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On the local Cauchy problem for nonlinear hyperbolic functional differential equations

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Abstract. We consider the local initial value problem for the hyperbolic partial functional differential equation of the first order

$$D_x z(x,y) = f(x,y,z(x,y),(Wz)(x,y),D_y z(x,y)) \quad \text{on } E, \tag{1}$$

$$z(x,y) = \phi(x,y) \quad \text{on } [-\tau_0,0]\times[-b,b], \tag{2}$$

where E is the Haar pyramid and $\tau_0 \in \mathbb{R}_+$, $b=(b_1,\dots,b_n)\in\mathbb{R}_+^n$. Using the method of bicharacteristics and the method of successive approximations for a certain functional integral system we prove, under suitable assumptions, a theorem on the local existence of weak solutions of the problem (1), (2).

1. Introduction. For any interval $I\subset\mathbb{R}$ let $L(I,\mathbb{R}_+)$ denote the class of all Lebesgue integrable functions from I to $\mathbb{R}_+=[0,\infty)$, and let $C(X,Y)$ denote the class of all continuous functions from X to Y, where X, Y are any metric spaces.

Suppose that the function $M=(M_1,\dots,M_n)\in C([0,a],\mathbb{R}_+^n)$, $a>0$, is nondecreasing and $M(0)=0$. Let E be the Haar pyramid

$$E=\{(x,y)\in\mathbb{R}^{1+n}: x\in[0,a],\ y=(y_1,\dots,y_n),\\ -b+M(x)\le y\le b-M(x)\},$$

where $b=(b_1,\dots,b_n)$ and $b_i>M_i(a)$ for $i=1,\dots,n$. Here and subsequently the inequality between two vectors means that the same inequalities hold between their corresponding components. Write $E_0=[-\tau_0,0]\times[-b,b]$, $\tau_0\in\mathbb{R}_+$, and

$$E_x=\{(t,s)=(t,s_1,\dots,s_n)\in E_0\cup E: t\le x\} \quad \text{for } x\in[0,a],$$
$$E_x^*=\{(t,s)=(t,s_1,\dots,s_n)\in E: t\le x\} \quad \text{for } x\in[0,a].$$

Put $I[x,y]=\{t:(t,y)\in E_x^*\}$ where $(x,y)\in[0,a]\times[-b,b]$.

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Given functions

$$f : E \times \mathbb{R}^{2+n} \to \mathbb{R}, \qquad \phi : E_0 \to \mathbb{R},$$

we consider the Cauchy problem

$$D_x z(x,y) = f(x,y,z(x,y),(Wz)(x,y),D_y z(x,y)) \quad \text{on } E, \tag{1}$$

$$z(x,y) = \phi(x,y) \quad \text{on } E_0. \tag{2}$$

where $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$ and $W : C(E_0 \cup E, \mathbb{R}) \to C(E, \mathbb{R})$ is some operator satisfying the *Volterra condition*. This means that for all $(x,y) \in E$ and $z, \overline{z} \in C(E_0 \cup E, \mathbb{R})$ if $z(t,s) = \overline{z}(t,s)$ for $(t,s) \in E_x$ then $(Wz)(x,y) = (W\overline{z})(x,y)$.

We will consider weak solutions of problem (1), (2). More precisely we call $z : E_c \to \mathbb{R}$, where $0 < c \le a$, a *solution* of (1), (2) if

(i) $z \in C(E_c, \mathbb{R})$ and the derivatives $D_y z(x,y) = (D_{y_1} z(x,y), \ldots, D_{y_n} z(x,y))$ exist for $(x,y) \in E_c^*$,

(ii) the function $z(\cdot, y) : I[c,y] \to \mathbb{R}$ is absolutely continuous for each $y \in (-b,b)$,

(iii) for each $y \in (-b,b)$ system (1) is satisfied for almost all $x \in I[c,y]$ and condition (2) holds.

In the theory of functional differential equations the existence results for initial value problems are obtained mainly by means of the method of successive approximations or the fixed point method. We mention the results of Myshkis and Slopak [18] and of Szarski [21] as classical references. From other results concerning classical (C^1) solutions we recall here those of Brandi and Ceppitelli [4, 5], Salvadori [20] and Jaruszewska-Walczak [15].

The existence result (global with respect to x) for generalized (in the "almost everywhere" sense) solutions of equations with deviated argument was obtained by Kamont and Zacharek [16] with the help of the difference method. An extension of this result to functional equations was given in [13]. For other concepts of a solution in non-functional setting we refer to Oleĭnik [19] with a survey on the best results obtained for distributional solutions of almost-linear problems and to Kiguradze [17] where a solution of a linear system is defined on the basis of Picone's canonical representation.

In this paper we use the method of bicharacteristics which was introduced and developed in non-functional setting by Cinquini Cibrario [10–12] for quasilinear as well as nonlinear problems. This method was adapted by Cesari [8, 9] and Bassanini [1, 2] to quasilinear systems in the second canonical form. Some extensions of Cesari's results to functional differential systems were given in [3, 14, 22]. The results obtained in the papers mentioned above by means of the method of bicharacteristics concern generalized solutions which are global with respect to the variable y. The local initial problem for non-functional semilinear systems in the second canonical form was investigated in [7]. Existence of generalized solutions to nonlinear functional differential equations was proved by Brandi, Kamont and Salvadori [6]. An existence result for such equations was also obtained by Brandi and Ceppitelli [5] by means of the method of successive approximations.

In this paper we deal with the problem in which the functional dependence of the differential equation is based on the use of an abstract operator of the Volterra type. Differential equations with a deviated argument and differential-integral equations are particular cases of (1). Note that since this equation is local with respect to y the model of functional dependence introduced in [6] is not suitable in our case. Analogously to [6] we use the method of bicharacteristics together with the method of successive approximations for a certain functional integral system.

2. Bicharacteristics. For $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we set $\|y\| = \sum_{i=1}^n |y_i|$ and for a matrix $A = [a_{ij}]_{i,j=1,\ldots,n}$ we put $\|A\| = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$. Let $C^{0,1}(E_x, \mathbb{R})$ be the set of all functions $z \in C(E_x, \mathbb{R})$ of the variables $(t, s) = (t, s_1, \ldots, s_n)$ such that the derivatives $D_s z = (D_{s_1} z, \ldots, D_{s_n} z)$ exist and are continuous on E_x. If $\|\cdot\|_{(x;0)}$ denotes the supremum norm in the space $C(E_x, \mathbb{R})$, where $0 < x \le a$, then the norm in $C^{0,1}(E_x, \mathbb{R})$ is defined by $\|z\|_{(x;1)} = \|z\|_{(x;0)} + \|D_s z\|_{(x;0)}$. For any $w \in C(E_x, \mathbb{R}^m)$ let

$$\|w\|_{(x;L)} = \sup\{\|z(t,s) - z(t,\overline{s})\| \cdot \|s - \overline{s}\|^{-1} : (t,s), (t,\overline{s}) \in E_x\}.$$

Putting $\|z\|_{(x;0.L)} = \|z\|_{(x;0)} + \|z\|_{(x;L)}$, $\|z\|_{(x;1.L)} = \|z\|_{(x;1)} + \|D_s z\|_{(x;L)}$, we denote by $C^{0,i+L}(E_x, \mathbb{R})$, $i = 0, 1$, the space of all functions $z \in C^{0,i}(E_x, \mathbb{R})$ such that $\|z\|_{(x;i.L)} < \infty$ with the norm $\|\cdot\|_{(x;i.L)}$. Analogously we define the spaces $C^{0,i}(E_x^*, \mathbb{R})$, $C^{0,i+L}(E_x^*, \mathbb{R})$, for $i = 0, 1$.

