κ_D for a domain D in $\mathbb{C}^n$ are defined as follows:

$$\widetilde{k}_D(x,y) = \inf\{\varrho(\xi,\eta) \mid \xi,\eta \in \Delta,\ \exists\varphi \in H(\Delta, D) \text{ such that } \varphi(\xi) = x,\ \varphi(\eta) = y\},$$

$$\kappa_D(z;X) = \inf\{\gamma(\lambda)|\alpha| \mid \exists\varphi \in H(\Delta, D),\ \exists\lambda \in \Delta \text{ such that } \varphi(\lambda) = z,\ \alpha\varphi'(\lambda) = X\},$$

where ϱ is the Poincaré distance on the unit disc Δ and $\gamma(\lambda) = 1/(1-|\lambda|^2)$.

Let D be a balanced pseudoconvex domain with Minkowski function h in $\mathbb{C}^n$. Then we have (Propositions 3.1.10 and 3.5.3 of Jarnicki and Pflug [3])

$$\widetilde{k}_D(0,x) = \varrho(0,h(x)) \quad \text{for any } x \text{ in } D, \tag{1}$$

$$\kappa_D(0,X) = h(X) \quad \text{for any } X \text{ in } \mathbb{C}^n. \tag{2}$$

Let f be a holomorphic map from D to D such that $f(0) = 0$. By (1) and the distance decreasing property of the Lempert functions, we have

$$\varrho(0,h(z)) = \widetilde{k}_D(0,z) \ge \widetilde{k}_D(0,f(z)) = \varrho(0,h(f(z))).$$

Since $\varrho(0,r)$ is increasing for $0 \le r < 1$, we obtain $h(f(z)) \le h(z)$. This is a generalization of part (i) of the Schwarz lemma to balanced pseudoconvex domains.

A boundary point x of D is said to be an *extreme point* of $\overline{D}$ if there is no non-constant holomorphic mapping $g : \Delta \to \overline{D}$ with $x = g(0)$. For example, C^2-smooth strictly pseudoconvex boundary points are extreme points (p. 257 of Jarnicki and Pflug [3]).

A mapping $\varphi \in H(\Delta, D)$ is said to be a *complex* $\widetilde{k}_D$*-geodesic* for (x, y) if there exist points $\xi, \eta \in \Delta$ such that $\varphi(\xi) = x$, $\varphi(\eta) = y$, and $\widetilde{k}_D(x, y) = \varrho(\xi, \eta)$.

A mapping $\varphi \in H(\Delta, D)$ is said to be a *complex* κ_D*-geodesic* for (z, X) if there exist $\lambda \in \Delta$ and $\alpha \in \mathbb{C}$ such that $\varphi(\lambda) = z$, $\alpha\varphi'(\lambda) = X$, and $\kappa_D(z, X) = \gamma(\lambda)|\alpha|$.

Using (1), (2) and complex $\widetilde{k}_D$-geodesics or κ_D-geodesics, we have the following proposition (cf. Vigué [7], [8], Hamada [2]).

PROPOSITION 1. *Let D_j be bounded balanced pseudoconvex domains with Minkowski functions h_j in $\mathbb{C}^{n_j}$ for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic map such that $f(0) = 0$. Let $f(z) = \sum_{m=1}^{\infty} P_m(z)$ be the development of f in vector-valued homogeneous polynomials P_m in a neighborhood of* 0, *where* $\deg P_m = m$ *for each m. Let $x \in D_1 \setminus \{0\}$. If one of the following conditions is satisfied, then we have $P_m(x) = 0$ for $m \geq 2$.*

- (H_1') *$h_2(f(x)) = h_1(x)$ and $f(x)/h_2(f(x))$ is an extreme point of $\overline{D}_2$.*
- (H_2') *$\widetilde{k}_{D_2}(f(0), f(x)) = \widetilde{k}_{D_1}(0, x)$ and $f(x)/h_2(f(x))$ is an extreme point of $\overline{D}_2$.*
- (H_3') *$\kappa_{D_2}(f(0), f'(0)x) = \kappa_{D_1}(0, x)$ and $f'(0)x/h_2(f'(0)x)$ is an extreme point of $\overline{D}_2$.*

