On definitions of the pluricomplex Green function

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Abstract. We give several definitions of the pluricomplex Green function and show their equivalence.

1. Introduction. We denote by $E$ the unit disc in $\mathbb{C}$. Let $D$ be a domain in $\mathbb{C}^n$. Put

$$g_D(a, z) := \sup \{ u(z) : u \in \text{PSH}(D), u < 0, \exists M, r > 0 : u(w) \leq M + \log \|w - a\|, w \in B(a, r) \subset D\}, \quad a, z \in D,$$

where $\text{PSH}(D)$ denotes the set of all plurisubharmonic functions on $D$ and $B(a, r)$ denotes the ball with center at $a$ and radius $r$. The function $g_D$ has been introduced by M. Klimek (cf. [K]) and is called the pluricomplex Green function.

In this paper we give several equivalent definitions of the pluricomplex Green function.

Following E. Poletsky (cf. [P-S], [P1], [P2]) for a domain $D \subset \mathbb{C}^n$ and $a, z \in D$, $a \neq z$, we define

$$g_D^1(a, z) := \inf \left\{ \sum_{\lambda \in \varphi^{-1}(a)} \text{ord}_\lambda(\varphi - a) \log |\lambda| : \varphi \in O(E, D), \ a \in \varphi(E), \ \varphi(0) = z \right\},$$

$$g_D^2(a, z) := \inf \left\{ \sum_{\lambda \in \varphi^{-1}(a)} \text{ord}_\lambda(\varphi - a) \log |\lambda| : \varphi \in O(E, D), \ a \in \varphi(E), \ \varphi(0) = z \right\},$$

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\[ g^3_D(a,z) := \inf \left\{ \sum_{\lambda \in \varphi^{-1}(a)} \log |\lambda| : \varphi \in \mathcal{O}(E,D), \ a \in \varphi(E), \ \varphi(0) = z \right\}, \]
\[ g^4_D(a,z) := \inf \left\{ \sum_{\lambda \in \varphi^{-1}(a)} \log |\lambda| : \varphi \in \mathcal{O}(E,D), \ a \in \varphi(E), \ \varphi(0) = z \right\}, \]

where \( \mathcal{O}(E,D) \) denotes the set of all holomorphic mappings \( E \to D \) and \( \text{ord}_\lambda(\varphi-a) \) denotes the order of vanishing of \( \varphi-a \) at \( \lambda \). Note that in the whole paper for any holomorphic mapping \( \varphi : E \to D \) by \( \varphi^{-1}(a) \) we mean \( \varphi^{-1}(a) \cap E \) and it is always a finite set provided \( \varphi \) is nonconstant.

We put \( g^1_D(a,a) = g^2_D(a,a) = g^3_D(a,a) = g^4_D(a,a) = -\infty \).

**Remarks.**

1. For any \( z \in D \setminus \{a\} \) there exists \( \varphi \in \mathcal{O}(E,D) \) such that \( \varphi(0) = z \) and \( a \in \varphi(E) \) (cf. [J-P], Remark 3.1.1). So, the above functions are well defined.

2. Note that \( g^1_D \leq g^2_D \leq g^3_D \leq g^4_D \), and \( g^1_D \leq g^3_D \). Define

\[ k_D(a,z) := \inf \{ \log \sigma : \exists \varphi \in \mathcal{O}(E,D) : \varphi(0) = a, \ \varphi(\sigma) = z, \ \sigma > 0 \}, \]
\[ g^5_D(a,z) := \inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} k_D(a,\varphi(e^{i\theta})) \, d\theta : \varphi \in \mathcal{O}(E,D), \ \varphi(0) = z \right\}, \quad a,z \in D. \]

Note that \( g^5_D(a,\cdot) \) is the envelope of \( k_D(a,\cdot) \) in the sense of Poletsky (see Theorem 11).

The main result of the paper is the following

**Theorem 1.** Let \( D \) be a domain in \( \mathbb{C}^n \). Then

\[ g_D = g^1_D = g^2_D = g^3_D = g^4_D = g^5_D. \]

**Remarks.** The most difficult problem in Theorem 1 is the equality \( g_D = g^2_D \). It was proved in [P1]. We present a much simpler and complete proof. The equality \( g_D = g^2_D \) was stated in [P2].

**2. Definitions and auxiliary results.** Let \( D \) be a domain in \( \mathbb{C}^n \) and let \( \varphi : \overline{E} \to D \) be a holomorphic mapping. For a point \( a \in D \) we define

\[ u_{(\varphi,a)}(\lambda) := \sum_{\zeta \in \varphi^{-1}(a)} \text{ord}_\zeta(\varphi-a) \log \left| \frac{\lambda - \zeta}{1 - \zeta \lambda} \right|, \quad \lambda \in E, \]
\[ H(\varphi, a) := u_{(\varphi,a)}(0). \]

For convenience we put \( \sum_{\emptyset} = 0 \) in the whole paper. For a constant mapping
\( \varphi \equiv a \) we put \( u(\varphi, a) \equiv -\infty \). In this notation we have
\[
g^D_D(a, z) = \inf \{ H(\varphi, a) : \varphi \in \mathcal{O}(E, D), \ \varphi(0) = z \}, \quad a, z \in D.
\]
For the functional \( H \) we have the following

**Lemma 2.** Let \( \varphi : \bar{E} \to D \) and \( h : \bar{E} \to \bar{E} \) be holomorphic mappings. Then for any \( a \in D \) such that \( \varphi \not\equiv a \) we have
\[
H(\varphi \circ h, a) = \int E \log |\xi| \Delta(u(\varphi, a) \circ h(\xi)).
\]