Assumption $H[\phi]$. (i) $\phi \in C(E_0, \mathbb{R})$ and the derivatives $D_y\phi = (D_{y_1}\phi, \ldots, D_{y_n}\phi)$ exist on E_0;

(ii) there are constants $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathbb{R}_+$ such that

$$|\phi(x,y)| \le \Lambda_0, \quad |D_y\phi(x,y)| \le \Lambda_1 \quad \text{on } E_0,$$
$$|D_y\phi(x,y) - D_y\phi(x,\overline{y})| \le \Lambda_2 |y - \overline{y}| \quad \text{for } (x,y), (x,\overline{y}) \in E_0.$$

We define two function spaces such that the solution z of (1) will belong to the first space, while $D_y z$ to the second.

Suppose that $c \in (0, a]$, $Q = (Q_0, Q_1, Q_2) \in \mathbb{R}_+^3$, where $Q_i \ge \Lambda_i$ for $i = 0, 1, 2$ and $\mu = (\mu_1, \mu_2) \in L([0, c], \mathbb{R}_+^2)$. If ϕ satisfies Assumption $H[\phi]$ then we denote by $C_{c.\phi}^{0,1+L}(Q, \mu)$ the set of all functions $z \in C^{0,1}(E_c, \mathbb{R})$ such that

(i) $z(x,y) = \phi(x,y)$ on E_0;

(ii) $|z(x,y)| \le Q_0$ and $\|D_y z(x,y)\| \le Q_1$ on E_c^*;

(iii) for $(x,y),(\overline{x},y),(\overline{x},\overline{y})\in E_c^*$ we have

$$|z(x,y)-z(\overline{x},y)|\le\Big|\int_x^{\bar{x}}\mu_1(\tau)\,d\tau\Big|,$$

$$\|D_yz(x,y)-D_yz(\overline{x},\overline{y})\|\le\Big|\int_x^{\bar{x}}\mu_2(\tau)\,d\tau\Big|+Q_2\|y-\overline{y}\|.$$

Suppose that $c\in(0,a]$, $P=(P_0,P_1)\in\mathbb{R}_+^2$, where $P_i\ge\Lambda_{i+1}$ for $i=0,1$, and $\nu\in L([0,c],\mathbb{R}_+)$. We denote by $C_c^{0,L}(P,\nu)$ the set of all functions $u\in C(E_c^*,\mathbb{R})$ such that

(i) $|u(x,y)|\le P_0$ on E_c^*;

(ii) for $(x,y),(\overline{x},\overline{y})\in E_c^*$ we have

$$|u(x,y)-u(\overline{x},\overline{y})|\le\Big|\int_x^{\bar{x}}\nu(\tau)\,d\tau\Big|+P_1\|y-\overline{y}\|.$$

Assumption $H[W]$. $W:C^{0,1+L}(E\cup E_0,\mathbb{R})\to C^{0,1+L}(E,\mathbb{R})$ and there are constants $A_i,B_i,L_j\in\mathbb{R}_+$, $i=0,1,2$, $j=0,1$, such that for all $z,\overline{z}\in C^{0,1+L}(E\cup E_0,\mathbb{R})$ and $x\in(0,a]$ we have

$$\|Wz\|_{(x;0)}\le A_0+B_0\|z\|_{(x;0)},\qquad \|D_y(Wz)\|_{(x;0)}\le A_1+B_1\|D_yz\|_{(x;0)},$$
$$\|D_y(Wz)\|_{(x;L)}\le A_2+B_2\|D_yz\|_{(x;L)},$$
$$\|Wz-W\overline{z}\|_{(x;0)}\le L_0\|z-\overline{z}\|_{(x;0)},$$
$$\|D_y(Wz)-D_y(W\overline{z})\|_{(x;0)}\le L_1\|D_yz-D_y\overline{z}\|_{(x;0)}.$$

Remark 2.1. From Assumption $H[W]$ it follows that W satisfies the Volterra condition. Although W is defined on the space $C^{0,1+L}(E\cup E_0,\mathbb{R})$, we may also define Wz for $z\in C(E_c,\mathbb{R})$, where $c\in(0,a]$, by the formula

$$(Wz)(x,y)=(W\widetilde{z})(x,y)\qquad\text{for }(x,y)\in E,$$

where $\widetilde{z}$ is any extension of z into the set $E\cup E_0$. It follows from the Volterra condition that the above definition does not depend on the extension of z.

Assumption $H[D_qf]$. The function $f:E\times\mathbb{R}^{2+n}\to\mathbb{R}$ of the variables (x,y,p,w,q) is such that

(i) the derivative $D_qf=(D_{q_1}f,\ldots,D_{q_n}f)$ exists on $E\times\mathbb{R}^{2+n}$ and for every $(y,p,w,q)\in[-b,b]\times\mathbb{R}^{2+n}$ we have $D_qf(\cdot,y,p,w,q)\in L(I[a,y],\mathbb{R}^n)$;

(ii) there is $\gamma=(\gamma_1,\ldots,\gamma_n)\in L([0,a],\mathbb{R}_+^n)$ such that

$$|D_{q_i}f(x,y,p,w,q)|\le\gamma_i(x),\qquad 1\le i\le n,\ \text{ on }E\times\mathbb{R}^{2+n};$$

(iii) there exists $\beta\in L([0,a],\mathbb{R}_+)$ such that

$$\|D_qf(x,y,p,w,q)-D_qf(x,\overline{y},\overline{p},\overline{w},\overline{q})\|$$
$$\le\beta(x)[\|y-\overline{y}\|+|p-\overline{p}|+|w-\overline{w}|+\|q-\overline{q}\|]$$

for $(x,y),(x,\overline{y})\in E_a$, $p,\overline{p},w,\overline{w}\in\mathbb{R}$, $q,\overline{q}\in\mathbb{R}^n$;

(iv) for $x \in [0, a]$ we have

$$M(x) \geq \int_0^x \gamma(\tau)\, d\tau.$$

We now give the notion of bicharacteristics for system (1). Suppose that Assumption $H[\phi]$ holds and that $z, \overline{z} \in C^{0,1+L}_{c.\phi}(Q, \mu)$, $u, \overline{u} \in C^{0,L}_{c}(P, \nu)$, where $c \in (0, a]$.

We consider the Cauchy problem

$$\text{(3)} \quad \eta'(t) = -D_q f(t, \eta(t), z(t, \eta(t)), (Wz)(t, \eta(t)), u(t, \eta(t))), \qquad \eta(x) = y,$$

where $(x, y) \in E^*_c$. Let $g[z, u](\cdot, x, y) = (g_1[z, u](\cdot, x, y), \ldots, g_n[z, u](\cdot, x, y))$ denote the solution of problem (3).

