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On definitions of the pluricomplex Green function

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Abstract. We give several definitions of the pluricomplex Green function and show their equivalence.

1. Introduction. We denote by E the unit disc in $\mathbb{C}$. Let D be a domain in $\mathbb{C}^n$. Put

$$g_D(a,z) := \sup\{u(z) : u \in PSH(D), u < 0, \\ \exists M, r > 0 : u(w) \le M + \log\|w-a\|,\ w \in B(a,r) \subset D\}, \quad a, z \in D,$$

where $\mathrm{PSH}(D)$ denotes the set of all plurisubharmonic functions on D and $B(a,r)$ denotes the ball with center at a and radius r. The function g_D has been introduced by M. Klimek (cf. [K]) and is called the *pluricomplex Green function*.

In this paper we give several equivalent definitions of the pluricomplex Green function.

Following E. Poletsky (cf. [P-S], [P1], [P2]) for a domain $D \subset \mathbb{C}^n$ and $a, z \in D$, $a \ne z$, we define

$$g_D^1(a,z) := \inf\Big\{ \sum_{\lambda\in\varphi^{-1}(a)} \mathrm{ord}_\lambda(\varphi - a)\log|\lambda| : \\ \varphi \in \mathcal{O}(E, D),\ a \in \varphi(E),\ \varphi(0) = z\Big\},$$

$$g_D^2(a,z) := \inf\Big\{ \sum_{\lambda\in\varphi^{-1}(a)} \mathrm{ord}_\lambda(\varphi - a)\log|\lambda| : \\ \varphi \in \mathcal{O}(\overline{E}, D),\ a \in \varphi(E),\ \varphi(0) = z\Big\},$$

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$$g_D^3(a,z) := \inf\Big\{ \sum_{\lambda\in\varphi^{-1}(a)} \log|\lambda| : \varphi \in \mathcal{O}(E,D),\ a\in\varphi(E),\ \varphi(0)=z\Big\},$$

$$g_D^4(a,z) := \inf\Big\{ \sum_{\lambda\in\varphi^{-1}(a)} \log|\lambda| : \varphi \in \mathcal{O}(\overline{E},D),\ a\in\varphi(E),\ \varphi(0)=z\Big\},$$

where $\mathcal{O}(E,D)$ denotes the set of all holomorphic mappings $E\to D$ and $\mathrm{ord}_\lambda(\varphi-a)$ denotes the order of vanishing of $\varphi-a$ at λ. Note that in the whole paper for any holomorphic mapping $\varphi:\overline{E}\to D$ by $\varphi^{-1}(a)$ we mean $\varphi^{-1}(a)\cap E$ and it is always a finite set provided φ is nonconstant.

We put $g_D^1(a,a)=g_D^2(a,a)=g_D^3(a,a)=g_D^4(a,a)=-\infty$.

Remarks. 1. For any $z\in D\setminus\{a\}$ there exists $\varphi\in\mathcal{O}(\overline{E},D)$ such that $\varphi(0)=z$ and $a\in\varphi(E)$ (cf. [J-P], Remark 3.1.1). So, the above functions are well defined.

2. Note that $g_D^1\le g_D^2$, $g_D^3\le g_D^4$, $g_D^1\le g_D^3$, and $g_D^2\le g_D^4$.

Define

$$k_D(a,z) := \inf\{\log\sigma : \exists\varphi\in\mathcal{O}(\overline{E},D) : \varphi(0)=a,\ \varphi(\sigma)=z,\ \sigma>0\},$$

$$g_D^5(a,z) := \inf\Big\{\frac{1}{2\pi}\int_0^{2\pi} k_D(a,\varphi(e^{i\theta}))\,d\theta : \varphi\in\mathcal{O}(\overline{E},D),\ \varphi(0)=z\Big\},\qquad a,z\in D.$$

Note that $g_D^5(a,\cdot)$ is the envelope of $k_D(a,\cdot)$ in the sense of Poletsky (see Theorem 11).

The main result of the paper is the following

THEOREM 1. *Let D be a domain in $\mathbb{C}^n$. Then*

$$g_D = g_D^1 = g_D^2 = g_D^3 = g_D^4 = g_D^5.$$

Remarks. The most difficult problem in Theorem 1 is the equality $g_D=g_D^2$. It was proved in [P1]. We present a much simpler and complete proof. The equality $g_D=g_D^4$ was stated in [P2].

2. Definitions and auxiliary results. Let D be a domain in $\mathbb{C}^n$ and let $\varphi:\overline{E}\to D$ be a holomorphic mapping. For a point $a\in D$ we define

$$u_{(\varphi,a)}(\lambda) := \sum_{\zeta\in\varphi^{-1}(a)} \mathrm{ord}_\zeta(\varphi-a)\log\left|\frac{\lambda-\zeta}{1-\overline{\zeta}\lambda}\right|,\qquad \lambda\in E,$$

$$H(\varphi,a) := u_{(\varphi,a)}(0).$$

For convenience we put $\sum_\emptyset = 0$ in the whole paper. For a constant mapping

$\varphi \equiv a$ we put $u_{(\varphi,a)} \equiv -\infty$. In this notation we have

$$g_D^2(a,z) = \inf\{H(\varphi,a) : \varphi \in \mathcal{O}(\overline{E}, D),\ \varphi(0) = z\}, \qquad a, z \in D.$$

For the functional H we have the following

LEMMA 2. *Let $\varphi : \overline{E} \to D$ and $h : \overline{E} \to \overline{E}$ be holomorphic mappings. Then for any $a \in D$ such that $\varphi \not\equiv a$ we have*

$$H(\varphi \circ h, a) = \iint\limits_E \log|\zeta| \Delta(u_{(\varphi,a)} \circ h(\zeta)).$$

