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On irreducible components of a Weierstrass-type variety

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Abstract. We give a characterization of the irreducible components of a Weierstrass-type (W-type) analytic (resp. algebraic, Nash) variety in terms of the orbits of a Galois group associated in a natural way to this variety. Since every irreducible variety of pure dimension is (locally) a component of a W-type variety, this description may be applied to any such variety.

1. Introduction. This work grew out of an attempt to provide an algebraic description of analytic varieties of constant dimension. We study Weierstrass type varieties introduced by Whitney in [6]. Since any analytic set of constant dimension is a sum of irreducible components of a W-type variety (see [6], p. 81), we can consider only the irreducible components of a Weierstrass-type set.

The main aim of this paper is to characterize the irreducible components of a W-type variety in terms of an action of a Galois group associated in a natural way with the given variety.

Let U be an open connected subset in $\mathbb{C}^n$. We denote by R one of the following rings:

- $\mathbb{C}[u]$ = ring of polynomials in n variables,
- $\mathcal{O}(U)$ = ring of holomorphic functions on U,
- $\mathcal{N}(U)$ = ring of Nash functions on U.

The ring $\mathcal{N}(U)$ is the algebraic closure of $\mathbb{C}[u]$ in $\mathcal{O}(U)$ (for further properties of Nash functions cf. [5]).

Let W be a W-type (Weierstrass type) n-dimensional variety in a connected open set $U \times \mathbb{C}^k \subset \mathbb{C}^{n+k}$:

$$W = \{(u, z_1, \ldots, z_k) \in U \times \mathbb{C}^k \mid p_i(u)(z_i) = 0,\ i = 1, \ldots, k\}$$

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where

$$p_i(u)(z) = z^{n_i} + \sum_{j=0}^{n_i-1} a_j^i z^j \in R[z],\ a_j^i \in R.$$

Let K be the field of fractions of the ring R, and L be a common splitting field over K of the defining polynomials $p_1, \ldots, p_k$ of W. Then L/K is a Galois extension. Let $X_i \subset L$ be the zero-set of the polynomial p_i in L. The Galois group $\mathrm{Gal}(L/K)$ acts in a natural way on $X_1 \times \ldots \times X_k$. We prove that the orbits of this action are in 1:1 correspondence with the irreducible components of W (Theorem 6.1).

In Section 2 we introduce some notation and state a few basic facts from Galois theory. In Section 3 we construct a covering space $\mathrm{Hom}(L, \mathcal{M})$ of $U' := U \backslash \{\text{branching locus of } W\}$, encoding the algebraic and topological structure of the problem. We investigate its relations with W, which will be exploited in the proof of the irreducibility of the variety associated with an orbit of $\mathrm{Gal}(L/K)$. In Section 4 the properties of $\mathrm{Hom}(L, \mathcal{M})$ are used to construct a homomorphism from the fundamental group $\pi_1(U')$ to the Galois group $\mathrm{Gal}(L/K)$.

In Section 5, certain finite subsets of L^k are shown to correspond to irreducible components of W. Finally, in Section 6, we state and prove our main result. The proof is based on the observation that every irreducible component of W can be obtained by the construction described in Section 5.

2. Algebraic preliminaries. We begin with two results from Galois theory.

Lemma 2.1 ([2], p. 42). *Let K be an algebraic extension of the field k contained in the algebraic closure $\overline{k}$ of k. Then the following statements are equivalent*:

1. *The field extension K/k is normal.*
2. *Every k-homomorphism $\sigma : K \to \overline{k}$ is onto K.*

Lemma 2.2. *Let L/K be an algebraic field extension*, *and $K \subset \Omega$ an arbitrary field extension. Then there is a K-homomorphism $\sigma : L \to \overline{\Omega}$.*

Let u be a point in U. We shall use the following notation:

- $\mathcal{O}_u$ = ring of germs of holomorphic functions at u,
- $\mathcal{M}_u$ = field of fractions of the ring $\mathcal{O}_u$,
- $\mathcal{M}(U)$ = field of fractions of the ring $\mathcal{O}(U)$,
- $\widetilde{\mathcal{N}}(U)$ = field of fractions of the ring $\mathcal{N}(U)$ of Nash functions.