**Proof.** Note that if \( \varphi(h(0)) = a \) then
\[
H(\varphi \circ h, a) = \int E \log |\xi| \Delta(u(\varphi, a) \circ h(\xi)) = -\infty.
\]
So, we may assume that \( \varphi(h(0)) \not\equiv a \). Put
\[
\psi_j(\lambda) := \frac{h(\lambda) - \lambda_j}{1 - \lambda_j h(\lambda)}, \quad \text{where } \lambda_j \in \varphi^{-1}(a).
\]
Note that \( \psi_j \in \mathcal{O}(E) \) and \( \psi_j(0) \neq 0 \). Hence using the Jensen formula (see [R], Theorem 15.18) we have
\[
\log |\psi_j(0)| = \sum_{m=1}^N \log |\alpha_m| + \frac{1}{2\pi} \int_0^{2\pi} \log |\psi_j(e^{i\theta})| d\theta,
\]
where \( \alpha_1, \ldots, \alpha_N \) are the zeros of \( \psi_j \) with multiplicities. But on the other hand by the Riesz representation we have
\[
\log |\psi_j(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |\psi_j(e^{i\theta})| d\theta + \int E \log |\xi| \Delta(\log |\psi_j(\xi)|).
\]
Hence,
\[
\sum_{m=1}^N \log |\alpha_m| = \int E \log |\xi| \Delta(\log |\psi_j(\xi)|).
\]
From this we derive the desired result. \( \blacksquare \)

**Lemma 3 (cf. [P1], Lemma 3.2).** Let \( v \) be a plurisubharmonic function in some neighborhood of \( \bar{E}^2 \) such that \( v(0, 0) \neq -\infty \) and \( v(0, e^{i\theta}) \neq -\infty, \ \theta \in [0, 2\pi) \). Then
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \int E \log |\xi| \Delta_\zeta(v(e^{i\alpha} \zeta, \zeta)) \right) d\alpha \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \int E \log |\xi| \Delta_\zeta(v(\zeta, e^{i\theta})) \right) d\theta.
\]
Therefore, there exists \( \alpha_0 \in [0, 2\pi) \) such that
\[
\int E \log |\xi| \Delta_\zeta(v(e^{i\alpha_0} \zeta, \zeta)) \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \int E \log |\xi| \Delta_\zeta(v(\zeta, e^{i\theta})) \right) d\theta.
\]
\[ v(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} v(0, e^{i\theta}) d\theta + \iint_E \log |\zeta| \Delta_\zeta v(0, \zeta) \]
\[ = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} v(e^{i\alpha}, e^{i\theta}) d\alpha d\theta \]
\[ + \frac{1}{2\pi} \int d\theta \left( \iint_E \log |\zeta| \Delta_\zeta v(\zeta, e^{i\theta}) + \iint_E \log |\zeta| \Delta_\zeta v(0, \zeta) \right). \]

Again by the Riesz representation for any fixed \( \alpha \in [0, 2\pi) \) we have
\[ v(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i(\alpha+\theta)}, e^{i\theta}) d\theta + \iint_E \log |\zeta| \Delta_\zeta v(e^{i\alpha} \zeta, \zeta). \]

Hence, integrating (1) in \( \alpha \in [0, 2\pi) \) we obtain
\[ v(0, 0) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} v(e^{i(\alpha+\theta)}, e^{i\theta}) d\theta d\alpha + \frac{1}{2\pi} \int_0^{2\pi} \left( \iint_E \log |\zeta| \Delta_\zeta v(e^{i\alpha} \zeta, \zeta) \right) d\alpha. \]

So,
\[ \frac{1}{2\pi} \int_0^{2\pi} \left[ \iint_E \log |\zeta| \Delta_\zeta v(e^{i\alpha} \zeta, \zeta) \right] d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \int d\theta \left( \iint_E \log |\zeta| \Delta_\zeta v(\zeta, e^{i\theta}) \right) \]
\[ + \iint_E \log |\zeta| \Delta_\zeta v(0, \zeta) \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \int d\theta \left( \iint_E \log |\zeta| \Delta_\zeta v(\zeta, e^{i\theta}) \right). \]

As a corollary we have the following

**Lemma 4.** Let \( \varphi : \overline{E} \to D \) and \( h : \overline{E}^2 \to \overline{E} \) be holomorphic mappings. Then for any \( a \in D \) such that \( a \notin \varphi(h(\{0\} \times \partial E)) \) and \( \varphi(h(0, 0)) \neq a \) there exists \( \alpha_0 \in [0, 2\pi) \) with
\[ H(\varphi \circ h(e^{i\alpha_0} \zeta, \zeta), a) \leq \frac{1}{2\pi} \int_0^{2\pi} H(\varphi \circ h(\zeta, e^{i\theta}), a) d\theta. \]

**Proof.** Take \( v := u_{(\varphi, a)} \circ h \). Then the result follows from Lemmas 2 and 3. \( \blacksquare \)

Recall that a holomorphic function \( \phi : E \to E \) is called **inner** if \( |\phi^*(\zeta)| = \lim_{r \to 1} |\phi(r\zeta)| = 1 \) for almost all \( \zeta \in \partial E \). Any Blaschke product is an inner function. A simple example of an inner function but not a Blaschke product
is the function \(e(\lambda, c) := e^{c(\lambda^{-1})/(\lambda+1)}, c > 0\). It plays an important role in our considerations. Put

\[l_k(\lambda, c) = \frac{\lambda + e^{-c/k}}{1 + e^{-c/k} \lambda}, \quad \lambda \in E, \ c > 0, \ k \in \mathbb{N}.
\]

We have

**Lemma 5.** (a) For fixed \(c > 0\) and \(\tau \in E \setminus \{0\}\) the function

\[\phi(\lambda) = \frac{e(\lambda, c) - \tau}{1 - \tau e(\lambda, c)}
\]

is a Blaschke product.

