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A set on which the Łojasiewicz exponent at infinity is attained

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Abstract. We show that for a polynomial mapping $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ the Łojasiewicz exponent $\mathcal{L}_\infty(F)$ of F is attained on the set $\{z \in \mathbb{C}^n : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$.

1. Introduction. The purpose of this paper is to prove that the Łojasiewicz exponent at infinity of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ is attained on a proper algebraic subset of $\mathbb{C}^n$ defined by the components of F (Thm. 1).

As a corollary we obtain a result of Z. Jelonek on testing sets for properness of polynomial mappings (Cor. 3) and a formula for the Łojasiewicz exponent at infinity of F in the case $n = 2$, $m \geq 2$, in terms of parametrizations of branches (at infinity) of zeroes of the components of F (Thm. 2). This result is a generalization of the authors' result for $n = m = 2$ ([CK], Main Theorem).

Before the main considerations we show some basic properties of the Łojasiewicz exponent at infinity for regular mappings, i.e. for polynomial mappings restricted to algebraic subsets of $\mathbb{C}^n$. We prove that the exponent is a rational number, that it is attained on a meromorphic curve (Prop. 1), and we give a condition equivalent to the properness of regular mappings (Cor. 2). These properties are analogous to ones, known in folklore, for polynomial mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$. We do not pretend to the originality of proof methods; we only want to fill gaps in the literature.

The results obtained by Z. Jelonek in [J] played an inspiring role in undertaking this research. On the other hand, the idea of the proof of the main theorem was taken from A. Płoski ([P_2], App.).

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2. The Łojasiewicz exponent. Let $F:\mathbb{C}^n\to\mathbb{C}^m$, $n\ge 2$, be a polynomial mapping and let $S\subset\mathbb{C}^n$ be an unbounded algebraic set. Put

$$N(F|S):=\{\nu\in\mathbb{R}:\exists A>0,\ \exists B>0,\ \forall z\in S\ (|z|>B\Rightarrow A|z|^\nu\le|F(z)|)\},$$

where $|\cdot|$ is the polycylindric norm. If $S=\mathbb{C}^n$ we define $N(F):=N(F|\mathbb{C}^n)$.

By the *Łojasiewicz exponent at infinity* of $F|S$ we mean $\mathcal{L}_\infty(F|S):=\sup N(F|S)$. Analogously $\mathcal{L}_\infty(F):=\sup N(F)$.

Before we pass to properties of the Łojasiewicz exponent we quote the known curve selection lemma at infinity (cf. [NZ], Lemma 2). We begin with a definition. A curve $\varphi:(R,+\infty)\to\mathbb{R}^k$ is called *meromorphic at* $+\infty$ if φ is the sum of a Laurent series of the form

$$\varphi(t)=\alpha_pt^p+\alpha_{p-1}t^{p-1}+\ldots,\qquad \alpha_i\in\mathbb{R}^k.$$

By $\|\cdot\|$ we denote the euclidian norm in $\mathbb{R}^k$.

LEMMA 1 (Curve Selection Lemma). *If $X\subset\mathbb{R}^k$ is an unbounded semi-algebraic set, then there exists a curve $\varphi:(R,+\infty)\to\mathbb{R}^k$, meromorphic at $+\infty$, such that $\varphi(t)\in X$ for $t\in(R,+\infty)$ and $\|\varphi(t)\|\to\infty$ as $t\to+\infty$.*

Notice that the Łojasiewicz exponent at infinity of a regular mapping $F|S$ does not depend on the norm in $\mathbb{C}^n$. So, in the rest of this section, we shall use the euclidian norm $\|\cdot\|$ in the definition of $N(F|S)$.

Let us introduce one more definition. A curve $\varphi=(\varphi_1,\ldots,\varphi_m):\{t\in\mathbb{C}:|t|>R\}\to\mathbb{C}^m$ is called *meromorphic at* ∞ if φ_i are meromorphic at ∞.

Let $F:\mathbb{C}^n\to\mathbb{C}^m$, $n\ge2$, be a polynomial mapping and let $S\subset\mathbb{C}^n$ be an unbounded algebraic set.