Let $\delta_i \in \mathcal{O}(U)$ be the discriminant of the polynomial $p_i(z) \in \mathcal{O}(U)[z]$ (for the definition and basic properties of the discriminant see e.g. [3], p. 25). Let Δ be the discriminant variety $\Delta := \{u \in U : \delta_1(u) \cdot \ldots \cdot \delta_k(u) = 0\}$. We shall denote its complement in U by U'. Then U' is the biggest open set such that $\pi : (U' \times \mathbb{C}^k) \cap W \ni (u,z) \mapsto u \in U'$ is an unbranched covering. Furthermore all the defining polynomials of W split in the local rings $\mathcal{O}_u$ for all points $u \in U'$. If V is a set in $\mathbb{C}^{n+k}$ then V' is defined to be $V' := V \cap \pi^{-1}(U')$ where $\pi : \mathbb{C}^{n+k} \to \mathbb{C}^n$ is the standard projection on the first n variables. The structure of a W-type variety is studied in detail in [6].

Recall that L is defined to be a common splitting field over K of the polynomials $p_1, \ldots, p_k$. Without loss of generality one can assume that L is contained in a fixed algebraic closure of $\mathcal{M}(U)$.

Remark 2.3. Let L be the common splitting field of the polynomials p_i. Take $u \in U'$ (i.e. outside the discriminant variety) and put $\Omega := \mathcal{M}_u$. Applying Lemma 2.2 we get a homomorphism $\sigma : L \to \mathcal{M}_u$ which maps the roots of the polynomials p_i to germs of holomorphic functions.

The following simple observation on K-homomorphisms of L to $\mathcal{M}_u$ will be used extensively.

LEMMA 2.4. *Let L/K be a Galois extension and let σ_1, $\sigma_2 : L \to \mathcal{M}_u$ be K-homomorphisms. Then there exists $g \in \mathrm{Gal}(L/K)$ such that $\sigma_1 = \sigma_2 \circ g$.*

In the sequel we use the Riemann extension theorem to extend holomorphic functions through the discriminant locus. The assumptions of the theorem are satisfied due to the lemma:

LEMMA 2.5 ([3], p. 86). *Let $s \in \mathbb{C}$ be a root of a monic polynomial*:

$$s^n + a_1 s^{n-1} + \ldots + a_n = 0.$$

Then

$$|s| \le 2 \max_{i=1,\ldots,n} |a_i|^{1/i}.$$

3. The covering $\mathrm{Hom}(L,\mathcal{M}) \to U'$.

In this section the set of all K-homomorphisms from L to $\mathcal{M}_u$ is endowed with the structure of a covering of U', and is used, in the next section, to define a homomorphism of the fundamental group $\pi_1(U')$ into $\mathrm{Gal}(L/K)$.

DEFINITION 3.1. Let $\mathrm{Hom}(L,\mathcal{M}_u)$ be the set of K-homomorphisms from L to $\mathcal{M}_u$. We define $\mathrm{Hom}(L,\mathcal{M}) := \coprod_{u \in U'} \mathrm{Hom}(L,\mathcal{M}_u)$ to be the disjoint sum of $\mathrm{Hom}(L,\mathcal{M}_u)$.

We introduce a topology on $\mathrm{Hom}(L,\mathcal{M})$ in a similar manner to the case of the sheaf of germs of holomorphic functions (cf. [4], p. 203).

Let $e_1, \ldots, e_m$ be a basis of the field extension L/K, and let $\sigma_u \in \mathrm{Hom}(L,\mathcal{M})$ be a K-homomorphism from L to $\mathcal{M}_u$. Let $\boldsymbol{f}_i := \sigma_u(e_i)$ be the images of basis elements of L in $\mathcal{M}_u$. Without loss of generality we can assume that the $\boldsymbol{f}_i$'s are germs of holomorphic functions. Let the pairs

(U_i, f_i) for $i = 1, \ldots, m$, where $f_i \in \mathcal{O}(U_i)$, be representatives of the germs $\boldsymbol{f}_i$ such that $U_1 \cap \ldots \cap U_m$ is a connected open set. As the basis of the topology in $\mathrm{Hom}(L, \mathcal{M})$ we take all the sets of the form

$$\{\sigma\ : e_i \mapsto (U_i, f_i)_{u'}\}_{u' \in U_1 \cap \ldots \cap U_m}$$

where $(U_i, f_i)_{u'}$ denotes the germ of f_i in the local ring $\mathcal{O}_{u'}$.