Proof. Note that if $\varphi(h(0)) = a$ then

$$H\big(\varphi \circ h, a\big) = \iint\limits_E \log|\zeta| \Delta(u_{(\varphi,a)} \circ h(\zeta)) = -\infty.$$

So, we may assume that $\varphi(h(0)) \neq a$. Put

$$\psi_j(\lambda) := \frac{h(\lambda) - \lambda_j}{1 - \overline{\lambda}_j h(\lambda)}, \qquad \text{where } \lambda_j \in \varphi^{-1}(a).$$

Note that $\psi_j \in \mathcal{O}(\overline{E})$ and $\psi_j(0) \neq 0$. Hence using the Jensen formula (see [R], Theorem 15.18) we have

$$\log|\psi_j(0)| = \sum_{m=1}^{N} \log|\alpha_m| + \frac{1}{2\pi} \int_0^{2\pi} \log|\psi_j(e^{i\theta})|\, d\theta,$$

where $\alpha_1, \ldots, \alpha_N$ are the zeros of ψ_j with multiplicities. But on the other hand by the Riesz representation we have

$$\log|\psi_j(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|\psi_j(e^{i\theta})|\, d\theta + \iint\limits_E \log|\zeta| \Delta(\log|\psi_j(\zeta)|).$$

Hence,

$$\sum_{m=1}^{N} \log|\alpha_m| = \iint\limits_E \log|\zeta| \Delta(\log|\psi_j(\zeta)|).$$

From this we derive the desired result. ∎

LEMMA 3 (cf. [P1], Lemma 3.2). *Let v be a plurisubharmonic function in some neighborhood of $\overline{E}^2$ such that $v(0,0) \neq -\infty$ and $v(0, e^{i\theta}) \neq -\infty$, $\theta \in [0, 2\pi)$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \Big(\iint\limits_E \log|\zeta| \Delta_\zeta(v(e^{i\alpha}\zeta, \zeta)) \Big)\, d\alpha \le \frac{1}{2\pi} \int_0^{2\pi} \Big(\iint\limits_E \log|\zeta| \Delta_\zeta v(\zeta, e^{i\theta}) \Big)\, d\theta.$$

Therefore, there exists $\alpha_0 \in [0, 2\pi)$ such that

$$\iint\limits_E \log|\zeta| \Delta_\zeta(v(e^{i\alpha_0}\zeta, \zeta)) \le \frac{1}{2\pi} \int_0^{2\pi} \Big(\iint\limits_E \log|\zeta| \Delta_\zeta v(\zeta, e^{i\theta}) \Big)\, d\theta.$$

Proof. By the Riesz representation we have

$$v(0,0)=\frac{1}{2\pi}\int_0^{2\pi} v(0,e^{i\theta})\,d\theta+\iint_E \log|\zeta|\Delta_\zeta v(0,\zeta)$$
$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} v(e^{i\alpha},e^{i\theta})\,d\alpha\,d\theta$$
$$+\frac{1}{2\pi}\int_0^{2\pi} d\theta\iint_E \log|\zeta|\Delta_\zeta v(\zeta,e^{i\theta})+\iint_E \log|\zeta|\Delta_\zeta v(0,\zeta).$$

Again by the Riesz representation for any fixed $\alpha\in[0,2\pi)$ we have

$$v(0,0)=\frac{1}{2\pi}\int_0^{2\pi} v(e^{i(\alpha+\theta)},e^{i\theta})\,d\theta+\iint_E \log|\zeta|\Delta_\zeta v(e^{i\alpha}\zeta,\zeta). \tag{1}$$

Hence, integrating (1) in $\alpha\in[0,2\pi)$ we obtain

$$v(0,0)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} v(e^{i(\alpha+\theta)},e^{i\theta})\,d\theta\,d\alpha+\frac{1}{2\pi}\int_0^{2\pi}\Big[\iint_E \log|\zeta|\Delta_\zeta v(e^{i\alpha}\zeta,\zeta)\Big]\,d\alpha.$$

So,

$$\frac{1}{2\pi}\int_0^{2\pi}\Big[\iint_E \log|\zeta|\Delta_\zeta v(e^{i\alpha}\zeta,\zeta)\Big]\,d\alpha=\frac{1}{2\pi}\int_0^{2\pi} d\theta\iint_E \log|\zeta|\Delta_\zeta v(\zeta,e^{i\theta})$$
$$+\iint_E \log|\zeta|\Delta_\zeta v(0,\zeta)$$
$$\le\frac{1}{2\pi}\int_0^{2\pi} d\theta\iint_E \log|\zeta|\Delta_\zeta v(\zeta,e^{i\theta}).\ ■$$

As a corollary we have the following

LEMMA 4. *Let $\varphi:\overline{E}\to D$ and h: $\overline{E}^2\to\overline{E}$ be holomorphic mappings. Then for any $a\in D$ such that $a\notin\varphi(h(\{0\}\times\partial E))$ and $\varphi(h(0,0))\neq a$ there exists $\alpha_0\in[0,2\pi)$ with*

$$H(\varphi\circ h(e^{i\alpha_0}\zeta,\zeta),a)\le\frac{1}{2\pi}\int_0^{2\pi} H(\varphi\circ h(\zeta,e^{i\theta}),a)\,d\theta.$$

Proof. Take $v:=u_{(\varphi,a)}\circ h$. Then the result follows from Lemmas 2 and 3. ■