(b) For fixed \(c > 0\) we have \(l_k(\lambda, c) \to 1\) and \(l_k(\lambda, c) \to e(\lambda, c)\) locally uniformly on \(E\) as \(k \to \infty\).

**Proof.** (a) Note that \(\phi\) is an inner function. By Theorem 2 in Chapter III of [N], any inner function which has no zero radial limits is a Blaschke product. By simple calculations we see that \(\phi\) has no zero radial limits.

(b) It is enough to note that

\[l_k(\lambda, c) = 1 + \frac{\lambda - 1}{1 + e^{-c/k} \lambda}.
\]

Recall the following approximation result:

**Lemma 6.** Let \(F \in C(V \times \partial E)\) and \(F(\cdot, \zeta) \in O(V), \ \zeta \in \partial E, \) where \(V\) is a domain in \(\mathbb{C}^m\). For \(\nu = 1, 2, \ldots\) put

\[F_{\nu}(\xi, \zeta) := \frac{1}{2\pi \nu} \sum_{j=0}^{\nu-1} \sum_{k=-j}^{j} \left( \int_0^{2\pi} F(\xi, e^{i\theta}) K_{\nu}(t-\theta) d\theta \right) \zeta^k.
\]

Then:

(1) \(F_{\nu}\) are holomorphic w.r.t. \(\xi \in V\) and rational w.r.t. \(\zeta\) with pole of order \(\leq \nu - 1\) at \(\zeta = 0\);

(2) \(\{F_{\nu}\}\) converges locally uniformly to \(F\) on \(V \times \partial E\);

(3) if \(F(0, \zeta) \equiv 0\), then \(F_{\nu}(0, \zeta) \equiv 0, \ \zeta \in \partial E\).

**Proof.** It is enough to prove (2), because (1) and (3) are evident. Put

\[K_{\nu}(x) := \frac{1}{\nu} \left( \frac{\sin \frac{\nu}{2} x}{\sin \frac{x}{2}} \right)^2.
\]

Then (see [H], Chapter II) \(\int_0^{2\pi} K_{\nu}(\theta) d\theta = 1\) and

\[F_{\nu}(\xi, e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} F(\xi, e^{i\theta}) K_{\nu}(t-\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} F(\xi, e^{i(t-\theta)}) K_{\nu}(\theta) d\theta.
\]
For $\delta > 0$ we have
\[
F_\nu(\xi, e^{it}) - F(\xi, e^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it}))K_\nu(\theta) d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\delta}^{\delta} (F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it}))K_\nu(\theta) d\theta
\]
\[
+ \frac{1}{2\pi} \int_{|\theta| \geq \delta} (F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it}))K_\nu(\theta) d\theta.
\]
Suppose that $K = L \times \partial E$, where $L \subseteq V$. Then
\[
|F_\nu(\xi, e^{it}) - F(\xi, e^{it})| \leq \sup_{-\delta < \theta < \delta} |F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it})| + 2\|F\|_K \sup_{|\theta| \geq \delta} K_\nu(\theta),
\]
where $\|F\|_K := \sup_{(\xi, \zeta) \in K} |F(\xi, \zeta)|$. Recall that $\lim_{\nu \to \infty} \sup_{|\theta| \geq \delta} K_\nu(\theta) = 0$. Since $F$ is a continuous mapping, we conclude the proof. \]

3. Proof of Theorem 1. We will prove Theorem 1 in several lemmas. We prove consecutively that $g_D^1 = g_D^2 = g_D^3 = g_D^4$ (Lemma 7), $g_D^5 \geq g_D^2$ (Lemma 9), $g_D = g_D^0$ (Lemma 10), and finally, $g_D \leq g_D^1$ (Lemma 12). In this way we will have proved Theorem 1. In the whole section we assume that the domain $D$ and points $a, z \in D$ are fixed. Note that if $a = z$ then the assertion of Theorem 1 is evident, because all the functions are equal to $-\infty$. So, we may assume that $a \neq z$.

**Lemma 7.** $g_D^1(a, z) = g_D^2(a, z) = g_D^3(a, z) = g_D^4(a, z)$.

**Proof.** It is enough to prove that

1. $g_D^1(a, z) = g_D^3(a, z)$,
2. $g_D^2(a, z) = g_D^4(a, z)$,
3. $g_D^2(a, z) = g_D^0(a, z)$.

(1)-(2) We know that $g_D^1(a, z) \leq g_D^2(a, z)$ (resp. $g_D^3(a, z) \leq g_D^4(a, z)$). Fix $A > g_D^1(a, z)$ (resp. $A > g_D^2(a, z)$).