PROPOSITION 3.2. (1) *The family $\mathcal{B}$ of all sets defined above is a basis of a topology.*

(2) $\mathrm{Hom}(L, \mathcal{M})$ *with the natural projection $\pi : \mathrm{Hom}(L, \mathcal{M}) \to U'$ is a topological covering, i.e. for every $u \in U'$ there exists an open, connected neighborhood V of u such that $\pi^{-1}(V) = \bigcup U_\alpha$ where U_α are disjoint open sets and $\pi|_{U_\alpha} : U_\alpha \to V$ is a surjective homeomorphism.*

Proof. The first part is obvious. We will give the proof of the second statement. Take $u \in U'$. Let σ_u be a K-homomorphism from L to $\mathcal{M}_u$ (i.e. a point lying in the fiber of u). Let $\{\sigma_{u'} : u' \in V\}$ be an element of the basis of neighborhoods of σ_u. Then for every $g \in \mathrm{Gal}(L/K)$ the set $\{\sigma_{u'} \circ g : u' \in V\}$ is a neighborhood of $\sigma_u \circ g$ lying above V.

In this way we obtain the inclusion $\pi^{-1}(V) \supset \bigcup_{g \in \mathrm{Gal}(L/K)} \{\sigma_u \circ g\}$. The sets $\{\sigma_u \circ g\}$ are disjoint (by the identity principle) and open. It remains to show that $\pi^{-1}(V) = \bigcup_{g \in \mathrm{Gal}(L/K)} \{\sigma_u \circ g\}$. But by Lemma 2.4, given any two K-homomorphisms σ_1, σ_2 to $\mathcal{M}_{u_0}$, there is a $g \in \mathrm{Gal}(L/K)$ such that $\sigma_1 = \sigma_2 \circ g$. So for any K-homomorphism σ lying above u we can find an element of the Galois group $\mathrm{Gal}(L/K)$ such that $\sigma \in \{\sigma_{u'} \circ g\}$. ∎

Since $\mathrm{Hom}(L, \mathcal{M})$ is a covering, given a path $\gamma : [0, 1] \to U'$ and any K-homomorphism σ_0 from L to $\mathcal{M}_{\gamma(0)}$, there is a unique lifting of γ to a path $\widetilde{\gamma} : [0, 1] \to \mathrm{Hom}(L, \mathcal{M})$ such that $\widetilde{\gamma}(0) = \sigma_0$. This lifting has good properties with respect to the action of $\mathrm{Gal}(L/K)$, namely we have

PROPOSITION 3.3. *Let γ be a path in U', and let $\widetilde{\gamma}$ be its lifting to* $\mathrm{Hom}(L, \mathcal{M})$ *such that $\widetilde{\gamma}(0) = \sigma_0$. Then the map $t \mapsto (\widetilde{\gamma}(t)) \circ g$ is a lifting of γ to* $\mathrm{Hom}(L, \mathcal{M})$ *starting from $\sigma_0 \circ g$, for $g \in \mathrm{Gal}(L/K)$.*

Furthermore, liftings of curves to W' can be constructed from a lifting to $\mathrm{Hom}(L, \mathcal{M})$. To define the ith component of the path in W', first we lift γ to $\mathrm{Hom}(L, \mathcal{M})$ obtaining a K-homomorphism from L to $\mathcal{M}_{\gamma(t)}$. We can evaluate it on s_i (a root of the polynomial p_i) to get a holomorphic function in the neighborhood of $\gamma(t)$. Finally, we take its value on $\gamma(t) \in U' \subset \mathbb{C}^n$.

PROPOSITION 3.4. *If $\widetilde{\gamma}$ is a lifting of γ to* $\mathrm{Hom}(L, \mathcal{M})$, *and $s_i \in X_i$, then the map*

$$t \mapsto (\gamma(t), (\widetilde{\gamma}(t)(s_1))(\gamma(t)), \ldots, (\widetilde{\gamma}(t)(s_k))(\gamma(t)))$$

is a lifting of γ to W'.