Recall that a holomorphic function $\phi:E\to E$ is called *inner* if $|\phi^*(\zeta)|=\lim_{r\to1}|\phi(r\zeta)|=1$ for almost all $\zeta\in\partial E$. Any Blaschke product is an inner function. A simple example of an inner function but not a Blaschke product is the function $e(\lambda, c) := e^{c(\lambda-1)/(\lambda+1)}$, $c > 0$. It plays an important role in our considerations. Put

$$l_k(\lambda, c) = \frac{\lambda + e^{-c/k}}{1 + e^{-c/k}\lambda}, \qquad \lambda \in E,\ c > 0,\ k \in \mathbb{N}.$$

We have

LEMMA 5. (a) *For fixed $c > 0$ and $\tau \in E \setminus \{0\}$ the function*

$$\phi(\lambda) = \frac{e(\lambda, c) - \tau}{1 - \overline{\tau} e(\lambda, c)}$$

is a Blaschke product.

(b) *For fixed $c > 0$ we have $l_k(\lambda, c) \to 1$ and $l_k^k(\lambda, c) \to e(\lambda, c)$ locally uniformly on E as $k \to \infty$.*

Proof. (a) Note that ϕ is an inner function. By Theorem 2 in Chapter III of [N], any inner function which has no zero radial limits is a Blaschke product. By simple calculations we see that ϕ has no zero radial limits.

(b) It is enough to note that

$$l_k(\lambda, c) = 1 + (1 - e^{-c/k})\frac{\lambda - 1}{1 + e^{-c/k}\lambda}. \ \blacksquare$$

Recall the following approximation result:

LEMMA 6. *Let $F \in \mathcal{C}(V \times \partial E)$ and $F(\cdot, \zeta) \in \mathcal{O}(V)$, $\zeta \in \partial E$, where V is a domain in $\mathbb{C}^m$. For $\nu = 1, 2, \ldots$ put*

$$F_\nu(\xi, \zeta) := \frac{1}{2\pi\nu} \sum_{j=0}^{\nu-1} \sum_{k=-j}^{j} \bigg(\int_0^{2\pi} \frac{F(\xi, e^{i\theta})}{e^{i\theta(k+1)}}\, d\theta \bigg) \zeta^k.$$

Then:

(1) *F_ν are holomorphic w.r.t. $\xi \in V$ and rational w.r.t. ζ with pole of order $\le \nu - 1$ at $\zeta = 0$;*

(2) *$\{F_\nu\}$ converges locally uniformly to F on $V \times \partial E$;*

(3) *if $F(0, \zeta) \equiv 0$, then $F_\nu(0, \zeta) \equiv 0$, $\zeta \in \partial E$.*

Proof. It is enough to prove (2), because (1) and (3) are evident.

Put

$$K_\nu(x) := \frac{1}{\nu} \left[\frac{\sin \frac{\nu}{2} x}{\sin \frac{1}{2} x} \right]^2.$$

Then (see [H], Chapter II) $\frac{1}{2\pi} \int_0^{2\pi} K_\nu(\theta)\, d\theta = 1$ and

$$F_\nu(\xi, e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} F(\xi, e^{i\theta}) K_\nu(t - \theta)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} F(\xi, e^{i(t-\theta)}) K_\nu(\theta)\, d\theta.$$

For $\delta > 0$ we have

$$F_\nu(\xi, e^{it}) - F(\xi, e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it}))K_\nu(\theta)\,d\theta$$
$$= \frac{1}{2\pi}\int_{-\delta}^{\delta}(F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it}))K_\nu(\theta)\,d\theta$$
$$+ \frac{1}{2\pi}\int_{\pi>|\theta|\ge\delta}(F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it}))K_\nu(\theta)\,d\theta.$$

Suppose that $K = L \times \partial E$, where $L \Subset V$. Then

$$|F_\nu(\xi, e^{it}) - F(\xi, e^{it})| \le \sup_{-\delta<\theta<\delta}|F(\xi, e^{i(t-\theta)}) - F(\xi, e^{it})| + 2\|F\|_K \sup_{\pi>|\theta|\ge\delta} K_\nu(\theta),$$

where $\|F\|_K := \sup_{(\xi,\zeta)\in K}|F(\xi,\zeta)|$. Recall that $\lim_{\nu\to\infty}\sup_{\pi>|\theta|\ge\delta} K_\nu(\theta) = 0$. Since F is a continuous mapping, we conclude the proof. ∎

3. Proof of Theorem 1. We will prove Theorem 1 in several lemmas. We prove consecutively that $g_D^1 = g_D^2 = g_D^3 = g_D^4$ (Lemma 7), $g_D^5 \ge g_D^2$ (Lemma 9), $g_D = g_D^5$ (Lemma 10), and finally, $g_D \le g_D^4$ (Lemma 12). In this way we will have proved Theorem 1. In the whole section we assume that the domain D and points $a, z \in D$ are fixed. Note that if $a = z$ then the assertion of Theorem 1 is evident, because all the functions are equal to $-\infty$. So, we may assume that $a \ne z$.

LEMMA 7. $g_D^1(a,z) = g_D^2(a,z) = g_D^3(a,z) = g_D^4(a,z)$.

Proof. It is enough to prove that

(1) $g_D^1(a,z) = g_D^2(a,z)$,
(2) $g_D^3(a,z) = g_D^4(a,z)$,
(3) $g_D^2(a,z) = g_D^4(a,z)$.

(1)–(2) We know that $g_D^1(a,z) \le g_D^2(a,z)$ (resp. $g_D^3(a,z) \le g_D^4(a,z)$). Fix $A > g_D^1(a,z)$ (resp. $A > g_D^3(a,z)$).