Write

$$R_1 = 1 + Q_1 + A_1 + B_1 Q_1 + P_1, \qquad \Upsilon(t, x) = \exp\Big\{R_1 \Big| \int_t^x \|\gamma(\tau)\|\, d\tau \Big|\Big\}.$$

LEMMA 2.2. *Suppose that Assumptions $H[\phi]$, $H[W]$ and $H[D_q f]$ are satisfied and that $z, \overline{z} \in C^{0,1+L}_{c.\phi}(Q, \mu)$ and $u, \overline{u} \in C^{0,L}_{c}(P, \nu)$, where $c \in (0, a]$. Then there exist unique solutions $g[z, u](\cdot, x, y)$ and $g[\overline{z}, \overline{u}](\cdot, x, y)$, which are defined on intervals $[0, c(x, y)]$ and $[0, \overline{c}(x, y)]$ such that*

$$(c(x, y), g[z, u](c(x, y), x, y)) \in \partial_0 E^*_c,$$
$$(\overline{c}(x, y), g[\overline{z}, \overline{u}](\overline{c}(x, y), x, y)) \in \partial_0 E^*_c,$$

where $\partial_0 E^*_c = \{(x, y) \in E^*_c : |y_i| = b_i - M_i(x)$ for some $i = 1, \ldots, n\}$. Moreover, we have the estimates

$$\text{(4)} \quad \|g[z, u](t, x, y) - g[z, u](t, \overline{x}, \overline{y})\| \leq \Upsilon(t, x)\Big\{\Big| \int_x^{\bar{x}} \|\gamma(\tau)\|\, d\tau \Big| + \|y - \overline{y}\|\Big\},$$

where $(x, y), (\overline{x}, \overline{y}) \in E^*_c$, $t \in [0, \min\{c(x, y), c(\overline{x}, \overline{y})\}]$ and

$$\text{(5)} \quad \|g[z, u](t, x, y) - g[\overline{z}, \overline{u}](t, x, y)\| \\ \leq \Upsilon(t, x) \Big| \int_x^t \beta(\tau)\{(1 + L_0)\|z - \overline{z}\|_{(\tau;0)} + \|u - \overline{u}\|_{(\tau;0)}\}\, d\tau \Big|,$$

where $(x, y) \in E^*_c$ and $t \in [0, \min\{c(x, y), \overline{c}(x, y)\}]$.

Proof. The existence and uniqueness of solutions of (3) follows from classical theorems since the right hand side of the system is Lipchitzian with respect to the unknown function and it satisfies the Carathéodory conditions.

If we transform (3) into an integral equation, then by Assumptions $H[D_q f]$ and $H[W]$ we have

$$
\begin{aligned}
&\|g[z,u](t,x,y)-g[z,u](t,\overline{x},\overline{y})\| \\
&\quad \le \|y-\overline{y}\| + \Big|\int_x^{\bar{x}} \|D_q f(P[z,u](\tau,\overline{x},\overline{y}))\|\,d\tau\Big| \\
&\qquad + \Big|\int_x^{t} \|D_q f(P[z,u](\tau,x,y)) - D_q f(P[z,u](\tau,\overline{x},\overline{y}))\|\,d\tau\Big| \\
&\quad \le \|y-\overline{y}\| + \Big|\int_x^{\bar{x}} \|\gamma(\tau)\|\,d\tau\Big| + \Big|\int_x^{t} \beta(\tau)\{\|g[z,u](\tau,x,y)-g[z,u](\tau,\overline{x},\overline{y})\| \\
&\qquad + |z(\tau,g[z,u](\tau,x,y)) - z(\tau,g[z,u](\tau,\overline{x},\overline{y}))| \\
&\qquad + |(Wz)(\tau,g[z,u](\tau,x,y)) - (Wz)(\tau,g[z,u](\tau,\overline{x},\overline{y}))| \\
&\qquad + \|u(\tau,g[z,u](\tau,x,y)) - u(\tau,g[z,u](\tau,\overline{x},\overline{y}))\|\}\,d\tau\Big| \\
&\quad \le |y-\overline{y}| + \Big|\int_x^{\bar{x}} \|\gamma(\tau)\|\,d\tau\Big| + \Big|\int_x^{t} \beta(\tau)R_1\|g[z,u](\tau,x,y)-g[z,u](\tau,\overline{x},\overline{y})\|\,d\tau\Big|
\end{aligned}
$$

for $(x,y),(\overline{x},\overline{y})\in E_c^*$ and $t\in[0,\min\{c(x,y),c(\overline{x},\overline{y})\}]$, where

$$
\begin{aligned}
(6)\qquad P[z,u](t,x,y) = {}&(t,g[z,u](t,x,y),z(t,g[z,u](t,x,y)),\\
&(Wz)(t,g[z,u](t,x,y)),u(t,g[z,u](t,x,y))).
\end{aligned}
$$

Thus (4) follows from the Gronwall lemma.

Analogously we get by Assumptions $H[W]$ and $H[D_q f]$ the estimate

$$
\begin{aligned}
&\|g[z,u](t,x,y)-g[\overline{z},\overline{u}](t,x,y)\| \\
&\qquad \le \Big|\int_x^{t} \beta(\tau)\{\|z-\overline{z}\|_{(\tau;0)} + L_0\|z-\overline{z}\|_{(\tau;0)} + \|u-\overline{u}\|_{(\tau;0)}\}\,d\tau\Big| \\
&\qquad\quad + \Big|\int_x^{t} \beta(\tau)R_1\|g[z,u](\tau,x,y)-g[\overline{z},\bar{u}](\tau,x,y)\|\,d\tau\Big|
\end{aligned}
$$

for $(x,y)\in E_c^*$ and $t\in[0,\min\{c(x,y),\overline{c}(x,y)\}]$. Now, again using the Gronwall lemma we get (5), which completes the proof of Lemma 2.2.

3. Integral operators and their properties. Now we formulate further assumptions on f.

Assumption $H[f]$. The function $f : E\times\mathbb{R}^{2+n}\to\mathbb{R}$ of the variables (x,y,p,w,q) satisfies Assumption $H[D_q f]$ and

(i) there exists $\delta\in L([0,a],\mathbb{R}_+)$ such that $|f(x,y,p,w,q)|\le\delta(x)$ on $E\times\mathbb{R}^{2+n}$;

(ii) the derivatives $D_y f = (D_{y_1} f, \dots, D_{y_n} f)$, $D_p f$, $D_w f$ exist on $E \times \mathbb{R}^{2+n}$ and for every $(y, p, w, q) \in [-b, b] \times \mathbb{R}^{2+n}$ we have $D_y f(\cdot, y, p, w, q) \in L(I[a, y], \mathbb{R}^n)$, $D_p f(\cdot, y, p, w, q) \in L(I[a, y], \mathbb{R})$ and $D_w f(\cdot, y, p, w, q) \in L(I[a, y], \mathbb{R})$;

(iii) there is $\alpha \in L([0, a], \mathbb{R}_+)$ such that

$$\|D_y f(x, y, p, w, q)\| \le \alpha(x), \qquad |D_p f(x, y, p, w, q)| \le \alpha(x),$$
$$|D_w f(x, y, p, w, q)| \le \alpha(x),$$

on $E \times \mathbb{R}^{2+n}$;

(iv) for $(x, y), (x, \overline{y}) \in E_a$, $p, \overline{p}, w, \overline{w} \in \mathbb{R}$ and $q, \overline{q} \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\|D_y f(x, y, p, w, q) - D_y f(x, \overline{y}, \overline{p}, \overline{w}, \overline{q})\| \\
&\qquad \le \beta(x)[\|y - \overline{y}\| + |p - \overline{p}| + |w - \overline{w}| + \|q - \overline{q}\|], \\
&|D_p f(x, y, p, w, q) - D_p f(x, \overline{y}, \overline{p}, \overline{w}, \overline{q})| \\
&\qquad \le \beta(x)[\|y - \overline{y}\| + |p - \overline{p}| + |w - \overline{w}| + \|q - \overline{q}\|], \\
&|D_w f(x, y, p, w, q) - D_w f(x, \overline{y}, \overline{p}, \overline{w}, \overline{q})| \\
&\qquad \le \beta(x)[\|y - \overline{y}\| + |p - \overline{p}| + |w - \overline{w}| + \|q - \overline{q}\|].
\end{aligned}$$