We have two natural covering spaces of U', namely the original W-type variety W' and the covering $\mathrm{Hom}(L,\mathcal{M})$. The latter can be seen as encoding the algebraic relations between the coordinate functions (projections) restricted to W':

$$z_i : U \times \mathbb{C}^k \supset W' \ni (u, z_1, \ldots, z_k) \mapsto z_i \in \mathbb{C}.$$

EXAMPLE 3.5. If $W = \{z_1^2 - u = 0, \ldots, z_k^2 - u = 0\} \subset \mathbb{C} \times \mathbb{C}^k$ then W' is a 2^k-sheeted covering of $U' \subset \mathbb{C}$. The fiber over each point $u \in U'$ of the covering $\mathrm{Hom}(L,\mathcal{M})$ is, by construction, bijective with the Galois group $\mathrm{Gal}(L/K)$. In this case L is the splitting field of the polynomial $z^2 - u$ over $\mathbb{C}(u)$. So the Galois group $\mathrm{Gal}(L/K)$ is $\mathbb{Z}_2$. The covering $\mathrm{Hom}(L,\mathcal{M})$ is therefore 2-sheeted. We see therefore that in general W' is not isomorphic to $\mathrm{Hom}(L,\mathcal{M})$.

EXAMPLE 3.6. If $W = \{z_1^2 - u_1 = 0, \ldots, z_k^2 - u_k = 0\} \subset \mathbb{C}^k \times \mathbb{C}^k$ then W' is also a 2^k-sheeted covering of $U' \subset \mathbb{C}^k$. But now L is the splitting field of the polynomial $(z^2 - u_1) \cdot \ldots \cdot (z^2 - u_k)$ over the field $\mathbb{C}(u_1, \ldots, u_k)$. The Galois group is now $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. Here the covering $\mathrm{Hom}(L,\mathcal{M})$ is also 2^k-sheeted, and one can show that it is in fact isomorphic to W'.

In some cases the structure of $\mathrm{Hom}(L,\mathcal{M})$ can be richer than that of the original variety W'.

EXAMPLE 3.7. Let $W = \{z^n + u_1 z^{n-1} + \ldots + u_n = 0\}$. Hence W' is a n-sheeted covering. But now the Galois group $\mathrm{Gal}(L/K)$ is isomorphic to the permutation group S_n. Therefore $\mathrm{Hom}(L,\mathcal{M})$ is a $n!$-sheeted covering of U'.

4. Homomorphism $\pi_1(U') \to \mathrm{Gal}(L/K)$. In this section we define a homomorphism of the fundamental group of U' to the Galois group $\mathrm{Gal}(L/K)$ and establish some of its properties.

Let $\sigma_0 : L \to \mathcal{M}_{u_0}$ be a K-homomorphism, and γ be some representative of $[\gamma] \in \pi_1(U', u_0)$. There is a unique lifting of γ to $\mathrm{Hom}(L,\mathcal{M})$ starting from σ_0. Since γ is a closed loop, $\widetilde{\gamma}(1)$ is also a K-homomorphism from L to $\mathcal{M}_{u_0}$. Then by Lemma 2.4 there is a unique $g \in \mathrm{Gal}(L/K)$ such that $\widetilde{\gamma}(1) = \sigma_0 \circ g$. By the homotopy lifting property of coverings, g is independent of the choice of a representative of $[\gamma] \in \pi_1(U', u_0)$.

DEFINITION 4.1. We shall denote by $\mathfrak{Gal}$ the mapping of the fundamental group $\pi_1(U', u_0)$ to the Galois group $\mathrm{Gal}(L/K)$ obtained by the above construction.

PROPOSITION 4.2. *Let* $\gamma_1, \gamma_2 \in \pi_1(U', u_0)$. *Then* $\mathfrak{Gal}(\gamma_2 \circ \gamma_1) = \mathfrak{Gal}(\gamma_2) \circ \mathfrak{Gal}(\gamma_1)$.