There exists a holomorphic mapping $\varphi : E \to D$ such that $\varphi(0) = z$, $a \in \varphi(E)$, and

$$\sum_{\lambda\in\varphi^{-1}(a)} \operatorname{ord}_\lambda(\varphi - a)\log|\lambda| < A \quad \Big(\text{resp. } \sum_{\lambda\in\varphi^{-1}(a)} \log|\lambda| < A\Big).$$

Let $\varphi^{-1}(a) = \{\lambda_j : j = 1, 2, \ldots\}$, where λ_j's are counted with multiplicities (resp. without multiplicities). We may assume that $|\lambda_1| \le |\lambda_2| \le \ldots$ There exists $N > 0$ such that $\sum_{j=1}^N \log|\lambda_j| < A$. Let $\widetilde{\varphi}(\lambda) = \varphi(R\lambda)$, where

$R \in (|\lambda_N|, 1)$. Note that $\widetilde{\varphi} \in \mathcal{O}(\overline{E}, D)$ and $\widetilde{\varphi}(0) = z$. Then we have

$$\sum_{\lambda \in \widetilde{\varphi}^{-1}(a)} \operatorname{ord}_\lambda(\widetilde{\varphi} - a) \log|\lambda| \le \sum_{j=1}^{N} (\log|\lambda_j| - \log R)$$

$$(\text{resp. } \sum_{\lambda \in \widetilde{\varphi}^{-1}(a)} \log|\lambda| \le \sum_{j=1}^{N} (\log|\lambda_j| - \log R)).$$

So, if R is close enough to 1 then

$$g_D^2(a, z) \le \sum_{\lambda \in \widetilde{\varphi}^{-1}(a)} \operatorname{ord}_\lambda(\widetilde{\varphi} - a) \log|\lambda| < A$$

$$(\text{resp. } g_D^4(a, z) \le \sum_{\lambda \in \widetilde{\varphi}^{-1}(a)} \log|\lambda| < A).$$

Hence, $g_D^2(a, z) \le g_D^1(a, z)$ (resp. $g_D^4(a, z) \le g_D^3(a, z)$).

(3) Let $\varphi : \overline{E} \to D$ be a holomorphic mapping such that $\varphi(0) = z \ne a$ and $a \in \varphi(E)$. Suppose that $\varphi(\mu) = a$ and $\operatorname{ord}_\mu(\varphi - a) = m$. Note that $\mu \ne 0$. Let

$$\psi(\lambda) := \frac{\varphi(\lambda) - a}{(\lambda - \mu)^m} (\lambda - \mu_1) \ldots (\lambda - \mu_m) + a, \quad \lambda \in E,$$

where $\mu_1, \ldots, \mu_m$ are pairwise different, $\mu_1 \ldots \mu_m = \mu^m$, and $\mu_1, \ldots, \mu_m$ are very close to μ ([1]). Note that if $\mu_1, \ldots, \mu_m$ are close enough to μ then $\psi \in \mathcal{O}(\overline{E}, D)$ and $\psi(0) = \varphi(0) = z$. Moreover, $\psi(\lambda_0) = a$ iff $\varphi(\lambda_0) = a$ and $\lambda_0 \ne \mu$, or $\lambda_0 \in \{\mu_1, \ldots, \mu_m\}$, and

$$\sum_{\lambda \in \varphi^{-1}(a)} \operatorname{ord}_\lambda(\varphi - a) \log|\lambda| = \sum_{\substack{\lambda \in \psi^{-1}(a) \\ \lambda \notin \{\mu_1, \ldots, \mu_m\}}} \operatorname{ord}_\lambda(\psi - a) \log|\lambda| + \sum_{j=1}^{m} \log|\mu_j|.$$

Note that the multiplicities of ψ at μ_j, $j = 1, \ldots, m$, are equal to 1. Applying this technique N times, where N is the number of zeros of $\varphi - a$ in E, we obtain the result. ∎

The following result is basic for the proof of Theorem 1.

LEMMA 8. *Let $\Phi : \overline{E} \to D$ be a holomorphic mapping such that $\Phi(0) = z$ and $a \notin \Phi(\partial E)$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} k_D(a, \Phi(e^{i\theta}))\, d\theta \ge g_D^2(a, z). \tag{2}$$

([1]) For instance, if $\mu = re^{i\theta}$ then let $\mu_j = re^{i\theta_j}$, $j = 1, \ldots, m$, where $\theta_1, \ldots, \theta_m$ are pairwise different, close to θ, and such that $\theta_1 + \ldots + \theta_m = m\theta$.

Remark. From the definitions we see that $k_D(a,w) \geq g_D^2(a,w)$, $w \in D$. So, a priori (2) states less than the subaverage property of the function $g_D^2(a,\cdot)$. But it turns out that (2) is sufficient to show that $g_D^2(a,\cdot)$ is a plurisubharmonic function, hence has the subaverage property. It is worth noting that we assume that $-\infty$ is a plurisubharmonic function.

Before we present the proof of Lemma 8 note the following immediate corollary.

LEMMA 9. $g_D^5(a,z) \geq g_D^2(a,z)$.