Proof. By (1), the conditions (H_1') and (H_2') are equivalent. Let

$$\varphi(\zeta) = \zeta \frac{x}{h_1(x)}.$$

Then φ is a complex $\widetilde{k}_{D_1}$-geodesic and κ_{D_1}-geodesic for $(0, x)$. Suppose that (H_1') or (H_2') is satisfied. Since

$$\widetilde{k}_{D_2}(f \circ \varphi(0), f \circ \varphi(h_1(x))) = \widetilde{k}_{D_2}(0, f(x)) = \widetilde{k}_{D_1}(0, x) = \varrho(0, h_1(x)),$$

$f \circ \varphi$ is a complex $\widetilde{k}_{D_2}$-geodesic for $(0, f(x))$. By Proposition 8.3.5(a) of Jarnicki and Pflug [3],

$$f \circ \varphi(\zeta) = \zeta \frac{f(x)}{h_2(f(x))}.$$

Since

$$f \circ \varphi(\zeta) = \sum P_m\left(\zeta \frac{x}{h_1(x)}\right) = \sum \left(\frac{\zeta}{h_1(x)}\right)^m P_m(x)$$

in a neighborhood of 0, $P_m(x) = 0$ for $m \geq 2$.

Suppose that (H_3') is satisfied. Since

$$\kappa_{D_2}(0, f'(0)x) = \kappa_{D_1}(0, x) = h_1(x) \quad \text{and} \quad h_1(x)(f \circ \varphi)'(0) = f'(0)x,$$

$f \circ \varphi$ is a complex κ_{D_2}-geodesic for $(0, f'(0)x)$. By Proposition 8.3.5(a) of Jarnicki and Pflug [3],

$$f \circ \varphi(\zeta) = \zeta e^{i\theta} \frac{f'(0)x}{h_2(f'(0)x)}$$

for some $\theta \in \mathbb{R}$. The rest of the argument is the same as above. This completes the proof.

The following proposition is a key for proving our theorem (Hamada [2]).

PROPOSITION 2. *Let U be an open subset of $\mathbb{C}^n$. Let M be a complex submanifold of U of dimension $n-1$. Assume that there exists a point a in M such that $a + T_a(M)$ does not contain the origin. Then there exists a neighborhood U_1 of a in $\mathbb{C}^n$ such that $U_1 \subset \mathbb{C}M = \{tx \mid t \in \mathbb{C},\ x \in M\}$.*

Proof. To prove this proposition, it is enough to prove the following claim.

CLAIM. *For any x in M, let $g(x)$ be the intersection point of $a + T_a(M)$ and the complex line through x and the origin O. Then g is a biholomorphic map from a neighborhood W_M of a in M onto a neighborhood W_T of a in $a + T_a(M)$.*

Assume the claim is proved. Since there exists an open neighborhood U_1 of a in $\mathbb{C}^n$ such that $U_1 \subset \mathbb{C}W_T$, we obtain $U_1 \subset \mathbb{C}M$.

Now we will prove the claim. By an affine coordinate change, we may assume that $a = 0$, $M = \{z_n = \psi(z')\}$ with $\psi(0) = 0$, $d\psi(0) = 0$, where $(z', z_n) \in \mathbb{C}^n$. Then $(z', \psi(z'))$ gives a local parametrization of M at a, $a + T_a(M) = \{z_n = 0\}$ and $O = (b_1, \ldots, b_n)$ with $b_n \neq 0$. Let $g(z', \psi(z')) = (g_1(z'), \ldots, g_{n-1}(z'), 0)$. Since

$$g_i(z', \psi(z')) = b_i + \frac{b_n}{b_n - \psi(z')}(z_i - b_i)$$

for sufficiently small z', we have

$$\frac{\partial g_i}{\partial z_j}(0) = \delta_{ij} \quad (1 \le i, j \le n-1).$$

Therefore g is biholomorphic in a neighborhood W_M of a. This completes the proof.

From now on, we assume that D is a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ which satisfies the following condition:

($*$) For any $1 \le j_1 < \ldots < j_k \le n$ $(0 \le k \le n-1)$, let

$$\widetilde{D} = D \cap \{z_{j_1} = \ldots = z_{j_k} = 0\}$$

be a domain in $\mathbb{C}^{n-k}$. Then every point of $\partial\widetilde{D}\cap(\mathbb{C}^*)^{n-k}$ is an extreme point of $\overline{\widetilde{D}}$.

By the above two propositions, we have the following theorem.

