A set on which the Lojasiewicz exponent at infinity is attained

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Abstract. We show that for a polynomial mapping $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ the Lojasiewicz exponent $L_\infty(F)$ of $F$ is attained on the set $\{z \in \mathbb{C}^n : f_1(z) \cdots f_m(z) = 0\}$.

1. Introduction. The purpose of this paper is to prove that the Lojasiewicz exponent at infinity of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ is attained on a proper algebraic subset of $\mathbb{C}^n$ defined by the components of $F$ (Thm. 1).

As a corollary we obtain a result of Z. Jelonek on testing sets for properness of polynomial mappings (Cor. 3) and a formula for the Lojasiewicz exponent at infinity of $F$ in the case $n = 2, m \geq 2$, in terms of parametrizations of branches (at infinity) of zeroes of the components of $F$ (Thm. 2). This result is a generalization of the authors’ result for $n = m = 2$ ([CK], Main Theorem).

Before the main considerations we show some basic properties of the Lojasiewicz exponent at infinity for regular mappings, i.e. for polynomial mappings restricted to algebraic subsets of $\mathbb{C}^n$. We prove that the exponent is a rational number, that it is attained on a meromorphic curve (Prop. 1), and we give a condition equivalent to the properness of regular mappings (Cor. 2). These properties are analogous to ones, known in folklore, for polynomial mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$. We do not pretend to the originality of proof methods; we only want to fill gaps in the literature.

The results obtained by Z. Jelonek in [J] played an inspiring role in undertaking this research. On the other hand, the idea of the proof of the main theorem was taken from A. Płoski ([P₃], App.).

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2. The Łojasiewicz exponent. Let $F : \mathbb{C}^n \to \mathbb{C}^m$, $n \geq 2$, be a polynomial mapping and let $S \subset \mathbb{C}^n$ be an unbounded algebraic set. Put

$$N(F|S) := \{ \nu \in \mathbb{R} : \exists A > 0, \exists B > 0, \forall z \in S \ (|z| > B \Rightarrow A|z|^{\nu} \leq |F(z)|) \},$$

where $|\cdot|$ is the polycylindric norm. If $S = \mathbb{C}^n$ we define $N(F) := N(F|\mathbb{C}^n)$.

By the Łojasiewicz exponent at infinity of a regular mapping $\varphi$, we denote the euclidian norm $\parallel \cdot \parallel$.

Before we pass to properties of the Łojasiewicz exponent we quote the known curve selection lemma at infinity (cf. [NZ], Lemma 2). We begin with

Lemma 1 (Curve Selection Lemma). If $X \subset \mathbb{R}^k$ is an unbounded semi-algebraic set, then there exists a curve $\varphi : (R, +\infty) \to \mathbb{R}^k$, meromorphic at $+\infty$, such that $\varphi(t) \in X$ for $t \in (R, +\infty)$ and $\parallel \varphi(t) \parallel \to \infty$ as $t \to +\infty$.

Notice that the Łojasiewicz exponent at infinity of a regular mapping $F|S$ does not depend on the norm in $\mathbb{C}^n$. So, in the rest of this section, we shall use the euclidian norm $\parallel \cdot \parallel$ in the definition of $N(F|S)$.

Let us introduce one more definition. A curve $\varphi = (\varphi_1, \ldots, \varphi_m) : \{ t \in \mathbb{C} : \parallel t \parallel > R \} \to \mathbb{C}^m$ is called meromorphic at $+\infty$ if $\varphi_i$ are meromorphic at $\infty$.

Let $F : \mathbb{C}^n \to \mathbb{C}^m$, $n \geq 2$, be a polynomial mapping and let $S \subset \mathbb{C}^n$ be an unbounded algebraic set.

Proposition 1. If $\#(F(S)^{-1}(0)) < +\infty$, then $\mathcal{L}_\infty(F|S) \in N(F|S) \cap Q$. Moreover, there exists a curve $\varphi : \{ t \in \mathbb{C} : \parallel t \parallel > R \} \to \mathbb{C}^m$, meromorphic at $+\infty$, such that $\varphi(t) \in S$, $\parallel \varphi(t) \parallel \to +\infty$ for $t \to +\infty$ and

$$\parallel F \circ \varphi(t) \parallel \sim \parallel \varphi(t) \parallel^{\mathcal{L}_\infty(F|S)} \text{ as } t \to +\infty.$$