Proof of Lemma 8. Take any $A \in \mathbb{R}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} k_D(a,\Phi(e^{i\theta}))\,d\theta < A.$$

It is sufficient to show that $g_D^2(a,z) \leq A$. Note that $k_D(a,\Phi(\cdot))$ is an upper semicontinuous function in $\overline{E}$ (see the proof of Lemma 10). Hence, we can find a continuous function $q : \partial E \to \mathbb{R}$ such that $k_D(a,\Phi(\xi)) < q(\xi)$, $\xi \in \partial E$, and

$$\frac{1}{2\pi}\int_0^{2\pi} q(e^{i\theta})\,d\theta < A.$$

For any $\xi \in \partial E$ there exist $\varphi_\xi \in \mathcal{O}(\overline{E},D)$ and $\sigma_\xi \in (0,1)$ such that $\varphi_\xi(0) = \Phi(\xi)$, $\varphi_\xi(\sigma_\xi) = a$, and

$$\log \sigma_\xi < q(\xi).$$

Note that for any $\xi \in \partial E$ there exists $t(\xi) > 0$ such that for any $\zeta \in B(\xi,t(\xi))$ we may define a mapping $\varphi_{\xi,\zeta} \in \mathcal{O}(\overline{E},D)$ as follows:

$$\varphi_{\xi,\zeta}(\lambda) := \varphi_\xi(\lambda) + (\Phi(\zeta) - \Phi(\xi))(1 - \lambda/\sigma_\xi), \qquad \lambda \in \overline{E}.$$

Observe that $\varphi_{\xi,\zeta}(0) = \Phi(\zeta)$ and $\varphi_{\xi,\zeta}(\sigma_\xi) = \varphi_\xi(\sigma_\xi) = a$. Taking smaller $t(\xi) > 0$ if necessary we have

$$\log \sigma_\xi < q(\zeta), \qquad \zeta \in B(\xi,t(\xi)),$$

and $\varphi_{\xi,\zeta}(\overline{E}) \Subset D$ for any $\zeta \in \partial E \cap B(\xi,t(\xi))$. Taking even smaller $t(\xi)$, we may choose $\xi_1,\ldots,\xi_l$ such that $\partial E \subset V_{\xi_1} \cup \ldots \cup V_{\xi_l}$ and $V_{\xi_k} \cap V_{\xi_j} = \emptyset$ if $1 < |k-j| < l-1$, $k,j = 1,\ldots,l$, where $V_{\xi_j} := B(\xi,t(\xi))$. We put $\delta := \min_{j=1,\ldots,l} \sigma_{\xi_j}$ and $C := \|q\|$.

Fix $\varepsilon > 0$. Note that there exists $r_1 > 1$ such that $\Phi, \varphi_{\xi_j,\zeta} \in \mathcal{O}(r_1E,D)$ for $\zeta \in V_{\xi_j}$, $j = 1,\ldots,l$. We may assume that $\log r_1 < \varepsilon$. Take $0 < t'(\xi_j) < t(\xi_j)$, $j = 1,\ldots,l$, such that for $I_j := \partial E \cap \overline{B(\xi_j,t'(\xi_j))}$ we have $I_j \cap I_k = \emptyset$ for $j \neq k$ and $m(\bigcup_{j=1}^l I_j) > 2\pi - \varepsilon$, where m denotes the Lebesgue measure on ∂E. Take a closed subset $\Gamma \subset \bigcup I_j$ and a continuous function $\tau : \partial E \to [0,1]$ such that $m(\Gamma) > 2\pi - \varepsilon$, $\tau = 1$ on Γ, and $\tau = 0$ outside $\bigcup I_j$.

For $\zeta \in \partial E$ put

$$\sigma(\zeta) := \begin{cases} \sigma_{\xi_j}/\tau(\zeta) & \text{if } \sigma_{\xi_j}/r_1 < \tau(\zeta) \text{ and } \zeta \in I_j, \\ r_1 & \text{otherwise.} \end{cases}$$

Note that σ is a continuous function on ∂E and if $\sigma(\zeta) < r_1$ then $\tau(\zeta)\sigma(\zeta) = \sigma_{\xi_j}$.

For $\lambda \in r_1 E$ and $\zeta \in \partial E$ we put

$$\psi(\lambda, \zeta) := \begin{cases} \varphi_{\xi_j,\zeta}(\tau(\zeta)\lambda) & \text{if } \zeta \in I_j, \\ \Phi(\zeta) & \text{if } \zeta \notin \bigcup_{j=1}^{l} I_j. \end{cases}$$

Note that $\psi(\lambda, \zeta)$ is holomorphic with respect to λ and continuous with respect to (λ, ζ). Moreover, $\psi(\cdot, \zeta) \in \mathcal{O}(r_1 E, D)$ and $\psi(0, \zeta) = \Phi(\zeta)$ when $\zeta \in \partial E$,

$$\psi(\sigma(\zeta), \zeta) = a \quad \text{if } \sigma(\zeta) < r_1, \tag{3}$$

and

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \log \sigma(e^{i\theta})\, d\theta &< \frac{1}{2\pi}\int_\Gamma \log \sigma(e^{i\theta})\, d\theta + \log r_1 \\ &< \frac{1}{2\pi}\int_0^{2\pi} q(e^{i\theta})\, d\theta + \varepsilon - \frac{1}{2\pi}\int_{[0,2\pi)\setminus\Gamma} q(e^{i\theta})\, d\theta \\ &< A + \varepsilon + C\varepsilon. \end{aligned}$$

Now we want to approximate ψ and σ by holomorphic (actually meromorphic) mappings. But applying Lemma 6 to ψ and σ we may loose the important relation (3). So, we "separate" in ψ the part related to (3). Namely, we have

$$\psi(\lambda, \zeta) = a\frac{\lambda}{\sigma(\zeta)} + \left(1 - \frac{\lambda}{\sigma(\zeta)}\right)\Phi(\zeta) + (\lambda - \sigma(\zeta))\psi_0(\lambda, \zeta),$$

where

$$\psi_0(\lambda, \zeta) := \frac{\psi(\lambda, \zeta) - a\frac{\lambda}{\sigma(\zeta)}}{\lambda - \sigma(\zeta)} + \frac{\Phi(\zeta)}{\sigma(\zeta)}.$$

Note that $\psi_0(\lambda, \zeta)$ extends as a continuous mapping in $r_1 E \times \partial E$ and holomorphic with respect to λ.