THEOREM 3. *Let D be a bounded balanced pseudoconvex domain with Minkowski function h in $\mathbb{C}^n$ which satisfies the condition $(*)$, and let $f : D \to D$ be a holomorphic map such that $f(0) = 0$. Let M be a connected complex submanifold of dimension $n-1$ of an open subset U of D such that $a+T_a(M)$ does not contain the origin for some a in M. Let V be a connected open subset of $T_0(D)$. If one of the following conditions is satisfied, then f is a linear automorphism of $\mathbb{C}^n$.*

(H_1'') $h(f(x)) = h(x)$ *on* M.
(H_2'') $\widetilde{k}_D(f(0), f(x)) = \widetilde{k}_D(0,x)$ *on* M.
(H_3'') $\kappa_D(f(0), f'(0)v) = \kappa_D(0,v)$ *on* V.

Proof. Suppose that (H_1'') or (H_2'') is satisfied. We may assume that for any $a \in M$, $a + T_a(M)$ does not contain the origin, the functions $f_1, \ldots, f_k$ do not vanish on M and the functions $f_{k+1}, \ldots, f_n$ are identically 0 on M for some k, $1 \le k \le n$. Let

$$\widetilde{D} = D \cap \{z_{k+1} = \ldots = z_n = 0\} \quad \text{and} \quad \widetilde{f} = (f_1, \ldots, f_k).$$

Then $\widetilde{D}$ is a bounded balanced pseudoconvex domain in $\mathbb{C}^k$ with Minkowski function $\widetilde{h} = h|\widetilde{D}$, and $\widetilde{f}$ is a holomorphic map from D to $\widetilde{D}$ with $\widetilde{f}(0) = 0$. Since the functions $f_1, \ldots, f_k$ do not vanish on M, $\widetilde{f}(x)/\widetilde{h}(\widetilde{f}(x))$ is an extreme point of $\overline{\widetilde{D}}$ for any $x \in M$. Let

$$\widetilde{f}(z) = \sum_{m=1}^{\infty} P_m(z)$$

be the development of $\widetilde{f}$ in vector-valued homogeneous polynomials P_m in a neighborhood of 0, where $\deg P_m = m$ for each m. Since $\widetilde{h}(\widetilde{f}(x)) = h(f(x)) = h(x)$ on M, we have $P_m(x) = 0$ on a nonempty open subset U_1 of D for $m \ge 2$ by Propositions 1 and 2. By the analytic continuation theorem, P_m is identically 0 for $m \ge 2$. Therefore $\widetilde{f}$ is linear. By Proposition 2, we have $\widetilde{h}(\widetilde{f}(x)) = h(x)$ on U_1. We can show that $\mathrm{Ker}(\widetilde{f}) = 0$ as in Vigué [7]. Then k must be n and f is a linear automorphism of $\mathbb{C}^n$.

Suppose that (H_3'') is satisfied. We may assume that $\partial f_1(0), \ldots, \partial f_k(0)$ are not 0 and $\partial f_{k+1}(0), \ldots, \partial f_n(0)$ are 0 for some k, $1 \le k \le n$. Let

$$\widetilde{D} = D \cap \{z_{k+1} = \ldots = z_n = 0\} \quad \text{and} \quad \widetilde{f} = (f_1, \ldots, f_k).$$

Then $\widetilde{D}$ is a bounded balanced pseudoconvex domain in $\mathbb{C}^k$ with Minkowski function $\widetilde{h} = h|\widetilde{D}$, and $\widetilde{f}$ is a holomorphic map from D to $\widetilde{D}$ with $\widetilde{f}(0) = 0$.

We may assume that $\partial f_1(0)\cdot(\sum v_j\partial/\partial z_j),\ldots,\partial f_k(0)\cdot(\sum v_j\partial/\partial z_j)$ do not vanish for any $v\in V$. Then $\widetilde{f}'(0)v/\widetilde{h}(\widetilde{f}'(0)v)$ is an extreme point of $\overline{\widetilde{D}}$ for any $v\in V$. The rest of the argument is the same as above. This completes the proof.

