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A set on which the local Łojasiewicz exponent is attained

by Jacek Chądzyński and Tadeusz Krasiński (Łódź)

Abstract. Let U be a neighbourhood of $0 \in \mathbb{C}^n$. We show that for a holomorphic mapping $F = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$, $F(0) = 0$, the Łojasiewicz exponent $\mathcal{L}_0(F)$ is attained on the set $\{z \in U : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$.

1. Introduction. In [CK$_2$] the authors showed that for a polynomial mapping $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$, $n \geq 2$, the Łojasiewicz exponent $\mathcal{L}_\infty(F)$ of F at infinity is attained on the set $\{z \in \mathbb{C}^n : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$. The purpose of this paper is to prove an analogous result for the Łojasiewicz exponent $\mathcal{L}_0(F)$, where $F : U \to \mathbb{C}^m$ is a holomorphic mapping, $F(0) = 0$ and U is a neighbourhood of $0 \in \mathbb{C}^n$ (Thm. 1). From this result we easily obtain a strict formula for $\mathcal{L}_0(F)$ in the case $n = 2$ and $m \geq 2$ in terms of multiplicities of some mappings from U into $\mathbb{C}^2$ defined by components of F (Thm. 2). It is a generalization of Main Theorem from [CK$_1$]. The proof of this theorem has been simplified by A. Płoski in [P]. His proof has been an inspiration to write this paper.

Theorem 1 is an important tool for investigation of the Łojasiewicz exponent for analytic curves having an isolated intersection point at $0 \in \mathbb{C}^m$. Using it, we shall give, in the next paper [CK$_3$], an effective formula for the Łojasiewicz exponent for such curves in terms of their parametrizations.

2. The Łojasiewicz exponent. Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a neighbourhood of the origin, $F : U \to \mathbb{C}^m$ a holomorphic mapping, and $S \subset U$ an analytic set in U. Assume that $0 \in \mathbb{C}^n$ is an accumulation point of S. Put

$$N(F|S) := \{\nu \in \mathbb{R}_+ : \exists A > 0,\ \exists B > 0,\ \forall z \in S,\ |z| < B \Rightarrow A|z|^\nu \leq |F(z)|\}.$$

Here $|\cdot|$ means the polycylindric norm. If $S = U$ we write $N(F)$ instead of $N(F|U)$.

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By the *Łojasiewicz exponent* of the mapping $F|S$ at 0 we mean

$$\mathcal{L}_0(F|S) := \inf N(F|S).$$

Analogously $\mathcal{L}_0(F) := \inf N(F)$.

The following can be shown (cf. [LT], §§5, 6):

PROPOSITION 1. *If $F|S$ has an isolated zero at $0 \in \mathbb{C}^n$, then $\mathcal{L}_0(F|S) \in N(F|S) \cap \mathbb{Q}$. Moreover, there exists an analytic complex curve $\varphi : \{t \in \mathbb{C} : |t| < r\} \to S$ such that $\varphi(0) = 0$ and*

$$|F \circ \varphi(t)| \sim |\varphi(t)|^{\mathcal{L}_0(F|S)} \quad \textit{as } t \to 0. \tag{1}$$

From the above proposition we easily get

COROLLARY 1. $\mathcal{L}_0(F|S) < \infty$ *if and only if $F|S$ has an isolated zero at $0 \in \mathbb{C}^n$.*

3. The main result. Now, we shall give the main result of this paper.

THEOREM 1. *Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a neighbourhood of the origin, and $F = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$ a holomorphic mapping with $F(0) = 0$. Define $S := \{z \in U : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$. Then*

$$\mathcal{L}_0(F) = \mathcal{L}_0(F|S). \tag{2}$$

The proof will be given in Section 4.

Immediately from Theorem 1 we obtain an effective formula for the Łojasiewicz exponent in the case $n = 2$, $m \geq 2$, generalizing an earlier result of the authors ([CK$_1$], Main Theorem).

Let us begin with some notations. Let U be a neighbourhood of $0 \in \mathbb{C}^2$. Then: $\mu(f, g)$ is the intersection multiplicity of a holomorphic mapping $(f, g) : U \to \mathbb{C}^2$; $\widehat{h}$ is the germ of a holomorphic function $h : U \to \mathbb{C}$ in the ring $\mathcal{O}^2$ of germs of holomorphic functions at $0 \in \mathbb{C}^2$; $\operatorname{ord} h$ stands for the order of h at $0 \in \mathbb{C}^2$.