There exists a holomorphic mapping $\varphi : E \to D$ such that $\varphi(0) = z$, $a \in \varphi(E)$, and
\[
\sum_{\lambda \in \varphi^{-1}(a)} \text{ord}_\lambda(\varphi - a) \log |\lambda| < A \quad \text{(resp. } \sum_{\lambda \in \varphi^{-1}(a)} \log |\lambda| < A).\]

Let $\varphi^{-1}(a) = \{\lambda_j : j = 1, 2, \ldots\}$, where $\lambda_j$’s are counted with multiplicities (resp. without multiplicities). We may assume that $|\lambda_1| \leq |\lambda_2| \leq \ldots$ There exists $N > 0$ such that $\sum_{j=1}^{N} \log |\lambda_j| < A$. Let $\tilde{\varphi}(\lambda) = \varphi(R\lambda)$, where
\[ R \in (|\lambda_N|, 1). \] Note that \( \tilde{\varphi} \in \mathcal{O}(E, D) \) and \( \tilde{\varphi}(0) = z \). Then we have
\[
\sum_{\lambda \in \tilde{\varphi}^{-1}(a)} \text{ord}_\lambda(\tilde{\varphi} - a) \log |\lambda| \leq \sum_{j=1}^{N} (\log |\lambda_j| - \log R)
\]
(resp. \[ \sum_{\lambda \in \tilde{\varphi}^{-1}(a)} \log |\lambda| \leq \sum_{j=1}^{N} (\log |\lambda_j| - \log R) \]).

So, if \( R \) is close enough to 1 then
\[
g_D^2(a, z) \leq \sum_{\lambda \in \tilde{\varphi}^{-1}(a)} \text{ord}_\lambda(\tilde{\varphi} - a) \log |\lambda| < A
\]
(resp. \[ g_D^4(a, z) \leq \sum_{\lambda \in \tilde{\varphi}^{-1}(a)} \log |\lambda| < A \]).

Hence, \( g_D^2(a, z) \leq g_D^1(a, z) \) (resp. \[ g_D^4(a, z) \leq g_D^3(a, z) \]).

(3) Let \( \varphi : E \to D \) be a holomorphic mapping such that \( \varphi(0) = z \neq a \) and \( a \in \varphi(E) \). Suppose that \( \varphi(\mu) = a \) and \( \text{ord}_\mu(\varphi - a) = m \). Note that \( \mu \neq 0 \). Let
\[
\psi(\lambda) := \frac{\varphi(\lambda) - a}{(\lambda - \mu)^m} (\lambda - \mu_1) \ldots (\lambda - \mu_m) + a, \quad \lambda \in E,
\]
where \( \mu_1, \ldots, \mu_m \) are pairwise different, \( \mu_1 \ldots \mu_m = \mu^m \), and \( \mu_1, \ldots, \mu_m \) are very close to \( \mu \) \(^{(1)}\). Note that if \( \mu_1, \ldots, \mu_m \) are close enough to \( \mu \) then \( \psi \in \mathcal{O}(E, D) \) and \( \psi(0) = \varphi(0) = z \). Moreover, \( \psi(\lambda_0) = a \) iff \( \varphi(\lambda_0) = a \) and \( \lambda_0 \neq \mu \), or \( \lambda_0 \in \{\mu_1, \ldots, \mu_m\} \), and
\[
\sum_{\lambda \in \varphi^{-1}(a)} \text{ord}_\lambda(\varphi - a) \log |\lambda| = \sum_{\lambda \in \psi^{-1}(a)} \text{ord}_\lambda(\psi - a) \log |\lambda| + \sum_{j=1}^{m} \log |\mu_j|.
\]
Note that the multiplicities of \( \psi \) at \( \mu_j \), \( j = 1, \ldots, m \), are equal to 1. Applying this technique \( N \) times, where \( N \) is the number of zeros of \( \varphi - a \) in \( E \), we obtain the result.

The following result is basic for the proof of Theorem 1.

**Lemma 8.** Let \( \Phi : E \to D \) be a holomorphic mapping such that \( \Phi(0) = z \) and \( a \notin \Phi(\partial E) \). Then
\[
(2) \quad \frac{1}{2\pi} \int_{0}^{2\pi} k_D(a, \Phi(e^{i\theta})) \, d\theta \geq g_D^2(a, z).
\]

\(^{(1)}\) For instance, if \( \mu = re^{i\theta} \) then let \( \mu_j = re^{i\theta_j} \), \( j = 1, \ldots, m \), where \( \theta_1, \ldots, \theta_m \) are pairwise different, close to \( \theta \), and such that \( \theta_1 + \ldots + \theta_m = m\theta \).
Remark. From the definitions we see that $k_D(a, w) \geq g^2_D(a, w)$, $w \in D$. So, a priori (2) states less than the subaverage property of the function $g^2_D(a, \cdot)$. But it turns out that (2) is sufficient to show that $g^2_D(a, \cdot)$ is a plurisubharmonic function, hence has the subaverage property. It is worth noting that we assume that $-\infty$ is a plurisubharmonic function.

Before we present the proof of Lemma 8 note the following immediate corollary.

**Lemma 9.** $g^5_D(a, z) \geq g^2_D(a, z)$.

**Proof of Lemma 8.** Take any $A \in \mathbb{R}$ such that

$$
\frac{1}{2\pi} \int_0^{2\pi} k_D(a, \Phi(e^{i\theta})) \, d\theta < A.
$$

It is sufficient to show that $g^2_D(a, z) \leq A$. Note that $k_D(a, \Phi(\cdot))$ is an upper semicontinuous function in $E$ (see the proof of Lemma 10). Hence, we can find a continuous function $q : \partial E \to \mathbb{R}$ such that $k_D(a, \Phi(\xi)) < q(\xi)$, $\xi \in \partial E$, and

$$
\frac{1}{2\pi} \int_0^{2\pi} q(e^{i\theta}) \, d\theta < A.
$$

For any $\xi \in \partial E$ there exist $\varphi_\xi \in \mathcal{O}(\overline{E}, D)$ and $\sigma_\xi \in (0, 1)$ such that $\varphi_\xi(0) = \Phi(\xi)$, $\varphi_\xi(\sigma_\xi) = a$, and

$$
\log \sigma_\xi < q(\xi).
$$

Note that for any $\xi \in \partial E$ there exists $t(\xi) > 0$ such that for any $\zeta \in B(\xi, t(\xi))$ we may define a mapping $\varphi_{\xi, \zeta} \in \mathcal{O}(\overline{E}, D)$ as follows:

$$
\varphi_{\xi, \zeta}(\lambda) := \varphi_\xi(\lambda) + (\Phi(\zeta) - \Phi(\xi))(1 - \lambda / \sigma_\xi), \quad \lambda \in \overline{E}.
$$