For $p=(p_1,\ldots,p_n)$ with $p_1,\ldots,p_n>0$, let

$$\mathcal{E}(p)=\Big\{(z_1,\ldots,z_n)\ \Big|\ \sum_{j=1}^{n}|z_j|^{2p_j}<1\Big\}.$$

Then $\mathcal{E}(p)$ is a bounded balanced pseudoconvex domain which satisfies the condition $(*)$ (cf. p. 264 of Jarnicki and Pflug [3]). Let f be a holomorphic map from $\mathcal{E}(p)$ to itself which satisfies the condition of Theorem 3. Then f is a linear automorphism of $\mathbb{C}^n$ by Theorem 3. Moreover, we can show that f is a linear automorphism of $\mathcal{E}(p)$ using the idea of Dini and Primicerio [1].

THEOREM 4. *Let f be a holomorphic map from $\mathcal{E}(p)$ to itself such that $f(0)=0$. Let M be a connected complex submanifold of dimension $n-1$ of an open subset U of $\mathcal{E}(p)$ such that $a+T_a(M)$ does not contain the origin for some a in M. Let V be a connected open subset of $T_0(D)$. If one of the following conditions is satisfied, then f is a linear automorphism of $\mathcal{E}(p)$.*

(H_1'') *$h(f(x))=h(x)$ on M, where h is the Minkowski function of $\mathcal{E}(p)$.*
(H_2'') *$\widetilde{k}_{\mathcal{E}(p)}(f(0),f(x))=\widetilde{k}_{\mathcal{E}(p)}(0,x)$ on M.*
(H_3'') *$\kappa_{\mathcal{E}(p)}(f(0),f'(0)v)=\kappa_{\mathcal{E}(p)}(0,v)$ on V.*

Proof. By Theorem 3 and its proof, we may assume that f is a linear automorphism of $\mathbb{C}^n$ and there exists an open set U_1 in $\mathcal{E}(p)$ such that on U_1, the functions $z_1,\ldots,z_n$, $f_1,\ldots,f_n$ do not vanish and $h(f(x))=h(x)$. Then there exists an open connected set U_2 in $\mathbb{C}^n$ such that:

1) $U_2\cap\partial\mathcal{E}(p)\neq\emptyset$,
2) the mapping $g=(z_1^{p_1},\ldots,z_n^{p_n})$ is well-defined and 1-1 on U_2 and $f(U_2)$,
3) $f(U_2\cap\partial\mathcal{E}(p))\subset\partial\mathcal{E}(p)$.

Then the map $F=g\circ f\circ g^{-1}$ is holomorphic and 1-1 on $g(U_2)$ and $F(g(U_2)\cap\partial\mathbb{B}^n)\subset\partial\mathbb{B}^n$. By the proof of Theorem 1.1 and Corollary 1.2 of Dini and Primicerio [1], f is a linear automorphism of $\mathcal{E}(p)$.

COROLLARY 1. *Let f be a holomorphic map from $D=\{(z_1,\ldots,z_n)\mid \|z\|_q^q=\sum_{j=1}^n|z_j|^q<1\}$ $(q\geq 1)$ to itself such that $f(0)=0$. Let M be a connected complex submanifold of dimension $n-1$ of an open subset U of D such that $a+T_a(M)$ does not contain the origin for some a in M. Let V be a connected open subset of $T_0(D)$. If one of the following conditions is satisfied, then f is a linear automorphism of D.*

(H_1'') $\|f(x)\|_q = \|x\|_q$ *on* M.
(H_2'') $\widetilde{k}_D(f(0), f(x)) = \widetilde{k}_D(0, x)$ *on* M.
(H_3'') $\kappa_D(f(0), f'(0)v) = \kappa_D(0, v)$ *on* V.

EXAMPLE 1. Let $f(z) = (z_1, \ldots, z_{n-1}, z_n^2)$. Then f maps $\mathcal{E}(p)$ into itself and $f(0) = 0$.

(i) Let $M = \{z_n = 0\}$. We have $h(f(z)) = h(z)$ on M. Since f is not linear, the condition that $a + T_a(M)$ does not contain the origin cannot be omitted.

(ii) For $k \geq 2$, let $M_{n-k} = \{z_{n-k+1} = b,\ z_{n-k+2} = \ldots = z_n = 0\}$, where $b \neq 0$. The complex dimension of M_{n-k} is $n - k$, and for any $a \in M_{n-k}$, $a + T_a(M)$ does not contain the origin. Since $h(f(z)) = h(z)$ on M_{n-k} and f is not linear, the condition that the complex dimension of M is $n-1$ cannot be omitted.

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