PROPOSITION 1. *If $\#(F|S)^{-1}(0)<+\infty$, then $\mathcal{L}_\infty(F|S)\in N(F|S)\cap\mathbb{Q}$. Moreover, there exists a curve $\varphi:\{t\in\mathbb{C}:|t|>R\}\to\mathbb{C}^m$, meromorphic at ∞, such that $\varphi(t)\in S$, $\|\varphi(t)\|\to+\infty$ for $t\to\infty$ and*

$$\|F\circ\varphi(t)\|\sim\|\varphi(t)\|^{\mathcal{L}_\infty(F|S)}\quad \text{as } t\to\infty. \tag{1}$$

Proof. Notice first that the set

$$\{(z,w)\in S\times S:\|F(z)\|^2\le\|F(w)\|^2\vee\|z\|^2\ne\|w\|^2\}$$

is semi-algebraic in $\mathbb{C}^n\times\mathbb{C}^n\cong\mathbb{R}^{4n}$. Then by the Tarski–Seidenberg theorem (cf. [BR], Rem. 3.8) the set

$$\begin{aligned}X&:=\{z\in S:\forall w\in S\ (\|F(z)\|^2\le\|F(w)\|^2\vee\|z\|^2\ne\|w\|^2)\}\\&=\{z\in S:\|F(z)\|=\min_{\|w\|=\|z\|}\|F(w)\|\}\end{aligned}$$

is also semi-algebraic and obviously unbounded in $\mathbb{C}^n\cong\mathbb{R}^{2n}$. So, by Lemma 1 there exists a curve $\widetilde\varphi:(R,+\infty)\to X$, meromorphic at $+\infty$, such that $\|\widetilde\varphi(t)\|\to+\infty$ as $t\to+\infty$. Then there exists a positive integer p such that

$\widetilde{\varphi}$ is the sum of a Laurent series

$$\widetilde{\varphi}(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots, \quad \alpha_i \in \mathbb{C}^n, \ \alpha_p \neq 0. \tag{2}$$

Since $\#(F|S)^{-1}(0) < \infty$, there exists an integer q such that $F \circ \widetilde{\varphi}$ is the sum of a Laurent series

$$F \circ \widetilde{\varphi}(t) = \beta_q t^q + \beta_{q-1} t^{q-1} + \ldots, \quad \beta_i \in \mathbb{C}^m, \ \beta_q \neq 0. \tag{3}$$

From (2) and (3) we have

$$\|F \circ \widetilde{\varphi}(t)\| \sim \|\widetilde{\varphi}(t)\|^\lambda \quad \text{as } t \to +\infty, \tag{4}$$

where $\lambda := q/p$. Let $\widetilde{\Gamma} := \{z \in \mathbb{C}^n : z = \widetilde{\varphi}(t), \ t \in (R, +\infty)\}$. Then from (4),

$$\|F(z)\| \sim \|z\|^\lambda \quad \text{as } \|z\| \to +\infty, \ z \in \widetilde{\Gamma}. \tag{5}$$

Now, we shall show that $\mathcal{L}_\infty(F|S) = \lambda$. From (5) we have $\mathcal{L}_\infty(F|S) \le \lambda$. Since $\widetilde{\Gamma} \subset X$ is unbounded, there exist positive constants A, B such that $\|F(z)\| \ge A\|z\|^\lambda$ for every $z \in S$ and $\|z\| > B$. Then $\lambda \in N(F|S)$ and in consequence $\mathcal{L}_\infty(F|S) \ge \lambda$. Summing up, $\mathcal{L}_\infty(F|S) = \lambda \in N(F|S) \cap \mathbb{Q}$.

Now, we shall prove the second part of the assertion. Let φ be an extension of $\widetilde{\varphi}$ to the complex domain, that is,

$$\varphi(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots, \tag{6}$$

where $t \in \mathbb{C}$ and $|t| > R$. Obviously, series (6) is convergent and, as above, $\alpha_i \in \mathbb{C}^n$, $\alpha_p \neq 0$. Hence φ is a curve, meromorphic at ∞, and clearly $\|\varphi(t)\| \to +\infty$ as $t \to \infty$. Moreover, $F \circ \varphi$ is an extension of $F \circ \widetilde{\varphi}$ to the complex domain and

$$F \circ \varphi(t) = \beta_q t^q + \beta_{q-1} t^{q-1} + \ldots, \tag{7}$$

where $t \in \mathbb{C}$ and $|t| > R$. Obviously, the series (7) is convergent and, as above, $\beta_i \in \mathbb{C}^m$, $\beta_q \neq 0$. From (6), (7) and the definition of λ we get (1). Since S is an algebraic subset of $\mathbb{C}^n$ and $\widetilde{\varphi}(t) \in S$ for $t \in (R, +\infty)$, also $\varphi(t) \in S$ for $t \in \mathbb{C}$, $|t| > R$.

This ends the proof of the proposition.

Let $F : \mathbb{C}^n \to \mathbb{C}^m$, $n \ge 2$, be a polynomial mapping and $S \subset \mathbb{C}^n$ an algebraic unbounded set.