Observe that $\varphi_{\xi, \zeta}(0) = \Phi(\zeta)$ and $\varphi_{\xi, \zeta}(\sigma_\xi) = \varphi_\xi(\sigma_\xi) = a$. Taking smaller $t(\xi) > 0$ if necessary we have

$$
\log \sigma_\xi < q(\zeta), \quad \zeta \in B(\xi, t(\xi)),
$$

and $\varphi_{\xi, \zeta}(\overline{E}) \in D$ for any $\zeta \in \partial E \cap B(\xi, t(\xi))$. Taking even smaller $t(\xi)$, we may choose $\xi_1, \ldots, \xi_l$ such that $\partial E \subset V_{\xi_1} \cup \ldots \cup V_{\xi_l}$ and $V_{\xi_j} \cap V_{\xi_l} = \emptyset$ if $1 < |k - j| < l - 1$, $k, j = 1, \ldots, l$, where $V_{\xi_j} := B(\xi_j, t(\xi))$. We put $\delta := \min_{j=1, \ldots, l} \sigma_{\xi_j}$ and $C := ||q||$.

Fix $\varepsilon > 0$. Note that there exists $r_1 > 1$ such that $\Phi, \varphi_{\xi_j, \zeta} \in \mathcal{O}(r_1 E, D)$ for $\zeta \in V_{\xi_j}, \, j = 1, \ldots, l$. We may assume that $\log r_1 < \varepsilon$. Take $0 < t'(\xi_j) < t(\xi_j), \, j = 1, \ldots, l$, such that for $I_j := B(\xi_j, t'(\xi_j))$ we have $I_j \cap I_k = \emptyset$ for $j \neq k$ and $m(\bigcup_{j=1}^l I_j) > 2\pi - \varepsilon$, where $m$ denotes the Lebesgue measure on $\partial E$. Take a closed subset $\Gamma \subset \bigcup I_j$ and a continuous function $\tau : \partial E \to [0, 1]$ such that $m(\Gamma) > 2\pi - \varepsilon$, $\tau = 1$ on $\Gamma$, and $\tau = 0$ outside $\bigcup I_j$. 


For $\zeta \in \partial E$ put
\[
\sigma(\zeta) := \begin{cases} \frac{\sigma_{\xi_j}}{r_1} & \text{if } \sigma_{\xi_j}/r_1 < \tau(\zeta) \text{ and } \zeta \in I_j, \\ r_1 & \text{otherwise.} \end{cases}
\]
Note that $\sigma$ is a continuous function on $\partial E$ and if $\sigma(\zeta) < r_1$ then $\tau(\zeta)\sigma(\zeta) = \sigma_{\xi_j}$.

For $\lambda \in r_1E$ and $\zeta \in \partial E$ we put
\[
\psi(\lambda, \zeta) := \begin{cases} \varphi_{\xi_j, \zeta}(\tau(\zeta)\lambda) & \text{if } \zeta \in I_j, \\ \Phi(\zeta) & \text{if } \zeta \notin \bigcup_{j=1}^{l} I_j. \end{cases}
\]
Note that $\psi(\lambda, \zeta)$ is holomorphic with respect to $\lambda$ and continuous with respect to $(\lambda, \zeta)$. Moreover, $\psi(\cdot, \zeta) \in \mathcal{O}(r_1E, D)$ and $\psi(0, \zeta) = \Phi(\zeta)$ when $\zeta \in \partial E$.

(3) \[
\psi(\sigma(\zeta), \zeta) = a \quad \text{if } \sigma(\zeta) < r_1,
\]
and
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log \sigma(e^{i\theta}) d\theta < \frac{1}{2\pi} \int_{\Gamma} \log \sigma(e^{i\theta}) d\theta + \log r_1
\]
\[
< \frac{1}{2\pi} \int_{0}^{2\pi} q(e^{i\theta}) d\theta + \varepsilon - \frac{1}{2\pi} \int_{[0,2\pi]\setminus\Gamma} q(e^{i\theta}) d\theta
\]
\[
< A + \varepsilon + C\varepsilon.
\]

Now we want to approximate $\psi$ and $\sigma$ by holomorphic (actually meromorphic) mappings. But applying Lemma 6 to $\psi$ and $\sigma$ we may loose the important relation (3). So, we “separate” in $\psi$ the part related to (3). Namely, we have
\[
\psi(\lambda, \zeta) = a \frac{\lambda}{\sigma(\zeta)} + \left( 1 - \frac{\lambda}{\sigma(\zeta)} \right) \Phi(\zeta) + (\lambda - \sigma(\zeta))\psi_0(\lambda, \zeta),
\]
where
\[
\psi_0(\lambda, \zeta) := \frac{\psi(\lambda, \zeta) - a \frac{\lambda}{\sigma(\zeta)}}{\lambda - \sigma(\zeta)} + \Phi(\zeta) \frac{\sigma(\zeta)}{\lambda - \sigma(\zeta)}.
\]
Note that $\psi_0(\lambda, \zeta)$ extends as a continuous mapping in $r_1E \times \partial E$ and holomorphic with respect to $\lambda$.