THEOREM 2. *Let $U \subset \mathbb{C}^2$ be a neighbourhood of the origin, and $F = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$ a holomorphic mapping with $F(0) = 0$. Put $f := f_1 \cdot \ldots \cdot f_m$. If $\widehat{f} \neq 0$ and $\widehat{f} = \widehat{h}_1 \cdot \ldots \cdot \widehat{h}_r$ is a factorization of $\widehat{f}$ into irreducible germs in $\mathcal{O}^2$, then*

$$\mathcal{L}_0(F) = \max_{i=1}^{r} \frac{1}{\operatorname{ord} h_i} \min_{j=1}^{m} \mu(h_i, f_j). \tag{3}$$

Proof. Since our considerations are local, we may assume that h_i are holomorphic in U and $f = h_1 \cdot \ldots \cdot h_r$ in U. Let $S := \{z \in U : f(z) = 0\}$ and $\Gamma_i := \{z \in U : h_i(z) = 0\}$. Hence

$$S = \Gamma_1 \cup \ldots \cup \Gamma_r. \tag{4}$$

Define $\lambda_i := (1/\operatorname{ord} h_i) \min_{j=1}^m \mu(h_i, f_j)$. If $\lambda_i = \infty$ for some i, then (3) holds. So, assume that $\lambda_i \neq \infty$, $i = 1, \ldots, r$. Then for every i we have

$$|F(z)| \sim |z|^{\lambda_i} \quad \text{ as } |z| \to 0,\ z \in \Gamma_i.$$

Hence and from (4), $\mathcal{L}_0(F|S) = \max_{i=1}^r \lambda_i$. This and Theorem 1 give (3), which ends the proof.

4. Proof of the main theorem. The proof is given in two steps. In the first one we give the proof under some additional assumptions, whereas in the second step we show that these assumptions do not restrict our considerations.

First we fix some notations. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $n \geq 2$, and for every $i \in \{1, \ldots, n\}$ we put $z_i' := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$. Additionally, we define $f := f_1 \cdot \ldots \cdot f_m$.

Step 1. We assume that

(i) $(F|S)^{-1}(0) = \{0\}$,

(ii) $\operatorname{ord} f < \infty$,

(iii) for every $i \in \{1, \ldots, n\}$, f is $(\operatorname{ord} f)$-regular with respect to z_i.

Obviously $N(F) \subset N(F|S)$. To show (2) it suffices to prove

$$N(F|S) \subset N(F). \tag{5}$$

It follows from (i) and Corollary 1 that $N(F|S)$ is not empty. Take an arbitrary $\nu \in N(F|S)$. Then there exist $A, B > 0$ such that

$$|F(\zeta)| \geq A|\zeta|^{\nu} \quad \text{ for } \zeta \in S,\ |\zeta| < B. \tag{6}$$

From the assumptions of the theorem we have $\operatorname{ord} f > 0$ and $\operatorname{ord} f_j > 0$, $j = 1, \ldots, m$. Then from (ii), (iii) we easily get $0 < \operatorname{ord} f_j < \infty$ and

$$\text{for every } i,\ j,\ f_j \text{ is } (\operatorname{ord} f_j)\text{-regular with respect to } z_i. \tag{7}$$

The Weierstrass Preparation Theorem and (iii) imply that for every $i \in \{1, \ldots, n\}$ there exists a Weierstrass polynomial with respect to z_i of degree $\operatorname{ord} f$, associated with f. Denote it by $p^{(i)}$. Analogously from (7) for every i, j there exists a Weierstrass polynomial with respect to z_i of degree $\operatorname{ord} f_j$, associated with f_j. Denote it by $p_j^{(i)}$. Then there exist $C, D > 0$ and a polycylinder $\{z \in \mathbb{C}^n : |z| < r\} \subset U$ such that

$$C|p^{(i)}(z)| \leq |f(z)| \leq D|p^{(i)}(z)| \quad \text{ for } |z| < r,\ i = 1, \ldots, n, \tag{8}$$

and

$$C|p_j^{(i)}(z)| \leq |f_j(z)| \leq D|p_j^{(i)}(z)| \quad \text{ for } |z| < r,\ i = 1, \ldots, n,\ j = 1, \ldots, m. \tag{9}$$

Put $P^{(i)} := (p_1^{(i)}, \ldots, p_m^{(i)})$. Then from (9) we get

$$C|P^{(i)}(z)| \le |F(z)| \le D|P^{(i)}(z)| \quad \text{for } |z| < r,\ i = 1, \ldots, n. \tag{10}$$