If Assumptions $H[\phi]$, $H[W]$ and $H[f]$ are satisfied then for given $z \in C^{0,1+L}_{c.\phi}(Q, \mu)$ and $u \in C^{0,L}_c(P, \nu)$ we define the operators $T[z, u]$ and $V_i[z, u]$, $i = 1, \dots, n$, by

$$\begin{aligned}
T[z, u](x, y) &= \phi(0, g[z, u](0, x, y)) + \int_0^x [f(P[z, u](\tau, x, y)) \\
&\quad - \sum_{j=1}^n D_{q_j} f(P[z, u](\tau, x, y)) u_j(\tau, g[z, u](\tau, x, y))]\, d\tau, \\
V_i[z, u](x, y) &= D_{y_i}\phi(0, g[z, u](0, x, y)) + \int_0^x [D_{y_i} f(P[z, u](\tau, x, y)) \\
&\quad + D_p f(P[z, u](\tau, x, y)) u_i(\tau, g[z, u](\tau, x, y)) \\
&\quad + D_w f(P[z, u](\tau, x, y)) D_{y_i}(Wz)(\tau, g[z, u](\tau, x, y))]\, d\tau
\end{aligned}$$

for $(x, y) \in E_c^*$, and

$$T[z, u](x, y) = \phi(x, y) \quad \text{for } (x, y) \in E_0,$$

where $g[z, u]$ is a solution of (3) and $P[z, u]$ is given by (6). We will consider the system of integral-functional equations

$$z = T[z, u], \qquad u = V[z, u], \tag{7}$$

where $V[z, u] = (V_1[z, u], \dots, V_n[z, u])$.

Remark 3.1. The integral-functional system (7) arises in the following way. We introduce an additional unknown function $u = D_y z$ in (1). Then we consider the linearization of (1) with respect to u, which yields

$$D_x z(x,y)=f(P)+\sum_{j=1}^{n} D_{q_j} f(P)(D_{y_j} z(x,y)-u_j(x,y)), \tag{8}$$

where $P=(x,y,z(x,y),(Wz)(x,y),u(x,y))$. Differentiating (1) with respect to y_i and substituting $u=D_y z$ we get

$$\begin{aligned} D_x u_i(x,y) &= D_{y_i} f(P)+D_p f(P) u_i(x,y) \\ &\quad + D_w f(P) D_{y_i}(Wz)(x,y)+\sum_{j=1}^{n} D_{q_j} f(P) D_{y_i} u_j(x,y), \quad i=1,\ldots,n. \end{aligned} \tag{9}$$

Making use of (3) we have

$$\begin{aligned} &\frac{d}{d\tau} z(\tau, g[z,u](\tau,x,y)) \\ &= D_x z(\tau, g[z,u](\tau,x,y))-\sum_{j=1}^{n} D_{q_j} f(P[z,u](\tau,x,y)) D_{y_j} z(\tau, g[z,u](\tau,x,y)). \end{aligned}$$

Substituting (8) in the above relation, integrating the result with respect to t on $[0,x]$ and taking into account that $z=\phi$ we get the first equation of (7) on E_c^*. Repeating these considerations for (9) we get the second equation of (7).

Under Assumptions $H[\phi]$, $H[W]$ and $H[f]$ we prove that the solution of (6) exists by the method of successive approximations. We define a sequence $\{z^{(m)}, u^{(m)}\}$ in the following way.

1. We put

$$z^{(0)}(x,y)=\begin{cases} \phi(x,y) & \text{on } E_0, \\ \phi(0,y) & \text{on } E_c^*, \end{cases} \qquad u^{(0)}(x,y)=D_y \phi(0,y) \quad \text{on } E_c^*; \tag{10}$$

then $z^{(0)} \in C_{c.\phi}^{0,1+L}(Q,\mu)$ and $u^{(0)} \in C_c^{0,L}(P,\nu)$.

2. If $z^{(m)} \in C_{c.\phi}^{0,1+L}(Q,\mu)$ and $u^{(m)} \in C_c^{0,L}(P,\nu)$ are already defined then $u^{(m+1)}$ is a solution of the equation

$$u=V^{(m)}[z^{(m)},u], \tag{11}$$

and

$$z^{(m+1)}=T[z^{(m)},u^{(m+1)}], \tag{12}$$

where $V^{(m)}[z^{(m)},u]=(V_1^{(m)}[z^{(m)},u],\ldots,V_n^{(m)}[z^{(m)},u])$ is defined by

(13)
$$
\begin{aligned}
V_i^{(m)}&[z^{(m)},u](x,y)\\
&= D_{y_i}\phi(0,g[z^{(m)},u](0,x,y)) + \int_0^x [D_{y_i} f(P[z^{(m)},u](\tau,x,y))\\
&\quad + D_p f(P[z^{(m)},u](\tau,x,y)) u_i^{(m)}(\tau,g[z^{(m)},u](\tau,x,y))\\
&\quad + D_w f(P[z^{(m)},u](\tau,x,y)) D_{y_i}(Wz^{(m)})(\tau,g[z^{(m)},u](\tau,x,y))]\,d\tau
\end{aligned}
$$

for $(x,y)\in E_c^*$.

Remark 3.2. Since $V[z^{(m)},\cdot]$ and $V^{(m)}[z^{(m)},\cdot]$ are not the same operator we explain how system (13) is obtained. If $z^{(m)}\in C^{0,1+L}_{c.\phi}(Q,\mu)$ and $u^{(m)}\in C^{0,L}_c(P,\nu)$ are known functions then replacing z with $z^{(m)}$ in (9) we get

$$
\begin{aligned}
D_x u_i(x,y) &= D_{y_i} f(P^{(m)}) + D_p f(P^{(m)}) D_{y_i} z^{(m)}(x,y)\\
&\quad + D_w f(P^{(m)}) D_{y_i}(Wz^{(m)})(x,y)\\
&\quad + \sum_{j=1}^n D_{q_j} f(P^{(m)}) D_{y_i} u_j(x,y), \qquad i=1,\dots,n,
\end{aligned}
$$

where $P^{(m)}=(x,y,z^{(m)}(x,y),(Wz^{(m)})(x,y),u(x,y))$. If we assume that $D_y z^{(m)}=u^{(m)}$ (see Theorem 5.1), then by integrating the above system along the bicharacteristic $g[z^{(m)},u](\cdot,x,y)$ on the interval $[0,x]$ we get (13).