PROPOSITION 4.3. (1) *If* $K = \mathbb{C}(u)$ *then* $\mathfrak{Gal}(\pi_1(U')) = \mathrm{Gal}(L/(L \cap \widetilde{\mathcal{N}}(U)))$ (*here* $\widetilde{\mathcal{N}}(U)$ *denotes the field of fractions of* $\mathcal{N}(U)$).

(2) *In the analytic or Nash case* ($R = \mathcal{O}(U)$ *or* $R = \mathcal{N}(U)$), *the mapping* $\mathfrak{Gal}$ *is an epimorphism.*

Proof. Take an element $f \in L$ invariant with respect to $\mathfrak{Gal}(\pi_1(U'))$. Since L is algebraic over K we have $L = K(s_1, \ldots, s_p) = K[s_1, \ldots, s_p]$, where s_i are the roots of the defining polynomials of W. Therefore we can write $f = \sum_{j=1}^{p} w_j t_j$, where $w_j \in K$ and t_j are monomials in $s_1, \ldots, s_p$. Multiplying by the denominators of w_j we get $F := hf = \sum_{j=1}^{p} v_j t_j$, where $h, v_j \in R$ (K is the field of fractions of R).

By Remark 2.3 we have $\sigma_u(F) \in \mathcal{O}_u$ for every $u \in U'$. We define $\widehat{F} : U' \to \mathbb{C}$ in the following way:

$$\widehat{F}(u) := \sigma_u(F)(u)$$

where σ_u is obtained from σ_0 by a lifting along some path. It is defined only up to the action of $g \in \mathfrak{Gal}(\pi_1(U'))$, but since F is $\mathfrak{Gal}(\pi_1(U'))$-invariant the images of F in $\mathcal{O}_u$ coincide.

$\widehat{F}$ is clearly holomorphic in U', and locally bounded in U. By the Riemann extension theorem it can be extended to a holomorphic function in U. Now $\widehat{F} = F$ since σ_{u_0} is a monomorphism. This shows that $f = F/h$ is an element of $\mathcal{M}(U)$.

Conversely, if $f \in L$ is holomorphic then it is $\mathfrak{Gal}(\pi_1(U'))$-invariant. ∎

5. Algebraic set associated with an orbit of $\mathrm{Gal}(L/K)$**.** Recall that L is the common splitting field of the defining polynomials $p_1, \ldots, p_k$ of W. Let $X_i \subset L$ be the zero-set of p_i in L. The group $\mathrm{Gal}(L/K)$ acts in a natural way on $X_1 \times \ldots \times X_k \subset L^k$. An orbit is therefore a finite set of points in L^k. Since L is an infinite field there exists a collection of polynomials in k variables whose common zeroes are precisely the given finite set of points.

In the sequel we shall use a standard choice of these polynomials, called the canonical equations.

DEFINITION 5.1. ([6], Appendix V, p. 369, Canonical equations). Let $S := \{(s_1^{(j)}, \ldots, s_k^{(j)})\}_{j=1,\ldots,m} \subset L^k$ be a set of m points. For every m-tuple $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{N}^m$ such that $|\mu| = \mu_1 + \ldots + \mu_m = m$, the polynomial $\Phi_\mu \in L[z_1, \ldots, z_k]$ is defined by

$$\Phi_\mu(z) = \sum_{\nu}^{(\mu)} (z_{\nu_1} - s_{\nu_1}^{(1)}) \cdot \ldots \cdot (z_{\nu_m} - s_{\nu_m}^{(m)})$$

where the summation is over all m-tuples ν such that every $j \in \{1, \ldots, k\}$ appears exactly μ_j times in $(\nu_1, \ldots, \nu_k)$. The set of common zeroes of the above defined polynomials is precisely the set S.

It is easy to see that the polynomials Φ_μ have coefficients in K when S is taken to be $\mathrm{Gal}(L/K)$-invariant. In fact, we shall use a more precise result.

PROPOSITION 5.2. *If* $K = \mathbb{C}(u)$ *then* $\Phi_\mu \in \mathbb{C}[u][z_1, \ldots, z_k]$.

Proof. For every irreducible polynomial $P \in \mathbb{C}[u]$ one can define a valuation $\nu_P : \mathbb{C}(u) - \{0\} \to \mathbb{Z}$ by

$$f = P^{\nu_P(f)} \cdot \frac{Q}{R}, \qquad f \in \mathbb{C}(u) - \{0\},$$

where P, Q and R are relatively prime (cf. [1], p. 139). Since $L/\mathbb{C}(u)$ is algebraic, one can extend this valuation to the field L (see [1], p. 144).