Proof. Notice first that the set

$$\{(z, w) \in S \times S : \parallel F(z) \parallel^2 \leq \parallel F(w) \parallel^2 \lor \parallel z \parallel^2 \neq \parallel w \parallel^2\}$$

is semi-algebraic in $\mathbb{C}^n \times \mathbb{C}^n \cong \mathbb{R}^{4n}$. Then by the Tarski–Seidenberg theorem (cf. [BR], Rem. 3.8) the set

$$X := \{ z \in S : \forall w \in S \ (\parallel F(z) \parallel^2 \leq \parallel F(w) \parallel^2 \lor \parallel z \parallel^2 \neq \parallel w \parallel^2) \}$$

$$= \{ z \in S : \parallel F(z) \parallel = \min_{\parallel w = \parallel z \parallel} \parallel F(w) \parallel \}$$

is also semi-algebraic and obviously unbounded in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. So, by Lemma 1 there exists a curve $\tilde{\varphi} : (R, +\infty) \to X$, meromorphic at $+\infty$, such that $\parallel \tilde{\varphi}(t) \parallel \to +\infty$ as $t \to +\infty$. Then there exists a positive integer $p$ such that
\( \tilde{\varphi} \) is the sum of a Laurent series

\[
(2) \quad \tilde{\varphi}(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots, \quad \alpha_i \in \mathbb{C}^n, \ \alpha_p \neq 0.
\]

Since \( \#(F|S)^{-1}(0) < \infty \), there exists an integer \( q \) such that \( F \circ \tilde{\varphi} \) is the sum of a Laurent series

\[
(3) \quad F \circ \tilde{\varphi}(t) = \beta_q t^q + \beta_{q-1} t^{q-1} + \ldots, \quad \beta_i \in \mathbb{C}^m, \ \beta_q \neq 0.
\]

From (2) and (3) we have

\[
(4) \quad \| F \circ \tilde{\varphi}(t) \| \sim \| \tilde{\varphi}(t) \|^\lambda \quad \text{as} \ t \to +\infty,
\]

where \( \lambda := q/p \). Let \( \tilde{T} := \{ z \in \mathbb{C}^n : z = \tilde{\varphi}(t), \ t \in (R, +\infty) \} \). Then from (4),

\[
(5) \quad \| F(z) \| \sim \| z \|^\lambda \quad \text{as} \ |z| \to +\infty, \quad z \in \tilde{T}.
\]

Now, we shall show that \( \mathcal{L}_\infty(F|S) = \lambda \). From (5) we have \( \mathcal{L}_\infty(F|S) \leq \lambda \).

Since \( \tilde{T} \subset X \) is unbounded, there exist positive constants \( A, B \) such that \( \| F(z) \| \geq A \| z \|^\lambda \) for every \( z \in S \) and \( \| z \| > B \). Then \( \lambda \in N(F|S) \) and in consequence \( \mathcal{L}_\infty(F|S) \geq \lambda \). Summing up, \( \mathcal{L}_\infty(F|S) = \lambda \in N(F|S) \cap \mathbb{Q} \).

Now, we shall prove the second part of the assertion. Let \( \varphi \) be an extension of \( \tilde{\varphi} \) to the complex domain, that is,

\[
(6) \quad \varphi(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots,
\]

where \( t \in \mathbb{C} \) and \( |t| > R \). Obviously, series (6) is convergent and, as above, \( \alpha_i \in \mathbb{C}^n, \ \alpha_p \neq 0 \). Hence \( \varphi \) is a curve, meromorphic at \( \infty \), and clearly \( \| \varphi(t) \| \to +\infty \) as \( t \to \infty \). Moreover, \( F \circ \varphi \) is an extension of \( F \circ \tilde{\varphi} \) to the complex domain and

\[
(7) \quad F \circ \varphi(t) = \beta_q t^q + \beta_{q-1} t^{q-1} + \ldots,
\]

where \( t \in \mathbb{C} \) and \( |t| > R \). Obviously, the series (7) is convergent and, as above, \( \beta_i \in \mathbb{C}^m, \ \beta_q \neq 0 \). From (6), (7) and the definition of \( \lambda \) we get (1).

Since \( S \) is an algebraic subset of \( \mathbb{C}^n \) and \( \tilde{\varphi}(t) \in S \) for \( t \in (R, +\infty) \), also \( \varphi(t) \in S \) for \( t \in \mathbb{C}, \ |t| > R \).

This ends the proof of the proposition.

Let \( F : \mathbb{C}^n \to \mathbb{C}^m, \ n \geq 2, \) be a polynomial mapping and \( S \subset \mathbb{C}^n \) an algebraic unbounded set.

Directly from Proposition 1 we get

**Corollary 1.** \( \mathcal{L}_\infty(F|S) > -\infty \) if and only if \( \#(F|S)^{-1}(0) < +\infty \).

From Proposition 1 we also easily get

**Corollary 2.** The mapping \( F|S \) is proper if and only if \( \mathcal{L}_\infty(F|S) > 0 \).

In fact, if \( \mathcal{L}_\infty(F|S) > 0 \), then obviously \( F|S \) is a proper mapping. If, in turn, \( \mathcal{L}_\infty(F|S) \leq 0 \) then from the second part of Proposition 1 and
Corollary 1 it follows that there exists a sequence \( z_n \in S \) such that \( \|z_n\| \to +\infty \) and the sequence \( F(z_n) \) is bounded. Hence \( F|S \) is not a proper mapping in this case.