Write

$$
\begin{aligned}
&\Gamma_0(x) = \Lambda_1 + \int_0^x \alpha(\tau)\,d\tau,\\
&\widetilde{\Gamma}_0(x) = \Lambda_1 \Upsilon(0,x) + P_0 + \int_0^x \{\alpha(\tau)S_1 + \beta(\tau)P_0R_1 + \|\gamma(\tau)\|2P_1\}\Upsilon(\tau,x)\,d\tau,\\
&\Gamma_1(x) = \Lambda_2\Upsilon(0,x) + \int_0^x \{\beta(\tau)R_1S_1 + \alpha(\tau)S_2\}\Upsilon(\tau,x)\,d\tau,\\
&G(x) = \Lambda_2\Upsilon(0,x)\beta(x) + [\beta(x)R_1S_1 + \alpha(x)S_2]\Upsilon(0,x)\int_0^x \beta(\tau)\,d\tau + \beta(x)S_1,
\end{aligned}
$$

where

$$S_1 = 1 + P_0 + A_1 + B_1Q_1, \qquad S_2 = P_1 + A_2 + B_2Q_2.$$

With the above notation we define

$$\mu_1(t) = \delta(t) + \widetilde{\Gamma}_0(c)\|\gamma(t)\|, \qquad \mu_2(t) = \nu(t) = \Gamma_1(c)\|\gamma(t)\| + S_1\alpha(t).$$

Assumption $H[Q,P]$. (i) $Q_i > \Lambda_i$ for $i=0,1,2$, and $P_i = Q_{i+1}$ for $i=0,1$;

(ii) the constant $c \in (0, a]$ is sufficiently small in order that

$$\Lambda_0 + \int_0^c [\delta(\tau) + \|\gamma(\tau)\| P_0]\, d\tau \le Q_0, \qquad \int_0^c G(\tau)\, d\tau < 1,$$

$$\Gamma_0(c) \le Q_1 = P_0, \qquad \Gamma_1(c) \le Q_2 = P_1.$$

4. Existence of successive approximations. The problem of the existence of the sequence $\{z^{(m)}, u^{(m)}\}$ is the main difficulty in our method. We prove that this sequence exists provided c, $0 < c \le a$, is sufficiently small.

THEOREM 4.1. *If Assumptions $H[\phi]$, $H[W]$, $H[f]$ and $H[Q,P]$ are satisfied then for any $m \in \mathbb{N}$,*

(I_m) *$z^{(m)}$, $u^{(m)}$ are defined on E_c, E_c^*, respectively, and we have $z^{(m)} \in C^{0,1+L}_{c.\phi}(Q, \mu)$, $u^{(m)} \in C^{0,L}_c(P, \nu)$;*

(II_m) *$D_y z^{(m)}(x, y) = u^{(m)}(x, y)$ on E_c^*.*

Proof. We prove (I_m) and (II_m) by induction. It follows from (10) that (I_0) and (II_0) are satisfied. Suppose that conditions (I_m) and (II_m) hold for some $m \in \mathbb{N}$. We first prove that there exists a solution $u^{(m+1)} : E_c^* \to \mathbb{R}^n$ of (11) and that $u^{(m+1)} \in C^{0,L}_c(P, \nu)$.

We claim that given $z^{(m)} \in C^{0,1+L}_{c.\phi}(Q, \mu)$ the operator $V[z^{(m)}, \cdot]$ maps $C^{0,L}_c(P, \nu)$ into itself. For simplicity of notation we ignore the dependence of g and P on $z^{(m)}$ and u. It follows from Assumptions $H[W]$, $H[f]$ and (4) that given $u \in C^{0,L}_c(P, \nu)$ we have for all $(x, y), (\overline{x}, \overline{y}) \in E_c^*$ the estimates

$$|V^{(m)}[z^{(m)}, u](x, y)| \le \Lambda_1 + \int_0^x \alpha(\tau) S_1\, d\tau$$

and

$$\begin{aligned}
&|V^{(m)}[z^{(m)}, u](x, y) - V^{(m)}[z^{(m)}, u](\overline{x}, \overline{y})| \\
&\quad \le \Lambda_2 \Upsilon(0, x) \Big\{ \Big| \int_x^{\bar{x}} \|\gamma(\tau)\|\, d\tau \Big| + \|y - \overline{y}\| \Big\} + \Big| \int_x^{\bar{x}} \alpha(\tau) S_1\, d\tau \Big| \\
&\qquad + \Big\{ \Big| \int_x^{\bar{x}} \|\gamma(\tau)\|\, d\tau \Big| + \|y - \overline{y}\| \Big\} \cdot \int_0^x \{\beta(\tau) R_1 S_1 + \alpha(\tau) S_2\} \Upsilon(\tau, x)\, d\tau.
\end{aligned}$$

Hence by Assumption $H[Q,P]$ we get

$$|V^{(m)}[z^{(m)}, u](x, y)| \le P_0,$$

$$|V^{(m)}[z^{(m)}, u](x, y) - V^{(m)}[z^{(m)}, u](\overline{x}, \overline{y})| \le \Big\{ \Big| \int_x^{\bar{x}} \nu(\tau)\, d\tau \Big| + P_1 |y - \overline{y}| \Big\} \tag{14}$$

for $(x, y), (\overline{x}, \overline{y}) \in E_c^*$. Thus $V^{(m)}[z^{(m)}, \cdot]$ maps $C^{0,L}_c(P, \nu)$ into itself.

If $u,\overline{u}\in C_c^{0,L}(P,\nu)$ then analogously by Assumptions $H[f]$, $H[W]$ and (5), we get

$$\|V^{(m)}[z^{(m)},u]-V^{(m)}[z^{(m)},\overline{u}]\|_{(c;0)}\le\int_0^c G(\tau)\|u-\overline{u}\|_{(\tau;0)}\,d\tau.$$

Thus Assumption $H[Q,P]$ yields that $V^{(m)}[z^{(m)},\cdot]$ is a contraction for the norm $\|\cdot\|_{(c;0)}$. By the Banach fixed point theorem there exists a unique solution $u\in C_c^{0,L}(P,\nu)$ of (11), which is $u^{(m+1)}$.

Our next goal is to prove that $z^{(m+1)}$ given by (12) satisfies (II_{m+1}). For $x\in[0,c]$ and $y,\overline{y}\in S_x$, where $S_x=[-b+M(x),b-M(x)]$, put

$$\Delta(x,y,\overline{y})=z^{(m+1)}(x,y)-z^{(m+1)}(x,\overline{y})-u^{(m+1)}(x,y)(y-\overline{y}).$$

By the Hadamard mean value theorem we have

$$\begin{aligned}
&\Delta(x,y,\overline{y})\\
&\quad=\phi(0,g(0,x,y))-\phi(0,g(0,x,\overline{y}))-D_y\phi(0,g(0,x,y))(y-\overline{y})\\
&\qquad+\int_0^x\int_0^1 D_yf(Q(s,\tau))[g(\tau,x,y)-g(\tau,x,\overline{y})]\,ds\,d\tau\\
&\qquad+\int_0^x\int_0^1 D_pf(Q(s,\tau))[z^{(m)}(\tau,g(\tau,x,y))-z^{(m)}(\tau,g(\tau,x,\overline{y}))]\,ds\,d\tau\\
&\qquad+\int_0^x\int_0^1 D_wf(Q(s,\tau))[(Wz^{(m)})(\tau,g(\tau,x,y))-(Wz^{(m)})(\tau,g(\tau,x,\overline{y}))]\,ds\,d\tau\\
&\qquad+\int_0^x\int_0^1 D_qf(Q(s,\tau))[u^{(m+1)}(\tau,g(\tau,x,y))-u^{(m+1)}(\tau,g(\tau,x,\overline{y}))]\,ds\,d\tau\\
&\qquad-\int_0^x\{D_qf(P(\tau,x,y))u^{(m+1)}(\tau,g(\tau,x,y))\\
&\qquad-D_qf(P(\tau,x,\overline{y}))u^{(m+1)}(\tau,g(\tau,x,\overline{y}))\}\,d\tau\\
&\qquad-\int_0^x\{D_yf(P(\tau,x,y))+D_pf(P(\tau,x,y))u^{(m)}(\tau,g(\tau,x,y))\\
&\qquad+D_wf(P(\tau,x,y))D_y(Wz^{(m)})(\tau,g(\tau,x,y))\}\,d\tau\,(y-\overline{y}),
\end{aligned}$$