Directly from Proposition 1 we get

COROLLARY 1. *$\mathcal{L}_\infty(F|S) > -\infty$ if and only if $\#(F|S)^{-1}(0) < +\infty$.*

From Proposition 1 we also easily get

COROLLARY 2. *The mapping $F|S$ is proper if and only if $\mathcal{L}_\infty(F|S) > 0$.*

In fact, if $\mathcal{L}_\infty(F|S) > 0$, then obviously $F|S$ is a proper mapping. If, in turn, $\mathcal{L}_\infty(F|S) \le 0$ then from the second part of Proposition 1 and

Corollary 1 it follows that there exists a sequence $z_n \in S$ such that $\|z_n\| \to +\infty$ and the sequence $F(z_n)$ is bounded. Hence $F|S$ is not a proper mapping in this case.

3. The main result.

Now, we formulate the main result of the paper.

THEOREM 1. *Let* $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$, $n \geq 2$, *be a polynomial mapping and* $S := \{z \in \mathbb{C}^n : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$. *If* $S \neq \emptyset$, *then*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|S). \tag{8}$$

The proof will be given in Section 4.

Directly from Theorem 1 and Corollary 2 we get

COROLLARY 3 ([J], Cor. 6.7). *If* $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$, $n \geq 2$, *is a polynomial mapping and* $S := \{z \in \mathbb{C}^n : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$ *is not empty*, *then* F *is proper if and only if* $F|S$ *is proper.*

Another corollary from Theorem 1 is an effective formula for the Łojasiewicz exponent, generalizing an earlier result of the authors ([CK], Main Theorem).

Let us introduce some notions. If $\Psi : \{z \in \mathbb{C} : |z| > R\} \to \mathbb{C}^k$ is the sum of a Laurent series of the form

$$\Psi(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots, \qquad \alpha_i \in \mathbb{C}^k,\ \alpha_p \neq 0,$$

then we put $\deg \Psi := p$. Additionally, $\deg \Psi := -\infty$ if $\Psi = 0$. For an algebraic curve in $\mathbb{C}^2$, the notions of its branches in a neighbourhood of ∞ and parametrizations of these branches are defined in [CK].

Let now $F = (f_1, \ldots, f_m) : \mathbb{C}^2 \to \mathbb{C}^m$ be a polynomial mapping and $S := \{z \in \mathbb{C}^2 : f_1(z) \cdot \ldots \cdot f_m(z) = 0\}$. Assume that $S \neq \emptyset$ and $S \neq \mathbb{C}^2$.

THEOREM 2. *If* $\Gamma_1, \ldots, \Gamma_s$ *are branches of the curve* S *in a neighbourhood* Y *of infinity and* $\Phi_i : U_i \to Y$, $i = 1, \ldots, s$, *are their parametrizations*, *then*

$$\mathcal{L}_\infty(F) = \min_{i=1}^{s} \frac{\deg F \circ \Phi_i}{\deg \Phi_i}. \tag{9}$$

P r o o f. Define $\lambda_i := \deg F \circ \Phi_i / \deg \Phi_i$. If $\lambda_i = -\infty$ for some i, then (9) holds. So, assume that $\lambda_i \neq -\infty$, $i = 1, \ldots, s$. Then

$$|F(z)| \sim |z|^{\lambda_i} \quad \text{as } |z| \to +\infty,\ z \in \Gamma_i.$$

Hence, taking into account the equality $S \cap Y = \Gamma_1 \cup \ldots \cup \Gamma_s$ we get (9).

4. Proof of the main theorem.

Let us begin with a lemma on polynomial mappings from $\mathbb{C}$ into $\mathbb{C}^m$. It is a generalization of a result by A. Płoski ([P_1], Lemma 3.1) and plays a key role in the proof of the main theorem.

LEMMA 2. *Let $\Phi = (\varphi_1, \ldots, \varphi_m) : \mathbb{C} \to \mathbb{C}^m$ be a polynomial mapping and $\varphi := \varphi_1 \cdot \ldots \cdot \varphi_m$. If φ is a polynomial of positive degree and T is its set of zeroes, then for every $t \in \mathbb{C}$,*

$$|\Phi(t)| \geq 2^{-\deg \Phi} \min_{\tau \in T} |\Phi(\tau)|.$$

Proof. Fix $t_0 \in \mathbb{C}$. Let $\min_{\tau \in T} |t_0 - \tau|$ be attained for some $\tau_0 \in T$. If φ_i is a polynomial of positive degree and has the form $\varphi_i(t) = c_i \prod_{j=1}^{\deg \varphi_i} (t - \tau_{ij})$, then we have

$$2|t_0 - \tau_{ij}| = |t_0 - \tau_{ij}| + |t_0 - \tau_{ij}| \geq |t_0 - \tau_0| + |t_0 - \tau_{ij}| \geq |\tau_0 - \tau_{ij}|.$$

Hence

$$2^{\deg \varphi_i} |\varphi_i(t_0)| \geq |\varphi_i(\tau_0)|.$$

Obviously, this inequality is also true for φ_i being a constant. Since $\deg \Phi \geq \deg \varphi_i$, from the above we get

$$2^{\deg \Phi} |\Phi(t_0)| \geq |\Phi(\tau_0)| \geq \min_{\tau \in T} |\Phi(\tau)|,$$

which ends the proof.