Clearly, for every $i \in \{1, \ldots, n\}$,

$$p^{(i)} = p_1^{(i)} \cdot \ldots \cdot p_m^{(i)}. \tag{11}$$

Put $B_1 := \min(B, r)$. Then from (8) we have for every $i \in \{1, \ldots, n\}$,

$$S \cap \{z \in \mathbb{C}^n : |z| < B_1\} = \{z \in \mathbb{C}^n : |z| < B_1,\ p^{(i)}(z) = 0\}.$$

Hence and from the Theorem on Continuity of Roots, applied to $p^{(i)}$, we find that there exists ϱ, $0 < \varrho \le B_1$, such that for every $i \in \{1, \ldots, n\}$ we have

$$\{z \in S : |z_i'| < \varrho, |z| < B_1\} = \{z \in \mathbb{C}^n : |z_i'| < \varrho,\ p^{(i)}(z) = 0\}. \tag{12}$$

Put $d := \max_{j=1}^m \operatorname{ord} f_j$, $A_2 := 2^{-d}A(C/D)$, $B_2 := \min(\varrho, B_1)$. Take an arbitrary $\mathring{z} \in \mathbb{C}^n$, $|\mathring{z}| < B_2$. There exists $i \in \{1, \ldots, n\}$ such that $|\mathring{z}| = |\mathring{z}_i'|$. Define $\varphi_j(t) := p_j^{(i)}(\mathring{z}_1, \ldots, \mathring{z}_{i-1}, t, \mathring{z}_{i+1}, \ldots, \mathring{z}_n)$, $\Phi := (\varphi_1, \ldots, \varphi_m)$. Clearly, $\Phi : \mathbb{C} \to \mathbb{C}^m$ is a polynomial mapping, $\deg \Phi := \max_{j=1}^m \deg \varphi_j = d$ and $\Phi(t) = P^{(i)}(\mathring{z}_1, \ldots, \mathring{z}_{i-1}, t, \mathring{z}_{i+1}, \ldots, \mathring{z}_n)$. Then by Lemma 2 of [CK$_2$] and (11) we have

$$|\Phi(\mathring{z}_i)| \ge 2^{-d} \min_{\tau \in T} |\Phi(\tau)|,$$

where $T := \{t \in \mathbb{C} : p^{(i)}(\mathring{z}_1, \ldots, \mathring{z}_{i-1}, t, \mathring{z}_{i+1}, \ldots, \mathring{z}_n) = 0\}$. Hence and by (10) we get

$$\begin{aligned} |F(\mathring{z})| &\ge C|P^{(i)}(\mathring{z})| = C|\Phi(\mathring{z}_i)| \ge C2^{-d}|\Phi(\tau_0)| \\ &= C2^{-d}|P^{(i)}(\mathring{\zeta})| \ge (C/D)2^{-d}|F(\mathring{\zeta})| \end{aligned} \tag{13}$$

for some $\mathring{\zeta} = (\mathring{z}_1, \ldots, \mathring{z}_{i-1}, \tau_0, \mathring{z}_{i+1}, \ldots, \mathring{z}_n)$ such that $p^{(i)}(\mathring{\zeta}) = 0$. Since $|\mathring{\zeta}_i'| = |\mathring{z}_i'| = |\mathring{z}| < B_2 \le \varrho$, from (12) we have $\mathring{\zeta} \in S$ and $|\mathring{\zeta}| < B_1$. In consequence, from (6) and (13) we get

$$|F(\mathring{z})| \ge (C/D)2^{-d}A|\mathring{\zeta}|^\nu \ge A_2|\mathring{\zeta}_i'|^\nu = A_2|\mathring{z}|^\nu.$$

Since $\mathring{z}$ is arbitrary we obtain $\nu \in N(F)$. This ends the proof of the theorem under assumptions (i)–(iii).

STEP 2. Now, we shall prove the theorem in the remaining cases. If (i) does not hold, then by Corollary 1 we have $\mathcal{L}_0(F|S) = \mathcal{L}_0(F) = \infty$, that is, (2) is satisfied. If (ii) does not hold, then $S = U$ and (2) is obvious. So, it suffices to consider the case when (i), (ii) hold but (iii) does not. Since $N(F)$ and $N(F|S)$ are invariant with respect to linear automorphisms of $\mathbb{C}^n$, so are $\mathcal{L}_0(F)$ and $\mathcal{L}_0(F|S)$. Since the case considered can be reduced to Step 1 by a linear automorphism of $\mathbb{C}^n$, (2) also holds in this case.

This ends the proof of the theorem.

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Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: jachadzy@imul.uni.lodz.pl
krasinsk@krysia.uni.lodz.pl

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