Let $s \in L$ be a zero of one of the defining polynomials of W. We will show that $\nu_P(s) \geq 0$. Suppose that $\nu_P(s) < 0$; then, since $s^n + \sum_i a_i s^i = 0$, we have

$$n\nu_P(s) = \nu_P\Big(-\sum_i a_i s^i\Big) \geq \min_{i=0,\ldots,n-1} (\nu_P(a_i) + i\nu_P(s)) > n\nu_P(s).$$

This is a contradiction. In the last inequality we used the fact that, since the a_i are polynomials, $\nu_P(a_i) \geq 0$.

The coefficients of the polynomials Φ_μ are linear combinations of monomials in s, therefore $\nu_P(\text{coefficients}) \geq 0$. The coefficients are elements of $\mathbb{C}(u)$, so the nonnegativity of ν_P for all irreducible polynomials P implies that the coefficients lie in $\mathbb{C}[u]$. ∎

Since the s_i are mapped locally to holomorphic functions in U' (locally bounded in U) by K-homomorphisms $\sigma \in \mathrm{Hom}(L, \mathcal{M})$, the coefficients are holomorphic in U' and can be extended by the Riemann extension theorem to the whole of U. Thus the Φ_μ always define an analytic set.

DEFINITION 5.3. Let $S \subset X_1 \times \ldots \times X_k$ be an orbit of $\mathrm{Gal}(L/K)$, consisting of m points. Define

$$V_S := \{(u, z) \in U \times \mathbb{C}^k \mid \Phi_\mu(u, z) = 0,\ |\mu| = m\},$$

where Φ_μ are the canonical equations of S.

LEMMA 5.4. *The algebraic* (*resp. analytic*, *Nash*) *variety* V_S *has the following properties*:

1. $V'_S := \pi^{-1}(U') \cap V_S$ *is a covering of* U' *and* $V'_S \subset W$,
2. V'_S *is connected*,
3. $V_S = \overline{V'_S} \subset W$.

Proof. (1) Choose $u \in U'$ and a K-homomorphism $\sigma_0 : L \to \mathcal{M}_u$. Since the $s_i \in S$ are mapped by σ_0 to holomorphic functions in a neighborhood of u, one can evaluate them at u. Taking $\widehat{S}$ to be the resulting set of points in $\mathbb{C}^k$ and repeating the construction of the polynomials Φ_μ one obtains $\pi^{-1}(u) \cap V_S = \widehat{S}$. It follows immediately that $V'_S \subset W'$, and that it is a subcovering.

(2) Let $u_0 \in U'$ and $\sigma_0 : L \to \mathcal{M}_{u_0}$ be as in the definition of the homomorphism $\mathfrak{Gal}$. We will show that any two points $P = (u, z_1, \ldots, z_k) \in V'_S$ and $P' = (u', z'_1, \ldots, z'_k) \in V'_S$ can be connected by a path in V'_S. Without loss of generality one can assume that $u = u_0$.

(a) Take $u' = u = u_0$. From the proof of (1) one can find $(s_1, \ldots, s_k)$, $(s'_1, \ldots, s'_k) \in S \subset X_1 \times \ldots \times X_k$ which map by σ_0 to the points P and P'. There is a $g \in \mathrm{Gal}(L/K)$ which takes $(s_1, \ldots, s_k)$ to $(s'_1, \ldots, s'_k)$. There is a corresponding loop $[\gamma] \in \pi_1(U', u_0)$ which maps to g. Proposition 3.4 gives a lifting of γ to $V'_S \subset W$ starting from P and ending at P'.

(b) General case $u' \neq u = u_0$. Transporting σ_0 from u_0 to u' gives a K-homomorphism $\sigma : L \to \mathcal{M}_{u'}$. One can find $(s'_1, \ldots, s'_k) \in S \subset X_1 \times \ldots \times X_k$ which σ maps to P'. Using Proposition 3.4 again, we obtain a path in V'_S joining P' and some point in the fiber over u_0. This reduces the proof to case (a).