We denote by $\sigma_\nu(\zeta)$ and $\psi_{0\nu}(\lambda, \zeta)$ the approximations of $\sigma(\zeta)$ and $\psi_0(\lambda, \zeta)$ given by Lemma 6 and define

$$\psi_\nu(\lambda, \zeta) := a\frac{\lambda}{\sigma_\nu(\zeta)} + \left(1 - \frac{\lambda}{\sigma_\nu(\zeta)}\right)\Phi(\zeta) + (\lambda - \sigma_\nu(\zeta))\psi_{0\nu}(\lambda, \zeta).$$

If ν is large enough, then

- $\min_{\zeta\in\partial E}|\sigma_\nu(\zeta)|>\delta/2$,
- $\psi_\nu(\cdot,\zeta)\in\mathcal{O}(r_2E,D)$ for $\zeta\in\partial E$, where $1<r_2<r_1$,
- $\max_{\zeta\in\Gamma}|\sigma_\nu(\zeta)|<1$,
- $\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\log|\sigma_\nu(e^{i\theta})|\,d\theta<\frac{1}{2\pi}\int_0^{2\pi}\log\sigma(e^{i\theta})\,d\theta+\varepsilon<A+2\varepsilon+C\varepsilon.$

We fix ν so large that the above conditions are satisfied.

Note that there exists $\varrho>1$ such that $\min_{1/\varrho<|\zeta|<\varrho}|\sigma_\nu(\zeta)|>\delta/2$, and, therefore $\psi_\nu(\sigma_\nu(\zeta),\zeta)=a$ if $1/\varrho<|\zeta|<\varrho$.

Let $\zeta_1,\zeta_2,\ldots$ be the zeros of σ_ν in E counted with multiplicity. Note that $|\zeta_j|<1/\varrho$ and it is a finite sequence. It is easy to see from Lemma 6 that

$$\zeta^{2\nu-2}\prod\left(\frac{\zeta-\zeta_j}{1-\overline{\zeta}_j\zeta}\right)\psi_\nu(\lambda,\zeta)$$

is a holomorphic mapping in $(r_3E)^2$, where $1<r_3<\min\{r_2,\varrho\}$. We know that $\psi_\nu(0,\zeta)=\Phi(\zeta)$ and, therefore, $\psi_\nu(0,\cdot)$ is a holomorphic mapping on r_3E. Hence, for any $k\ge 2\nu-2$,

$$f(\lambda,\zeta):=\psi_\nu\left(\lambda\zeta^k\prod\left(\frac{\zeta-\zeta_j}{1-\overline{\zeta}_j\zeta}\right),\zeta\right)$$

is a holomorphic mapping in $(r_4E)^2$, where $1<r_4<r_3$ is such that

$$\lambda\zeta^k\prod\left(\frac{\zeta-\zeta_j}{1-\overline{\zeta}_j\zeta}\right)\in r_3E\quad\text{ for }(\lambda,\zeta)\in(r_4E)^2.$$

Note that r_4 depends on k. We want to show that we can take k so large that $f\in\mathcal{O}((r_4E)^2,D)$. Note that there exists a neighborhood $W_1\subset\mathbb{C}$ of ∂E such that $\psi_\nu(r_3E\times W_1)\subset D$ and a neighborhood $W_2\subset\mathbb{C}$ of 0 such that $\psi_\nu(W_2\times r_3E)\subset D$. We can take k so large that

$$\left(\lambda\zeta^k\prod\left(\frac{\zeta-\zeta_j}{1-\overline{\zeta}_j\zeta}\right),\zeta\right)\in(r_3E\times W_1)\cup(W_2\times r_3E)\quad\text{ if }(\lambda,\zeta)\in(r_4E)^2.$$

For such fixed k we have $f\in\mathcal{O}((r_4E)^2,D)$. Put

$$\widetilde{\sigma}(\zeta):=\frac{\sigma_\nu(\zeta)}{\zeta^k\prod\left(\frac{\zeta-\zeta_j}{1-\overline{\zeta}_j\zeta}\right)}.$$

Let us collect the facts that we have just proved and that we shall need in the sequel (we change the notation, putting σ in place of $\widetilde{\sigma}$ and r_0 in place of r_4).

There exist a holomorphic mapping $f:(r_0E)^2\to D$, $r_0>1$, *and a holomorphic function* $\sigma\in\mathcal{O}(r_0E\setminus(1/r_0)\overline{E})$ *such that*

- $\frac{1}{2\pi}\int\limits_0^{2\pi} \log|\sigma(e^{i\theta})|\,d\theta < A + 2\varepsilon + C\varepsilon$,
- $f(\sigma(\zeta),\zeta) = a$ *if* $|\sigma(\zeta)| < r_0$ *and* $1/r_0 < |\zeta| < r_0$,
- $\min_{\zeta\in\partial E}|\sigma(\zeta)| > \delta/2$,
- $\max_{\zeta\in\Gamma}|\sigma(\zeta)| < 1$,
- $f(0,\zeta) = \Phi(\zeta)$, $\zeta \in r_0E$.