where $Q(s,\tau)=sP(\tau,x,y)+(1-s)P(\tau,x,\overline{y})$. Define

$$\begin{aligned}
\Delta_0(x,y,\overline{y})&=\phi(0,g(0,x,y))-\phi(0,g(0,x,\overline{y}))\\
&\quad-D_y\phi(0,g(0,x,y))[g(0,x,y)-g(0,x,\overline{y})],
\end{aligned}$$

$$\begin{aligned}
\Delta_1(x,y,\overline{y}) = {} & \int_0^x\int_0^1 [D_y f(Q(s,\tau)) - D_y f(P(\tau,x,y))] \\
& \cdot [g(\tau,x,y) - g(\tau,x,\overline{y})]\,ds\,d\tau, \\
\Delta_2(x,y,\overline{y}) = {} & \int_0^x\int_0^1 [D_p f(Q(s,\tau)) - D_p f(P(\tau,x,y))] \\
& \cdot [z^{(m)}(\tau,g(\tau,x,y)) - z^{(m)}(\tau,g(\tau,x,\overline{y}))]\,ds\,d\tau, \\
\Delta_3(x,y,\overline{y}) = {} & \int_0^x\int_0^1 [D_w f(Q(s,\tau)) - D_w f(P(\tau,x,y))] \\
& \cdot [(Wz^{(m)})(\tau,g(\tau,x,y)) - (Wz^{(m)})(\tau,g(\tau,x,\overline{y}))]\,ds\,d\tau, \\
\Delta_4(x,y,\overline{y}) = {} & \int_0^x\int_0^1 [D_q f(Q(s,\tau)) - D_q f(P(\tau,x,\overline{y}))] \\
& \cdot [u^{(m+1)}(\tau,g(\tau,x,y)) - u^{(m+1)}(\tau,g(\tau,x,\overline{y}))]\,ds\,d\tau, \\
\Delta_5(x,y,\overline{y}) = {} & \int_0^x D_p f(P(\tau,x,y))[z^{(m)}(\tau,g(\tau,x,y)) - z^{(m)}(\tau,g(\tau,x,\overline{y})) \\
& - u^{(m)}(\tau,g(\tau,x,y))[g(\tau,x,y) - g(\tau,x,\overline{y})]]\,d\tau, \\
\Delta_6(x,y,\overline{y}) = {} & \int_0^x D_w f(P(\tau,x,y))[(Wz^{(m)})(\tau,g(\tau,x,y)) \\
& - (Wz^{(m)})(\tau,g(\tau,x,\overline{y})) \\
& - D_y(Wz^{(m)})(\tau,g(\tau,x,y))[g(\tau,x,y) - g(\tau,x,\overline{y})]]\,d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Delta}_0(x,y,\overline{y}) = {} & D_y\phi(0,g(0,x,y))[g(0,x,y) - g(0,x,\overline{y}) - (y-\overline{y})], \\
\widetilde{\Delta}_1(x,y,\overline{y}) = {} & \int_0^x D_y f(P(\tau,x,y))[g(\tau,x,y) - g(\tau,x,\overline{y}) - (y-\overline{y})]\,d\tau \\
& + \int_0^x D_p f(P(\tau,x,y))u^{(m)}(\tau,g(\tau,x,y)) \\
& \cdot [g(\tau,x,y) - g(\tau,x,\overline{y}) - (y-\overline{y})]\,d\tau \\
& + \int_0^x D_w f(P(\tau,x,y))D_y(Wz^{(m)})(\tau,g(\tau,x,y)) \\
& \cdot [g(\tau,x,y) - g(\tau,x,\overline{y}) - (y-\overline{y})]\,d\tau, \\
\widetilde{\Delta}_2(x,y,\overline{y}) = {} & -\int_0^x [D_q f(P(\tau,x,y)) - D_q f(P(\tau,x,\overline{y}))]u^{(m+1)}(\tau,g(\tau,x,y))\,d\tau.
\end{aligned}$$

With the above definitions we have

$$\Delta(x,y,\overline{y}) = \sum_{i=0}^{6} \Delta_i(x,y,\overline{y}) + \sum_{i=0}^{2} \widetilde{\Delta}_i(x,y,\overline{y}). \tag{15}$$

Since $g(\cdot,x,y)$ is a solution of (3) we see that

$$g(\tau,x,y) - g(\tau,x,\overline{y}) - (y-\overline{y}) = \int_{\tau}^{x} [D_q f(P(\xi,x,y)) - D_q f(P(\xi,x,\overline{y}))]\,d\xi.$$

Substituting the above relation in $\widetilde{\Delta}_1$ and in $\widetilde{\Delta}_0$ with $\tau = 0$, and changing the order of integrals where necessary, we get

$$\begin{aligned}
&\sum_{i=0}^{2} \widetilde{\Delta}_i(x,y,\overline{y}) \\
&\quad= \int_0^x [D_q f(P(\tau,x,y)) - D_q f(P(\tau,x,\overline{y}))]\Big[D_y\phi(0,g(0,x,y)) \\
&\qquad+ \int_0^\tau D_y f(P(\xi,x,y))\,d\xi + \int_0^\tau D_p f(P(\xi,x,y))u^{(m)}(\xi,g(\xi,x,y))\,d\xi \\
&\qquad+ \int_0^\tau D_w f(P(\xi,x,y))D_y(Wz^{(m)})(\xi,g(\xi,x,y))\,d\xi \\
&\qquad- u^{(m+1)}(\tau,g(\tau,x,y))\Big]\,d\tau \\
&\quad= \int_0^x [D_q f(P(\tau,x,y)) - D_q f(P(\tau,x,\overline{y}))] \\
&\qquad\cdot [V^{(m)}[z^{(m)},u^{(m+1)}](\tau,g(\tau,x,y)) - u^{(m+1)}(\tau,g(\tau,x,y))]\,d\tau = 0,
\end{aligned}$$

from which and from (15) we get $\Delta(x,y,\overline{y}) = \sum_{i=0}^{6} \Delta_i(x,y,\overline{y})$. In the above transformations we have used the following group property:

$$g(\xi,\tau,g(\tau,x,y)) = g(\xi,x,y) \quad \text{for } (x,y) \in E_c^*,\ \tau,\xi \in [0,c(x,y)].$$

Assumptions $H[f]$, $H[W]$, (4) and the existence of the derivatives $D_y\phi$, $D_y z^{(m)} = u^{(m)}$ and $D_y(Wz^{(m)})$ yield that for $x \in [0,c]$, $i = 0,5,6$, we have

$$\frac{1}{|y-\overline{y}|}\Delta_i(x,y,\overline{y}) \to 0 \quad \text{as } |y-\overline{y}| \to 0. \tag{16}$$

From Assumptions $H[f]$, $H[W]$ and (4) we get the existence of some constants C_i, $i = 1,2,3,4$, such that

$$|\Delta_i(x,y,\overline{y})| \le C_i|y-\overline{y}|^2, \quad x \in [0,c],\ y,\overline{y} \in S_x,\ i = 1,2,3,4.$$

This means that (16) also holds for $i = 1, 2, 3, 4$, which completes the proof of (II_{m+1}).