We denote by $\sigma_{\nu}(\zeta)$ and $\psi_{0\nu}(\lambda, \zeta)$ the approximations of $\sigma(\zeta)$ and $\psi_0(\lambda, \zeta)$ given by Lemma 6 and define
\[
\psi_{\nu}(\lambda, \zeta) := a \frac{\lambda}{\sigma_{\nu}(\zeta)} + \left( 1 - \frac{\lambda}{\sigma_{\nu}(\zeta)} \right) \Phi(\zeta) + (\lambda - \sigma_{\nu}(\zeta))\psi_{0\nu}(\lambda, \zeta).
\]
If \( \nu \) is large enough, then
- \( \min_{\zeta \in \partial E} |\sigma_\nu(\zeta)| > \delta/2 \),
- \( \psi_\nu(\cdot, \zeta) \in \mathcal{O}(r_2 E, D) \) for \( \zeta \in \partial E \), where \( 1 < r_2 < r_1 \),
- \( \max_{\zeta \in E} |\sigma_\nu(\zeta)| < 1 \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |\sigma_\nu(e^{i\theta})| \, d\theta < \frac{1}{2\pi} \int_0^{2\pi} \log (e^{i\theta}) \, d\theta + \varepsilon < A + 2\varepsilon + C\varepsilon.
\]

We fix \( \nu \) so large that the above conditions are satisfied.

Note that there exists \( \varrho > 1 \) such that \( \min_{1/\varrho < |\zeta| < \varrho} |\sigma_\nu(\zeta)| > \delta/2 \), and, therefore \( \psi_\nu(\sigma_\nu(\zeta), \zeta) = a \) if \( 1/\varrho < |\zeta| < \varrho \).

Let \( \zeta_1, \zeta_2, \ldots \) be the zeros of \( \sigma_\nu \) in \( E \) counted with multiplicity. Note that \( |\zeta_j| < 1/\varrho \) and it is a finite sequence. It is easy to see from Lemma 6 that

\[
\zeta^{2\nu-2} \prod \left( \frac{\zeta - \zeta_j}{1 - \zeta_j \zeta} \right) \psi_\nu(\lambda, \zeta)
\]

is a holomorphic mapping in \( (r_3)E^2 \), where \( 1 < r_3 < \min\{r_2, \varrho\} \). We know that \( \psi_\nu(0, \zeta) = \Phi(\zeta) \) and, therefore, \( \psi_\nu(0, \cdot) \) is a holomorphic mapping on \( r_3 E \). Hence, for any \( k \geq 2\nu - 2 \),

\[
f(\lambda, \zeta) := \psi_\nu \left( \lambda \zeta^k \prod \left( \frac{\zeta - \zeta_j}{1 - \zeta_j \zeta} \right), \zeta \right)
\]

is a holomorphic mapping in \( (r_4)E^2 \), where \( 1 < r_4 < r_3 \) is such that

\[
\lambda \zeta^k \prod \left( \frac{\zeta - \zeta_j}{1 - \zeta_j \zeta} \right) \in r_3 E \quad \text{for} \quad (\lambda, \zeta) \in (r_4)E^2.
\]

Note that \( r_4 \) depends on \( k \). We want to show that we can take \( k \) so large that \( f \in \mathcal{O}((r_4)E^2, D) \). Note that there exists a neighborhood \( W_1 \subset E \) of \( \partial E \) such that \( \psi_\nu(r_3 E \times W_1) \subset D \) and a neighborhood \( W_2 \subset E \) of \( 0 \) such that \( \psi_\nu(W_2 \times r_3 E) \subset D \). We can take \( k \) so large that

\[
\left( \lambda \zeta^k \prod \left( \frac{\zeta - \zeta_j}{1 - \zeta_j \zeta} \right), \zeta \right) \in (r_3 E \times W_1) \cup (W_2 \times r_3 E) \quad \text{if} \quad (\lambda, \zeta) \in (r_4)E^2.
\]

For such fixed \( k \) we have \( f \in \mathcal{O}((r_4)E^2, D) \). Put

\[
\tilde{\sigma}(\zeta) := \frac{\sigma_\nu(\zeta)}{\zeta^k \prod \left( \frac{\zeta - \zeta_j}{1 - \zeta_j \zeta} \right)}.
\]

Let us collect the facts that we have just proved and that we shall need in the sequel (we change the notation, putting \( \bar{\sigma} \) in place of \( \tilde{\sigma} \) and \( r_0 \) in place of \( r_4 \)).