Note that $f(0,0) = \Phi(0) = z$ and $a \notin f(\{0\}\times\partial E)$, hence there exists $\varrho_0 > 0$ such that $a \neq f(\xi,\zeta)$ for any $\xi \in \varrho_0 E$ and any $\zeta \in \mathbb{C}$ such that $1-\varrho_0 < |\zeta| < 1+\varrho_0$.

Fix $\zeta_0 \in \Gamma$ and $\eta_0 \in \partial E$. For $c > 0$ consider the function

$$\phi_c(\lambda) := \frac{\eta_0 e(\lambda,c) - \sigma(\zeta_0)}{1-\overline{\sigma}(\zeta_0)\eta_0 e(\lambda,c)}.$$

We have $|\sigma(\zeta_0)| < 1$, so ϕ_c is holomorphic in E. But also $\sigma(\zeta_0) \neq 0$, hence by Lemma 5, ϕ_c is a Blaschke product. Therefore $|\phi_c(0)| = \prod_{j=1}^{\infty}|\lambda_j|$, where the λ_j are the zeros of ϕ_c counted with multiplicity. Note that

$$|\phi_c(0)| = \left|\frac{\eta_0 e^{-c} - \sigma(\zeta_0)}{1-\overline{\sigma}(\zeta_0)\eta_0 e^{-c}}\right| \to |\sigma(\zeta_0)| \quad \text{as } c\to\infty.$$

So, there exists $c > 0$ such that $\log|\phi_c(0)| < \log|\sigma(\zeta_0)| + \varepsilon$ and $e^{-c} < \varrho_0$. Fix such a $c > 0$. We can take $s \in \mathbb{N}$ so large that

$$\sum_{j=1}^{s}\log|\lambda_j| < \log|\sigma(\zeta_0)| + \varepsilon.$$

We may find $r < 1$ such that

$$\sum_{j=1}^{s}\log\frac{|\lambda_j|}{r} < \log|\sigma(\zeta_0)| + \varepsilon,$$

and $\max_{j=1,\dots,s}|\lambda_j| < r < 1$. Fix such an $r < 1$.

There is a neighborhood U_0 of ζ_0 such that $|\sigma(\zeta)| < 1$ for $\zeta \in U_0$. By Lemma 5 for large enough k we have $\zeta_0 l_k(r\xi,c) \in U_0$. Therefore, for $\xi \in \partial E$ we have

$$f(\sigma(\zeta_0 l_k(r\xi,c)),\zeta_0 l_k(r\xi,c)) = a. \tag{4}$$

Consider the functions $g_k(\xi) = \eta_0 l_k^k(r\xi,c) - \sigma(\zeta_0 l_k(r\xi,c))$ and $g_\infty(\xi) = \eta_0 e(r\xi,c) - \sigma(\zeta_0)$ for $\xi \in \overline{E}$. Note that $g_k \to g_\infty$ uniformly on E. We know that $g_\infty(\lambda_j/r) = 0$, $j = 1,\dots,s$. By the Hurwitz theorem for large enough k we know that g_k has zeros $\lambda_1'/r,\dots,\lambda_s'/r$ close to $\lambda_1/r,\dots,\lambda_s/r$ such that

$$\sum_{j=1}^{s}\log\frac{|\lambda_j'|}{r} < \log|\sigma(\zeta_0)| + \varepsilon.$$

So, $f(\eta_0 l_k^k(\lambda_j', c), \zeta_0 l_k(\lambda_j', c)) = a$, $j = 1, \ldots, s$ (use (4)). Therefore, for large enough k it follows that $1 - \varrho_0 < e^{-c/k}$ and

$$H(f(\eta_0 l_k^k(r\xi, c), \zeta_0 l_k(r\xi, c)), a) < \log|\sigma(\zeta_0)| + \varepsilon. \tag{5}$$

Hence, for any fixed $\zeta_0 \in \Gamma$ and $\eta_0 \in \partial E$ there exist $k \in \mathbb{N}$ and $r < 1$, $c > 0$ such that (5) is satisfied. Therefore we may find $k \in \mathbb{N}$, $r < 1$, $c > 0$, and $Q \subset \partial E \times \Gamma$ such that $m(Q) > 4\pi^2 - 4\pi\varepsilon$ and for any $(\eta, \zeta) \in Q$, (5) is satisfied, $e^{-c} < \varrho_0$, and $1 - \varrho_0 < e^{-c/k}$.

Let Q^* denote the image of Q under the mapping $(\eta, \zeta) \to (\eta\zeta^{-k}, \zeta)$. The Jacobian of this mapping is equal to 1 on $\partial E \times \partial E$, hence $m(Q^*) = m(Q)$. So, there exists $\nu \in \partial E$ such that

$$m(\{\zeta \in \partial E : (\nu, \zeta) \in Q^*\}) > 2\pi - 2\varepsilon.$$

Note that

$$H(f(\nu\zeta^k l_k^k(r\xi, c), \zeta l_k(r\xi, c)), a) < \log|\sigma(\zeta)| + \varepsilon$$

on $S := \{\zeta \in \partial E : (\nu, \zeta) \in Q^*\} \subset \Gamma$ and $m(S) > 2\pi - 2\varepsilon$. Consider the mapping $\varphi(\xi) := f(\nu\xi^k, \xi)$, $\xi \in \overline{E}$. Note that $\varphi(0) = f(0,0) = \Phi(0) = z$. Put

$$h(\xi, \zeta) = \zeta l_k(r\xi, c) = \zeta \frac{r\xi + e^{-c/k}}{1 + re^{-c/k}\xi}, \quad \xi, \zeta \in \partial E.$$