In the sequel, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $n \geq 2$, and for every $i \in \{1, \ldots, n\}$ we put $z_i' := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$.

We state an easy lemma without proof.

LEMMA 3. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial function and S its set of zeroes. If $\deg f = \deg_{z_i} f$ for every $i \in \{1, \ldots, n\}$, then there exist constants $C \geq 1$, $D > 0$ such that for every $i \in \{1, \ldots, n\}$,*

$$|z_i| \leq C|z_i'| \quad \text{for } z \in S \text{ and } |z_i'| > D.$$

Proof of Theorem 1. Without loss of generality we may assume that

(i) $S \neq \mathbb{C}^n$,
(ii) $\#(F|S)^{-1}(0) < \infty$.

In fact, if (i) does not hold then (8) is obvious, whereas if (ii) does not hold then (8) follows from Corollary 1.

Obviously $N(F) \subset N(F|S)$. So, to prove (8) it suffices to show

$$N(F|S) \subset N(F). \tag{10}$$

Put $f := f_1 \cdot \ldots \cdot f_m$. From (i) we have $\deg f > 0$. Since the sets $N(F|S)$ and $N(F)$ are invariant with respect to linear changes of coordinates in $\mathbb{C}^n$ we may assume that

$$\deg f = \deg_{z_i} f, \quad i = 1, \ldots, n. \tag{11}$$

This obviously implies

$$\deg f_j = \deg_{z_i} f_j, \quad j = 1, \ldots, m,\ i = 1, \ldots, n. \tag{12}$$

It follows from (ii) and Corollary 1 that $N(F|S)$ is not empty. Take $\nu \in N(F|S)$. Then there exist $A > 0$, $B > 0$ such that

$$|F(\zeta)| \geq A|\zeta|^{\nu} \quad \text{for } \zeta \in S,\ |\zeta| > B. \tag{13}$$

By (11) and Lemma 3 there exist $C \geq 1$, $D > 0$ such that for every $i \in \{1, \ldots, n\}$,

$$|z_i| \leq C|z_i'| \quad \text{for } z \in S,\ |z_i'| > D. \tag{14}$$

Put $A_1 := 2^{-\deg F} A \min(1, C^{\nu})$ and $B_1 := \max(B, D)$. Take arbitrary $\mathring{z} \in \mathbb{C}^n$ such that $|\mathring{z}| > B_1$. Clearly, $|\mathring{z}| = |\mathring{z}_i'|$ for some i. Define $\varphi_j(t) := f_j(\mathring{z}_1, \ldots, \mathring{z}_{i-1}, t, \mathring{z}_{i+1}, \ldots, \mathring{z}_n)$, $\Phi := (\varphi_1, \ldots, \varphi_m)$. Then from (12) we have

$$\deg F = \deg \Phi. \tag{15}$$

Moreover, from (11) it follows that $\varphi := \varphi_1 \cdot \ldots \cdot \varphi_m$ is a polynomial of positive degree. Then, from Lemma 2 (T is defined as in Lemma 2) and (15) we have

$$|F(\mathring{z})| = |\Phi(\mathring{z}_i)| \geq 2^{-\deg \Phi} \min_{\tau \in T} |\Phi(\tau)| = 2^{-\deg F} |F(\mathring{\zeta})| \tag{16}$$

for some $\mathring{\zeta} = (\mathring{z}_1, \ldots, \mathring{z}_{i-1}, \tau_0, \mathring{z}_{i+1}, \ldots, \mathring{z}_n)$, $\tau_0 \in T$. So, $\mathring{\zeta} \in S$. Since $|\mathring{z}| > B_1$ and $|\mathring{\zeta}| \geq |\mathring{z}_i'| = |\mathring{z}|$, from (16) and (13) we get

$$|F(\mathring{z})| \geq 2^{-\deg F} A |\mathring{\zeta}|^{\nu}, \tag{17}$$

whereas from (14),

$$|\mathring{z}| \leq |\mathring{\zeta}| \leq C|\mathring{z}|. \tag{18}$$

Considering two cases, when $\nu \geq 0$ and $\nu < 0$, from (17) and (18) we easily get

$$|F(\mathring{z})| \geq A_1 |\mathring{z}|^{\nu}.$$

Since $\mathring{z}$ is arbitrary we have $\nu \in N(F)$.

This ends the proof of the theorem.

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