**There exist a holomorphic mapping** \( f : (r_0)E^2 \to D \), \( r_0 > 1 \), and a holomorphic function \( \sigma \in \mathcal{O}(r_0 E \setminus (1/r_0)E) \) such that
Lemma 5 for large enough
\( k \)
we have
\( g \)
Consider the functions
\( 1 - \eta > 0 \) such that
\( \Phi(0) = z \) and \( a \not\in f(\{0\} \times \partial E) \), hence there exists \( g_0 > 0 \) such that \( a \not= f(\xi, \zeta) \) for any \( \xi \in g_0E \) and any \( \zeta \in \mathbb{C} \) such that
\( 1 - g_0 < |\zeta| < 1 + g_0 \).
Fix \( \zeta_0 \in \Gamma \) and \( \eta_0 \in \partial E \). For \( c > 0 \) consider the function
\[
\phi_c(\lambda) := \frac{\eta_0 e(\lambda, c) - \sigma(\zeta_0)}{1 - \overline{\sigma(\zeta_0)} \eta_0 e(\lambda, c)}.
\]
We have \( |\sigma(\zeta_0)| < 1 \), so \( \phi_c \) is holomorphic in \( E \). But also \( \sigma(\zeta_0) \not= 0 \), hence by Lemma 5, \( \phi_c \) is a Blaschke product. Therefore \( |\phi_c(0)| = \prod_{j=1}^{\infty} |\lambda_j| \), where the \( \lambda_j \) are the zeros of \( \phi_c \) counted with multiplicity. Note that
\[
|\phi_c(0)| = \left| \frac{\eta_0 e^{-c} - \sigma(\zeta_0)}{1 - \overline{\sigma(\zeta_0)} \eta_0 e^{-c}} \right| \to |\sigma(\zeta_0)| \quad \text{as } c \to \infty.
\]
So, there exists \( c > 0 \) such that \( \log |\phi_c(0)| < \log |\sigma(\zeta_0)| + \varepsilon \) and \( e^{-c} < g_0 \).
Fix such a \( c > 0 \). We can take \( s \in \mathbb{N} \) so large that
\[
\sum_{j=1}^{s} \log |\lambda_j| < \log |\sigma(\zeta_0)| + \varepsilon.
\]
We may find \( r < 1 \) such that
\[
\sum_{j=1}^{s} \frac{\log |\lambda_j|}{r} < \log |\sigma(\zeta_0)| + \varepsilon,
\]
and \( \max_{j=1,\ldots,s} |\lambda_j| < r < 1 \). Fix such an \( r < 1 \).
There is a neighborhood \( U_0 \) of \( \zeta_0 \) such that \( |\sigma(\zeta)| < 1 \) for \( \zeta \in U_0 \). By Lemma 5 for large enough \( k \) we have \( \zeta_0 l_k(r\xi, c) \in U_0 \). Therefore, for \( \xi \in \partial E \) we have
\[
(4) \quad f(\sigma(\zeta_0 l_k(r\xi, c)), \zeta_0 l_k(r\xi, c)) = a.
\]
Consider the functions \( g_k(\xi) = \eta_0 l_k^1(r\xi, c) - \sigma(\zeta_0 l_k(r\xi, c)) \) and \( g_\infty(\xi) = \eta_0 e(r\xi, c) - \sigma(\zeta_0) \) for \( \xi \in E \). Note that \( g_k \to g_\infty \) uniformly on \( E \). We know that \( g_\infty(\lambda_j/r) = 0, j = 1, \ldots, s \). By the Hurwitz theorem for large enough \( k \) we know that \( g_k \) has zeros \( \lambda_k^1/r, \ldots, \lambda_k^s/r \) close to \( \lambda_1/r, \ldots, \lambda_s/r \) such that
\[
\sum_{j=1}^{s} \frac{\log |\lambda_j^j|}{r} < \log |\sigma(\zeta_0)| + \varepsilon.
\]
So, \( f(\eta_0 l_k^j(\lambda_j', \lambda_j, \zeta_0 k(\lambda_j, c) = a, j = 1, \ldots, s \) (use (4)). Therefore, for large enough \( k \) it follows that \( 1 - \varrho_0 < e^{-c/k} \) and
\[
(5) \quad H(f(\eta_0 l_k^k(\rho \xi, c), \zeta_0 l_k(r \xi, c), a) < \log |\sigma(\zeta_0)| + \varepsilon.
\]
Hence, for any fixed \( \varepsilon_0 \in \Gamma \) and \( \eta_0 \in \partial E \) there exist \( k \in \mathbb{N} \) and \( r < 1, c > 0 \) such that (5) is satisfied. Therefore we may find \( k \in \mathbb{N}, r < 1, c > 0, \) and \( Q \subset \partial E \times \Gamma \) such that \( m(Q) > 4\pi^2 - 4\pi \varepsilon \) and for any \( (\eta, \zeta) \in Q, \) (5) is satisfied, \( e^{-c} < \varrho_0, \) and \( 1 - \varrho_0 < e^{-c/k}. \)

Let \( Q^* \) denote the image of \( Q \) under the mapping \( (\eta, \zeta) \to (\eta \zeta^{-k}, \zeta). \) The Jacobian of this mapping is equal to 1 on \( \partial E \times \partial E, \) hence \( m(Q^*) = m(Q). \) So, there exists \( \nu \in \partial E \) such that
\[
m(\{ \zeta \in \partial E : (\nu, \zeta) \in Q^* \}) > 2\pi - 2\varepsilon.
\]