Note that $h(\xi, \zeta) \in \mathcal{O}(\overline{E}^2)$, $a \notin \varphi(h(\{0\} \times \partial E))$, and $\varphi(h(0,0)) = z \neq a$. Therefore, by Lemma 4 there exists $\alpha_0 \in [0, 2\pi)$ such that

$$H(\varphi \circ h(e^{i\alpha_0}\zeta, \zeta), a) \le \frac{1}{2\pi} \int_0^{2\pi} H(\varphi \circ h(\zeta, e^{i\theta}), a)\, d\theta.$$

Put $\widetilde{\varphi}(\xi) := \varphi(h(e^{i\alpha_0}\xi, \xi))$. Then $\widetilde{\varphi} \in \mathcal{O}(\overline{E}, D)$, $\widetilde{\varphi}(0) = z$, and

$$\begin{aligned}
H(\widetilde{\varphi}, a) &= H(\varphi \circ h(e^{i\alpha_0}\xi, \xi), a) \le \frac{1}{2\pi} \int_0^{2\pi} H(\varphi \circ h(\xi, e^{i\theta}), a)\, d\theta \\
&\le \frac{1}{2\pi} \int_S H(\varphi \circ h(\xi, e^{i\theta}), a)\, d\theta < \frac{1}{2\pi} \int_S \log|\sigma(e^{i\theta})|\, d\theta + \varepsilon \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log|\sigma(e^{i\theta})|\, d\theta + \varepsilon - \frac{1}{2\pi} \int_{[0,2\pi)\setminus S} \log|\sigma(e^{i\theta})|\, d\theta \\
&< A + 3\varepsilon + C\varepsilon - \frac{\varepsilon}{\pi} \log \frac{\delta}{2}.
\end{aligned}$$

Hence, $g_D^2(z) < A + 3\varepsilon + C\varepsilon - (\varepsilon/\pi)\log(\delta/2)$. Since $\varepsilon > 0$ was arbitrary the proof is complete. ∎

LEMMA 10. $g_D(a, z) = g_D^5(a, z)$.

Before we go into the proof of Lemma 10 recall the following result (see [P2]):

THEOREM 11 (Poletsky). *Let G be a domain in $\mathbb{C}^n$ and let u be an upper semicontinuous function in G. Then*

$$\widetilde{u}(w) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} u(\phi(e^{i\theta}))\,d\theta : \phi \in \mathcal{O}(\overline{E}, G),\ \phi(0) = w\right\},\qquad w \in G,$$

is a plurisubharmonic function in G. Moreover, it is equal to the supremum of all plurisubharmonic functions v such that $v \le u$.

Proof of Lemma 10. Let us show first that for any $a \in D$ the function $k_D(a,\cdot)$ is upper semicontinuous in D.

Let $z_0 \ne a$ and $k_D(a, z_0) < A$. There exists a holomorphic mapping $\varphi : \overline{E} \to D$ such that $\varphi(0) = z_0$, $\varphi(\sigma) = a$, $\sigma > 0$, and $\log\sigma < A$. Let

$$\varphi_w(\lambda) := \varphi(\lambda) + (w - z_0)(1 - \lambda/\sigma),\qquad \lambda \in \overline{E}.$$

For some neighborhood V of z_0 we have $\varphi_w(\overline{E}) \subset D$, $w \in V$. Note that $\varphi_w(0) = w$ and $\varphi_w(\sigma) = a$. Hence,

$$k_D(a, w) < A,\qquad w \in V.$$

Assume now that $z_0 = a$. Then $k_D(a, z_0) = -\infty$. Fix $A < 0$ and let $\varphi_w(\lambda) := w + \lambda e^{-A}(a - w)$. Note that $\varphi_w(0) = w$ and $\varphi_w(e^A) = a$. For some neighborhood V of a we have $\varphi_w(\overline{E}) \subset D$, $w \in V$. Hence, $k_D(a, w) \le \log e^A = A$, $w \in V$.

Hence, by Theorem 11, we conclude that g_D^5 is a plurisubharmonic function which is a supremum over all plurisubharmonic functions not greater than k_D. But so is g_D, because $g_D(a, w) \le k_D(a, w) \le \log\|w - a\| - \log R$, $w \in B(a, R)$, where R is such that $B(a, R) \subset D$. ∎

LEMMA 12. *$g_D(a, z) \le g_D^4(a, z)$.*

Proof. Let $u \in \mathrm{PSH}(D)$, $u < 0$, be such that for some $M > 0$ we have

$$u(w) \le M + \log\|w - a\|\quad \text{for } w \text{ near } a.$$

Take $\varphi \in \mathcal{O}(\overline{E}, D)$ with $\varphi(0) = z$ and $a \in \varphi(E)$. Let λ_j, $j = 1, \ldots, N$, denote the solutions in E of the equation $\varphi(\lambda) = a$ without multiplicity (if one takes solutions with multiplicities then one will get the inequality $g_D(a, z) \le g_D^2(a, z)$, cf. [J-P], Chapter 4). Define

$$f(\lambda) := \prod_{j=1}^{N} \frac{\lambda - \lambda_j}{1 - \overline{\lambda}_j\lambda}.$$

Put $v := u \circ \varphi - \log|f|$. It is clear that v is a subharmonic function in $E \setminus \{\lambda_1, \ldots, \lambda_N\}$ and v is locally bounded above on E. Hence v extends subharmonically to E. By the maximum principle $v \leq 0$. In particular,

$$u(z) = u(\varphi(0)) \leq \log|f(0)| = \sum_{j=1}^{N} \log|\lambda_j|.$$

Hence $g_D(a,z) \leq g_D^4(a,z)$. ∎

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