3. The main result. Now, we formulate the main result of the paper.

**Theorem 1.** Let \( F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m, n \geq 2, \) be a polynomial mapping and \( S := \{ z \in \mathbb{C}^n : f_1(z) \cdot \ldots \cdot f_m(z) = 0 \} \). If \( S \neq \emptyset \), then

\[
\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|S).
\]

The proof will be given in Section 4.

Directly from Theorem 1 and Corollary 2 we get

**Corollary 3 (\cite{J}, Cor. 6.7).** If \( F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m, n \geq 2, \) is a polynomial mapping and \( S := \{ z \in \mathbb{C}^n : f_1(z) \cdot \ldots \cdot f_m(z) = 0 \} \) is not empty, then \( F \) is proper if and only if \( F|S \) is proper.

Another corollary from Theorem 1 is an effective formula for the Łojasiewicz exponent, generalizing an earlier result of the authors (\cite{CK}, Main Theorem).

Let us introduce some notions. If \( \Psi : \{ z \in \mathbb{C} : |z| > R \} \to \mathbb{C}^k \) is the sum of a Laurent series of the form

\[
\Psi(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots, \quad \alpha_i \in \mathbb{C}^k, \; \alpha_p \neq 0,
\]

then we put \( \deg \Psi := p \). Additionally, \( \deg \Psi := -\infty \) if \( \Psi = 0 \). For an algebraic curve in \( \mathbb{C}^2 \), the notions of its branches in a neighbourhood of \( \infty \) and parametrizations of these branches are defined in \cite{CK}.

Let now \( F = (f_1, \ldots, f_m) : \mathbb{C}^2 \to \mathbb{C}^m \) be a polynomial mapping and \( S := \{ z \in \mathbb{C}^2 : f_1(z) \cdot \ldots \cdot f_m(z) = 0 \} \). Assume that \( S \neq \emptyset \) and \( S \neq \mathbb{C}^2 \).

**Theorem 2.** If \( \Gamma_1, \ldots, \Gamma_s \) are branches of the curve \( S \) in a neighbourhood \( Y \) of infinity and \( \Phi_i : U_i \to Y, i = 1, \ldots, s, \) are their parametrizations, then

\[
\mathcal{L}_\infty(F) = \min_{i=1}^s \deg F \circ \Phi_i / \deg \Phi_i.
\]

**Proof.** Define \( \lambda_i := \deg F \circ \Phi_i / \deg \Phi_i \). If \( \lambda_i = -\infty \) for some \( i \), then (9) holds. So, assume that \( \lambda_i \neq -\infty, i = 1, \ldots, s \). Then

\[
|F(z)| \sim |z|^{\lambda_i} \quad \text{as} \; |z| \to +\infty, \; z \in \Gamma_i.
\]

Hence, taking into account the equality \( S \cap Y = \Gamma_1 \cup \ldots \cup \Gamma_s \) we get (9).

4. Proof of the main theorem. Let us begin with a lemma on polynomial mappings from \( \mathbb{C} \) into \( \mathbb{C}^m \). It is a generalization of a result by A. Płoski (\cite{P1}, Lemma 3.1) and plays a key role in the proof of the main theorem.
Lemma 2. Let \( \Phi = (\varphi_1, \ldots, \varphi_m) : \mathbb{C} \to \mathbb{C}^m \) be a polynomial mapping and \( \varphi := \varphi_1 \cdot \ldots \cdot \varphi_m \). If \( \varphi \) is a polynomial of positive degree and \( T \) is its set of zeroes, then for every \( t \in \mathbb{C} \),

\[
|\Phi(t)| \geq 2^{-\deg\Phi} \min_{\tau \in T} |\Phi(\tau)|.
\]

Proof. Fix \( t_0 \in \mathbb{C} \). Let \( \min_{\tau \in T} |t_0 - \tau| \) be attained for some \( \tau_0 \in T \). If \( \varphi_i \) is a polynomial of positive degree and has the form \( \varphi_i(t) = c_i \prod_{j=1}^{\deg\varphi_i} (t - \tau_{ij}) \), then we have

\[
2|t_0 - \tau_{ij}| = |t_0 - \tau_{ij}| + |t_0 - \tau_i| \geq |t_0 - \tau_0| + |t_0 - \tau_{ij}| \geq |\tau_0 - \tau_{ij}|.
\]

Hence

\[
2^{\deg\varphi_i} |\varphi_i(t_0)| \geq |\varphi_i(\tau_0)|.
\]

Obviously, this inequality is also true for \( \varphi_i \) being a constant. Since \( \deg\Phi \geq \deg\varphi_i \), from the above we get

\[
2^{\deg\Phi} |\Phi(t_0)| \geq |\Phi(\tau_0)| \geq \min_{\tau \in T} |\Phi(\tau)|,
\]

which ends the proof.