Finally, we prove that $z^{(m+1)}$ defined by (12) belongs to the class $C^{0,1+L}_{c.\phi}(Q,\mu)$. Since $D_y z^{(m+1)} = u^{(m+1)}$ it follows from (14) and from Assumption $H[Q,P]$ that

$$|D_y z^{(m+1)}(x,y)| \le Q_1,$$
$$|D_y z^{(m+1)}(x,y) - D_y z^{(m+1)}(\overline{x},\overline{y})| \le \Big|\int_x^{\bar{x}} \mu_2(\tau)\,d\tau\Big| + Q_2|y-\overline{y}|$$

for $(x,y),(\overline{x},\overline{y}) \in E^*_c$. By Assumptions $H[f]$, $H[W]$ and $H[Q,P]$ we easily get

$$|z^{(m+1)}(x,y)| \le Q_0, \quad |z^{(m+1)}(x,y) - z^{(m+1)}(\overline{x},y)| \le \Big|\int_x^{\bar{x}} \mu_1(\tau)\,d\tau\Big|$$

for $(x,y),(\overline{x},y) \in E^*_c$. This together with the relation $z^{(m+1)} = \phi$ on E_0 gives $z^{(m+1)} \in C^{0,1+L}_{c.\phi}(Q,\mu)$, which completes the proof of (I_{m+1}). Thus Theorem 4.1 follows by induction.

5. The existence theorem. Write

$$H(t) = (1+L_0)H^*(t) + H^*(t)\exp\Big\{\int_0^t G(\xi)\,d\xi\Big\}\int_0^t G^*(\xi)\,d\xi,$$

where

$$G^*(t) = \max\{(1+L_0)G(t),(1+L_1)\alpha(t)\},$$
$$H^*(t) = \Lambda_1\Upsilon(0,t)\beta(t) + [\beta(t)R_1P_0 + \alpha(t)S_1 + \|\gamma(t)\|2P_1]\Upsilon(0,t)\int_0^t \beta(\xi)\,d\xi$$
$$+ \alpha(t) + \beta(t)P_0 + \|\gamma(t)\|.$$

THEOREM 5.1. *If Assumptions $H[\phi]$, $H[f]$, $H[W]$ and $H[Q,P]$ are satisfied then the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ are uniformly convergent on E_c, E^*_c, respectively.*

Proof. For any $t \in [0,c]$ and $m \in \mathbb{N}$ we put

$$Z^{(m)}(t) = \sup\{|z^{(m)}(x,y) - z^{(m-1)}(x,y)| : (x,y) \in E_t\},$$
$$U^{(m)}(t) = \sup\{|u^{(m)}(x,y) - u^{(m-1)}(x,y)| : (x,y) \in E^*_t\}.$$

Using the same technique as in the proof of Theorem 4.1 we get by Assumptions $H[f]$, $H[W]$ and (5), for any $x \in [0,c]$ and $m \in \mathbb{N}$, the estimate

$$U^{(m+1)}(x) \le \int_0^x G(\tau)U^{(m+1)}(\tau)\,d\tau + \int_0^x [G(\tau)(1+L_0)Z^{(m)}(\tau) + \alpha(\tau)(1+L_1)U^{(m)}(\tau)]\,d\tau.$$

Making use of the Gronwall lemma we have

$$U^{(m+1)}(x) \le \exp\Big\{\int_0^x G(\tau)\,d\tau\Big\}\int_0^x G^*(\tau)[Z^{(m)}(\tau)+U^{(m)}(\tau)]\,d\tau. \tag{17}$$

By Assumptions $H[f]$, $H[W]$, (7) and (17) we get the estimate

$$\begin{aligned} Z^{(m+1)}(x) &\le \int_0^x H^*(\tau)[(1+L_0)Z^{(m)}(\tau)+U^{(m+1)}(\tau)]\,d\tau \\ &\le \int_0^x H(\tau)[Z^{(m)}(\tau)+U^{(m)}(\tau)]\,d\tau, \qquad x\in[0,c]. \end{aligned} \tag{18}$$

Thus if we take

$$M_c = \exp\Big\{\int_0^c G(\xi)d\xi\Big\}\int_0^c G^*(\xi)\,d\xi + H(c),$$

then using (17), (18) for any $x \in [0,c]$ we have

$$Z^{(m+1)}(x)+U^{(m+1)}(x) \le M_c\int_0^x [Z^{(m)}(\tau)+U^{(m)}(\tau)]\,d\tau.$$

Now, by induction it is easy to get

$$Z^{(m)}(x)+U^{(m)}(x) \le \frac{M_c^{m-1}x^{m-1}}{(m-1)!}[Z^{(1)}(c)+U^{(1)}(c)], \qquad x\in[0,c],$$

and consequently

$$\sum_{i=k}^{m}[Z^{(i)}(c)+U^{(i)}(c)] \le [Z^{(1)}(c)+U^{(1)}(c)]\sum_{i=k-1}^{m-1}\frac{M_c^i c^i}{i!}. \tag{19}$$

Since the series $\sum_{i=1}^{\infty} M_c^i c^i/i!$ is convergent it follows from (19) that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ satisfy the uniform Cauchy condition on E_c, E_c^*, respectively, which means that they are uniformly convergent. This completes the proof of Theorem 5.1.

THEOREM 5.2. *If Assumptions $H[\phi]$, $H[f]$, $H[W]$ and $H[Q,P]$ are satisfied then there is a solution of the problem* (1), (2).

P r o o f. It follows from Theorem 5.1 that there exist functions $\overline{z}, \overline{u}$ such that $\{z^{(m)}\}$, $\{u^{(m)}\}$ are uniformly convergent to $\overline{z}, \overline{u}$, respectively. Furthermore, $D_y\overline{z}$ exists on E_c^* and $D_y\overline{z} = \overline{u}$. We prove that $\overline{z}$ is a solution of (1).

From (9) it follows that for any $(x,y) \in E_c^*$, we have

$$\overline{z}(x,y) = \phi(0,\overline{g}(0,x,y)) + \int_0^x \Big[f(P[\overline{z}, D_y\overline{z}](\tau,x,y)) - \sum_{j=1}^n D_{q_j} f(P[\overline{z}, D_y\overline{z}](\tau,x,y)) D_{y_j}\overline{z}(\tau,x,y)\Big]\,d\tau, \tag{20}$$

where $\overline{g} = g[\overline{z}, D_y\overline{z}]$.

For a fixed x we define the transformation $y \mapsto \overline{g}(0,x,y) = \xi$. Then by the group property $\overline{g}(t,x,y) = \overline{g}(t,0,\xi)$ and by (20) we get

$$\begin{aligned}
&\overline{z}(x,\overline{g}(x,0,\xi)) = \phi(0,\xi) \\
&\quad + \int_0^x \Big[f(\tau,\overline{g}(\tau,0,\xi), \overline{z}(\tau,\overline{g}(\tau,0,\xi)), (W\overline{z})(\tau,\overline{g}(\tau,0,\xi)), D_y\overline{z}(\tau,\overline{g}(\tau,0,\xi))) \\
&- \sum_{j=1}^n D_{q_j} f(\tau,\overline{g}(\tau,0,\xi), \overline{z}(\tau,\overline{g}(\tau,0,\xi)), (Wz)(\tau,\overline{g}(\tau,0,\xi)), D_y\overline{z}(\tau,\overline{g}(\tau,0,\xi))) \\
&\qquad\qquad D_{y_j}\overline{z}(\tau,\overline{g}(\tau,0,\xi))\Big]\,d\tau.
\end{aligned}$$

Differentiating the above relation with respect to x and making use of the inverse transformation $\xi \mapsto \overline{g}(x,0,\xi) = y$, we see that $\overline{z}$ satisfies (1) for almost all x with fixed y on E_c^*. Since obviously $\overline{z}$ satisfies (2), the proof of Theorem 5.2 is complete.