Note that
\[
H(f(\nu \zeta \xi^k(\rho \xi, c), \zeta l_k(\rho \xi, c), a) < \log |\sigma(\zeta)| + \varepsilon
\]
on \( S := \{ \zeta \in \partial E : (\nu, \zeta) \in Q^* \} \subset \Gamma \) and \( m(S) > 2\pi - 2\varepsilon. \) Consider the mapping \( \varphi(\xi) := f(\nu \xi^k, \xi), \xi \in \overline{E}. \) Note that \( \varphi(0) = f(0, 0) = \Phi(0) = z. \) Put
\[
h(\xi, \zeta) = \zeta l_k(\rho \xi, c) = \xi \frac{\rho \xi + e^{-c/k}}{1 + r e^{-c/k} \xi}, \quad \xi, \zeta \in \partial E.
\]
Note that \( h(\xi, \zeta) \in \mathcal{O}(E), \) \( a \not\in \varphi(h(\{0\} \times \partial E)), \) and \( \varphi(h(0, 0)) = z \neq a. \) Therefore, by Lemma 4 there exists \( \alpha_0 \in [0, 2\pi) \) such that
\[
H(\varphi \circ h(e^i\alpha_0 \xi, \zeta), a) \leq \frac{1}{2\pi} \int_0^{2\pi} H(\varphi \circ h(\xi, e^{i\theta}), a) \, d\theta.
\]
Put \( \tilde{\varphi}(\xi) := \varphi(h(e^i\alpha_0 \xi, \xi)). \) Then \( \tilde{\varphi} \in \mathcal{O}(E, D), \) \( \tilde{\varphi}(0) = z, \) and
\[
H(\tilde{\varphi}, a) = H(\varphi \circ h(e^i\alpha_0 \xi, \zeta), a) \leq \frac{1}{2\pi} \int_0^{2\pi} H(\varphi \circ h(\xi, e^{i\theta}), a) \, d\theta
\]
\[
\leq \frac{1}{2\pi} \int_S H(\varphi \circ h(\xi, e^{i\theta}), a) \, d\theta < \frac{1}{2\pi} \int_S \log |\sigma(e^{i\theta})| \, d\theta + \varepsilon
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \log |\sigma(e^{i\theta})| \, d\theta + \varepsilon - \frac{1}{2\pi} \int_{[0, 2\pi) \setminus S} \log |\sigma(e^{i\theta})| \, d\theta
\]
\[
< A + 3\varepsilon + C\varepsilon - \frac{(\varepsilon / \pi) \log(\delta/2).}{\pi}
\]
Hence, \( g_D^2(\varepsilon) < A + 3\varepsilon + C\varepsilon - (\varepsilon / \pi) \log(\delta/2). \) Since \( \varepsilon > 0 \) was arbitrary the proof is complete. 

\textbf{Lemma 10}. \( g_D(a, z) = g_D^2(a, z). \)
Before we go into the proof of Lemma 10 recall the following result (see [P2]):

**Theorem 11 (Poletsky).** Let \( G \) be a domain in \( \mathbb{C}^n \) and let \( u \) be an upper semicontinuous function in \( G \). Then

\[
\tilde{u}(w) = \inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(\phi(e^{i\theta})) \, d\theta : \phi \in \mathcal{O}(E, G), \, \phi(0) = w \right\}, \quad w \in G,
\]

is a plurisubharmonic function in \( G \). Moreover, it is equal to the supremum of all plurisubharmonic functions \( v \) such that \( v \leq u \).

**Proof of Lemma 10.** Let us show first that for any \( a \in D \) the function \( k_D(a, \cdot) \) is upper semicontinuous in \( D \).

Let \( z_0 \neq a \) and \( k_D(a, z_0) < A \). There exists a holomorphic mapping \( \varphi : E \to D \) such that \( \varphi(0) = z_0 \), \( \varphi(\sigma) = a \), \( \sigma > 0 \), and \( \log \sigma < A \). Let

\[
\varphi_w(\lambda) := \varphi(\lambda) + (w - z_0)(1 - \lambda/\sigma), \quad \lambda \in E.
\]

For some neighborhood \( V \) of \( z_0 \) we have \( \varphi_w(E) \subset D \), \( w \in V \). Note that \( \varphi_w(0) = w \) and \( \varphi_w(\sigma) = a \). Hence,

\[
k_D(a, w) < A, \quad w \in V.
\]

Assume now that \( z_0 = a \). Then \( k_D(a, z_0) = -\infty \). Fix \( A < 0 \) and let \( \varphi_w(\lambda) := w + \lambda e^{-A}(a - w) \). Note that \( \varphi_w(0) = w \) and \( \varphi_w(e^A) = a \). For some neighborhood \( V \) of \( a \) we have \( \varphi_w(E) \subset D \), \( w \in V \). Hence, \( k_D(a, w) \leq \log e^A = A \), \( w \in V \).

Hence, by Theorem 11, we conclude that \( g_D^a \) is a plurisubharmonic function which is a supremum over all plurisubharmonic functions not greater than \( k_D \). But so is \( g_D \), because \( g_D(a, w) \leq k_D(a, w) \leq \log \|w - a\| - \log R \), \( w \in B(a, R) \), where \( R \) is such that \( B(a, R) \subset D \).

**Lemma 12.** \( g_D(a, z) \leq g_D^a(a, z) \).

**Proof.** Let \( u \in \text{PSH}(D) \), \( u < 0 \), be such that for some \( M > 0 \) we have

\[
u(w) \leq M + \log \|w - a\| \quad \text{for} \, w \text{ near} \, a.
\]

Take \( \varphi \in \mathcal{O}(E, D) \) with \( \varphi(0) = z \) and \( a \in \varphi(E) \). Let \( \lambda_j, j = 1, \ldots, N \), denote the solutions in \( E \) of the equation \( \varphi(\lambda) = a \) without multiplicity (if one takes solutions with multiplicities then one will get the inequality \( g_D(a, z) \leq g_D^a(a, z) \), cf. [J-P], Chapter 4). Define

\[
f(\lambda) := \prod_{j=1}^{N} \frac{\lambda - \lambda_j}{1 - \lambda_j \lambda}.
\]

Put \( v := u \circ \varphi - \log |f| \). It is clear that \( v \) is a subharmonic function in \( E \setminus \{\lambda_1, \ldots, \lambda_N\} \) and \( v \) is locally bounded above on \( E \). Hence \( v \) extends
subharmonically to $E$. By the maximum principle $v \leq 0$. In particular,

$$u(z) = u(\varphi(0)) \leq \log |f(0)| = \sum_{j=1}^{N} \log |\lambda_j|.$$ 

Hence $g_D(a, z) \leq g_D^4(a, z)$. ■

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**References**