In the sequel, \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n, n \geq 2 \), and for every \( i \in \{1, \ldots, n\} \) we put \( z_i' := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \).

We state an easy lemma without proof.

Lemma 3. Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a non-constant polynomial function and \( S \) its set of zeroes. If \( \deg f = \deg z_i \) for every \( i \in \{1, \ldots, n\} \), then there exist constants \( C \geq 1, D > 0 \) such that for every \( i \in \{1, \ldots, n\} \),

\[
|z_i| \leq C|z_i'| \text{ for } z \in S \text{ and } |z_i'| > D.
\]

Proof of Theorem 1. Without loss of generality we may assume that

(i) \( S \neq \mathbb{C}^n \),
(ii) \( \#(F|S)^{-1}(0) < \infty \).

In fact, if (i) does not hold then (8) is obvious, whereas if (ii) does not hold then (8) follows from Corollary 1.

Obviously \( N(F) \subset N(F|S) \). So, to prove (8) it suffices to show

\[
(10) \quad N(F|S) \subset N(F).
\]

Put \( f := f_1 \cdot \ldots \cdot f_m \). From (i) we have \( \deg f > 0 \). Since the sets \( N(F|S) \) and \( N(F) \) are invariant with respect to linear changes of coordinates in \( \mathbb{C}^n \) we may assume that

\[
(11) \quad \deg f = \deg z_i, \quad i = 1, \ldots, n.
\]

This obviously implies

\[
(12) \quad \deg f_j = \deg z_i, \quad j = 1, \ldots, m, \quad i = 1, \ldots, n.
\]
It follows from (ii) and Corollary 1 that \( N(F|S) \) is not empty. Take \( \nu \in N(F|S) \). Then there exist \( A > 0, B > 0 \) such that
\[
|F(\zeta)| \geq A|\zeta|^\nu \quad \text{for } \zeta \in S, \ |\zeta| > B.
\]
By (11) and Lemma 3 there exist \( C \geq 1, D > 0 \) such that for every \( i \in \{1, \ldots, n\} \),
\[
|z_i| \leq C|z'_i| \quad \text{for } z \in S, \ |z'_i| > D.
\]
Put \( A_1 := 2^{-\deg F} A \min(1, C\nu) \) and \( B_1 := \max(B, D) \). Take arbitrary \( \hat{z} \in \mathbb{C}^n \) such that \( |\hat{z}| > B_1 \). Clearly,
\[
|\hat{z}| = |\hat{z}'| \quad \text{for some } i.
\]
Define \( \varphi_j(t) := f_j(\hat{z}_1, \ldots, \hat{z}_{i-1}, t, \hat{z}_{i+1}, \ldots, \hat{z}_n) \), \( \Phi := (\varphi_1, \ldots, \varphi_m) \). Then from (12) we have
\[
\deg F = \deg \Phi.
\]
Moreover, from (11) it follows that \( \varphi := \varphi_1 \cdot \cdots \cdot \varphi_m \) is a polynomial of positive degree. Then, from Lemma 2 \( (T \) is defined as in Lemma 2) and (15) we have
\[
|F(\hat{z})| = |\Phi(\hat{z}_i)| \geq 2^{-\deg \Phi} \min_{\tau \in T} |\Phi(\tau)| = 2^{-\deg F} |F(\hat{z}')|
\]
for some \( \hat{\zeta} = (\hat{z}_1, \ldots, \hat{z}_{i-1}, \tau_0, \hat{z}_{i+1}, \ldots, \hat{z}_n), \tau_0 \in T \). So, \( \hat{\zeta} \in S \). Since \( |\hat{z}| > B_1 \) and \( |\hat{\zeta}| \geq |\hat{z}'_i| = |\hat{z}| \), from (16) and (13) we get
\[
|F(\hat{z})| \geq 2^{-\deg F} A|\hat{\zeta}|^\nu,
\]
whereas from (14),
\[
|\hat{z}| \leq |\hat{\zeta}| \leq C|\hat{z}|.
\]
Considering two cases, when \( \nu \geq 0 \) and \( \nu < 0 \), from (17) and (18) we easily get
\[
|F(\hat{z})| \geq A_1 |\hat{z}|^\nu.
\]
Since \( \hat{z} \) is arbitrary we have \( \nu \in N(F) \).

This ends the proof of the theorem.

References


Lojasiewicz exponent


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