R e m a r k 5.3. Note that Assumptions $H[\phi]$, $H[W]$ and $H[f]$ in Theorem 5.2 ensure the local existence of a solution of (1), (2) while Assumption $H[Q,P]$ gives only the estimate of the domain on which this solution exists.

R e m a r k 5.4. If in Theorem 5.2 we assume that f and its derivatives $D_y f$, $D_p f$, $D_w f$, $D_q f$ are continuous then we get a theorem on the existence of classical solutions of the problem (1), (2), which extends classical results for differential equations (cf. [23]) to functional differential equations.

R e m a r k 5.5. Theorem 5.2 will still be valid if we consider the following Cauchy problem for weak coupled functional differential systems:

$$D_x z_i(x,y) = f_i(x,y,z(x,y),(Wz)(x,y),D_y z_i(x,y)) \quad \text{on } E,$$
$$z_i(x,y) = \phi_i(x,y) \quad \text{on } E_0, \qquad i = 1,\ldots,m,$$

where $f = (f_1,\ldots,f_m) : E\times\mathbb{R}^{2m+n} \to \mathbb{R}^m$ and $\phi = (\phi_1,\ldots,\phi_m) : E_0 \to \mathbb{R}^m$

are given functions and $W : C(E_0 \cup E, \mathbb{R}^m) \to C(E, \mathbb{R}^m)$ is a Volterra operator.

We give two examples of equations that can be derived from (1) by specializing the operator W.

Example 1. Suppose that $\alpha : [0, a] \to \mathbb{R}$ and $\beta : E \to \mathbb{R}^n$ are given functions such that $(\alpha(t), \beta(t, s)) \in E_x$ for $(t, s) \in E_x$ and $x \in (0, a]$. If for any $z \in C^{0,1+L}(E \cup E_0, \mathbb{R})$ we put

$$(Wz)(x, y) = z(\alpha(x), \beta(x, y)), \qquad (x, y) \in E,$$

then equation (1) reduces to the equation with deviated argument

$$D_x z(x, y) = f(x, y, z(x, y), z(\alpha(x), \beta(x, y)), D_y z(x, y)).$$

Example 2. For any $z \in C^{0,1+L}(E \cup E_0, \mathbb{R})$ we define

$$(Wz)(x, y) = \int_{E_x} z(t, s)\, dt\, ds, \qquad (x, y) \in E.$$

Now equation (1) becomes the integral differential equation

$$D_x z(x, y) = f\Big(x, y, z(x, y), \int_{E_x} z(t, s)\, dt\, ds, D_y z(x, y)\Big).$$

References

[1] P. Bassanini, *On a recent proof concerning a boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form*, Boll. Un. Mat. Ital. A (5) 14 (1977), 325–332.

[2] —, *Iterative methods for quasilinear hyperbolic systems*, Boll. Un. Mat. Ital. B (6) 1 (1982), 225–250.

[3] P. Bassanini and J. Turo, *Generalized solutions of free boundary problems for hyperbolic systems of functional partial differential equations*, Ann. Mat. Pura Appl. 156 (1990), 211–230.

[4] P. Brandi and R. Ceppitelli, *On the existence of solutions of a nonlinear functional partial differential equation of the first order*, Atti Sem. Mat. Fis. Univ. Modena 29 (1980), 166–186.

[5] —, —, *Existence, uniqueness and continuous dependence for a hereditary nonlinear functional partial differential equations*, Ann. Polon. Math. 47 (1986), 121–136.

[6] P. Brandi, Z. Kamont and A. Salvadori, *Existence of weak solutions for partial differential-functional equations*, to appear.

[7] M. G. Cazzani-Nieri, *Un problema ai limiti per sistemi integrodifferenziali non lineari di tipo iperbolico*, Ann. Mat. Pura Appl. 157 (1994), 351–387.

[8] L. Cesari, *A boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form*, Ann. Scuola Norm. Sup. Pisa (4) 1 (1974), 311–358.

[9] —, *A boundary value problem for quasilinear hyperbolic systems*, Riv. Mat. Univ. Parma 3 (1974), 107–131.

[10] M. Cinquini Cibrario, *Nuove ricerche sui sistemi di equazioni non lineari a derivate parziali in più variabili indipendenti*, Rend. Sem. Mat. Fis. Univ. Milano 52 (1982), 531–550.

[11] —, *Teoremi di esistenza per sistemi di equazioni non lineari a derivate parziali in più variabili indipendenti*, Rend. Ist. Lombardo 104 (1970), 795–829.

[12] —, *Sopra una classe di sistemi di equazioni non lineari a derivate parziali in più variabili indipendenti*, Ann. Mat. Pura Appl. 140 (1985), 223–253.

[13] T. Człapiński, *On the existence of generalized solutions of nonlinear first order partial differential-functional equations in two independent variables*, Czechoslovak Math. J. 41 (1991), 490–506.

[14] —, *On the Cauchy problem for quasilinear hyperbolic systems of partial differential-functional equations of the first order*, Z. Anal. Anwendungen 10 (1991), 169–182.

[15] D. Jaruszewska-Walczak, *Existence of solutions of first order partial differential-functional equations*, Boll. Un. Mat. Ital. B (7) 4 (1990), 57–82.

[16] Z. Kamont and S. Zacharek, *On the existence of weak solutions of nonlinear first order partial differential equations in two independent variables*, ibid. (6) 5 (1986), 851–879.

[17] T. Kiguradze, *Some boundary value problems for systems of linear partial differential equations of hyperbolic type*, Mem. Differential Equations Math. Phys. 1 (1994), 1–144.

[18] A. D. Myshkis and A. S. Slopak, *A mixed problem for systems of differential-functional equations with partial derivatives and with operators of Volterra type*, Mat. Sb. 41 (1957), 239–256 (in Russian).

[19] O. A. Oleĭnik, *Discontinuous solutions of nonlinear differential equations*, Uspekhi Mat. Nauk 12 (3) (1957), 3–73 (in Russian).

[20] A. Salvadori, *Sul problema di Cauchy per una struttura ereditaria di tipo iperbolico. Esistenza, unicità e dipendenza continua*, Atti Sem. Mat. Fis. Univ. Modena 32 (1983), 329–356.

[21] J. Szarski, *Characteristics and Cauchy problems for nonlinear partial differential functional equations of first order*, Univ. Kansas, Lawrence, Kan., 1959.

[22] J. Turo, *On some class of quasilinear hyperbolic systems of partial differential-functional equations of the first order*, Czechoslovak Math. J. 36 (1986), 185–197.

[23] T. Ważewski, *Sur l'appréciation du domaine d'existence des intégrales de l'équation aux dérivées partielles du premier ordre*, Ann. Soc. Polon. Math. 14 (1935), 149–177.

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