(3) The inclusion $V_S \supset \overline{V'_S}$ is obvious. Take $u \in \Delta$. We construct the set $\widehat{S} \subset \mathbb{C}^k$ in the following way. Take $u_0 \in U'$, $\sigma_0 : L \to \mathcal{M}_{u_0}$ and $(s_1, \ldots, s_k) \in S$. For some path $\gamma : [0,1] \to U$ such that $\gamma(0) = u_0$, $\gamma(1) = u$ and $\gamma([0,1)) \subset U'$ define

$$\widehat{s}_i := \lim_{x \to 1} (\widetilde{\gamma}(x))(s_i)(\gamma(x))$$

where $\widetilde{\gamma}$ is a lifting of γ to $\mathrm{Hom}(L, \mathcal{M})$ starting from σ_0.

The expression $(\widetilde{\gamma}(x))(s_i)(\gamma(x))$ in the above limit is a lifting of γ to W projected onto the ith variable. Since W is bounded there is an accumulation point. Since W is closed there are at most a finite number of such points. But since $(\widetilde{\gamma}(x))(s_i)(\gamma(x))$ is a continuous function in x, the limit exists.

Repeating the construction of Φ_μ using the set $\widehat{S}$ gives $\pi^{-1}(u) \cap V_S \subset \widehat{S} \subset \overline{V'_S}$.

To obtain the inclusion $V_S \subset W$ note that since $V'_S \subset W'$, we have $V_S = \overline{V'_S} \subset \overline{W'} = W$. ∎

6. The main theorem. Now, we are in a position to prove our main theorem.

Let us recall the following notation and basic definitions. An algebraic (resp. Nash, analytic) W-type variety in a connected open set $U \times \mathbb{C}^k \subset \mathbb{C}^{n+k}$ is a set of the form

$$W = \{(u, z_1, \ldots, z_k) \in U \times \mathbb{C}^k \mid p_i(u)(z_i) = 0,\ i = 1, \ldots, k\}$$

where

$$p_i(u)(z) = z^{n_i} + \sum_{j=0}^{n_i-1} a_j^i z^j \in R[z], \qquad a_j^i \in R.$$

Here R denotes the ring $\mathbb{C}[u]$ (resp. $\mathcal{N}(U)$, $\mathcal{O}(U)$), and K the field of fractions of R. L is the common splitting field of the polynomials $p_1, \ldots, p_k$ over K. And finally $X_i \subset L$ is the zero-set of the polynomial $p_i \in R[z]$ in L.

THEOREM 6.1. *Suppose that one of the following holds*:

1. *W is a W-type algebraic variety in* $\mathbb{C}^n \times \mathbb{C}^k$,
2. *W is a W-type Nash variety in a connected open set* $U \times \mathbb{C}^k \subset \mathbb{C}^{n+k}$,
3. *W is a W-type analytic variety in a connected open set* $U \times \mathbb{C}^k \subset \mathbb{C}^{n+k}$.

Then the irreducible components of W are in 1:1 *correspondence with the orbits of* $\mathrm{Gal}(L/K)$ *in the set* $X_1 \times \ldots \times X_k$.

Proof. Let $V_S = \{\Phi_\mu = 0\}$ be the set associated with an orbit as in Definition 5.3. Then by Lemma 5.4, $V_S = \overline{V_S'}$ where $V_S' = \pi^{-1}(U') \cap V_S$. Since V_S' is a connected submanifold (Lemma 5.4), V_S is irreducible. Here we have used the simple observation that V_S' is dense in the regular part of V_S.

Let V be an irreducible component of W. Then by ([3], p. 215), V is the closure of a connected component of the regular part $\operatorname{Reg} W$ of W, $V = \overline{Z}$. Let $(u,z) \in Z'$. Then $(u,z) \in V_S$ for some orbit of $\mathrm{Gal}(L/K)$. By Lemma 5.4, V_S' is connected, so $V_S' \subset Z \subset V$. Taking closures, and using the fact that $\overline{V_S'} = V_S$, we obtain $\overline{V_S} \subset \overline{Z} = V$.

Since $\dim V_S = \dim V$, the inclusion is indeed an equality. ■

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