Univalent harmonic mappings II

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Abstract. Let \( a < 0 < b \) and \( \Omega(a, b) = \mathbb{C} - ((-\infty, a] \cup [b, +\infty)) \) and \( U = \{ z : |z| < 1 \} \). We consider the class \( S_H(U, \Omega(a, b)) \) of functions \( f \) which are univalent, harmonic and sense-preserving with \( f(U) = \Omega \) and satisfying \( f(0) = 0, f_z(0) > 0 \) and \( f_z(0) = 0 \).

1. Introduction. Let \( S_H \) be the class of functions \( f \) which are univalent, sense-preserving, harmonic mappings of the unit disk \( U = \{ z : |z| < 1 \} \) and satisfy \( f(0) = 0 \) and \( f_z(0) > 0 \). Let \( F \) and \( G \) be analytic in \( U \) with \( F(0) = G(0) = 0 \) and \( \text{Re} \ f(z) = \text{Re} \ F(z) \) and \( \text{Im} \ f(z) = \text{Re} \ G(z) \) for \( z \) in \( U \). Then \( h = (F + iG)/2 \) and \( g = (F - iG)/2 \) are analytic in \( U \) and \( f = h + \overline{g} \). \( f \) is locally one-to-one and sense-preserving if and only if \( |g'(z)| < |h'(z)| \) for \( z \) in \( U \) (cf. [4]). If \( h(z) = a_1 z + a_2 z^2 + \ldots, a_1 > 0 \), and \( g(z) = b_1 z + b_2 z^2 + \ldots \) for \( z \) in \( U \), it follows that \( |b_1| < a_1 \) and hence \( a_1 f - b_1 \overline{f} \) also belongs to \( S_H \). Thus consideration is often restricted to the subclass \( S'_H \) of \( S_H \) consisting of those functions in \( S_H \) with \( f_z(0) = 0 \).

Various authors have studied subclasses of \( S'_H \) consisting of functions mapping \( U \) onto a specific simply connected domain. See for example Hengartner and Schober [5], Abu-Muhanna and Schober [1], and Cima and the author [2], [3]. Recently the author [7] studied the subclass of functions mapping \( U \) onto the plane with the interval \( (-\infty, a], a < 0 \), removed. See also Hengartner and Schober [6]. In the present paper we consider the case when \( f(U) \) is \( C - ((-\infty, a] \cup [b, +\infty)) \), \( a < 0 < b \).

Let \( a < 0 < b \) and \( \Omega(a, b) = \mathbb{C} - ((-\infty, a] \cup [b, +\infty)) \). Then \( S_H(U, \Omega(a, b)) \) is the class of functions \( f \) in \( S'_H \) with \( f(U) = \Omega(a, b) \). Without loss of generality, we assume that \( a + b \geq 0 \).

In the sequel \( F \) and \( G \) will be functions analytic in \( U \) with \( F(0) = G(0) = 0 \), \( \text{Re} \ F(z) = \text{Re} \ F(z) \) and \( \text{Im} \ f(z) = \text{Re} \ G(z) \) for \( z \) in \( U \). If \( h = (F + iG)/2 \) and \( g = (F - iG)/2 \), then \( f = h + \overline{g} \) and \( |g'(z)| < |h'(z)| \) for \( z \) in \( U \).

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2. Preliminary lemmas. Let \( \mathcal{P} \) be the class of functions \( P(z) \) which are analytic in \( U \) with \( P(0) = 1 \) and \( \text{Re} \, P(z) > 0 \) for \( z \) in \( U \). To get an integral representation of functions in \( S_H(U, \Omega(a, b)) \) we require a few lemmas.

**Lemma 1.** Let

\[
T(x) = \frac{1}{0} \left( \frac{a(1+t)^2}{(1+xt+t^2)^2} + \frac{b(1-t)^2}{(1-xt+t^2)^2} \right) dt
\]

and

\[
S(x) = \frac{1}{0} \left( \frac{a(1-t)^2}{(1+xt+t^2)^2} + \frac{b(1+t)^2}{(1-xt+t^2)^2} \right) dt
\]

where \( a < 0 < b, \ a + b \geq 0 \) and \( -2 < x < 2 \). There exist unique numbers \( c_1 \) and \( c_2 \) with \( -2 < c_1 < 0 < c_2 < 2 \) so that \( S(c_1) = T(c_2) = 0 \). Moreover, \( T(x) \leq 0 \leq S(x) \) if and only if \( c_1 \leq x \leq c_2 \).

**Proof.** We note that

\[
S(x) - T(x) = \frac{1}{0} \left( \frac{-2at}{(1+xt+t^2)^2} + \frac{2bt}{(1-xt+t^2)^2} \right) dt \geq 0.
\]

Thus \( T(x) \leq S(x) \) for \(-2 < x < 2\). Also, it is easily checked that \( T'(x) > 0 \) and \( S'(x) > 0 \) for \(-2 < x < 2\). Thus \( T(x) \) and \( S(x) \) are both strictly increasing. Since \( \lim_{x \to -2} T(x) = \lim_{x \to -2} S(x) = -\infty \) and \( \lim_{x \to 2} T(x) = \lim_{x \to 2} S(x) = +\infty \), it follows that there exist unique \( c_1 \) and \( c_2 \) so that \( S(c_1) = T(c_2) = 0 \) and that \( c_1 < c_2 \). Moreover, \( S(0) > 0 \), thus \( c_1 < 0 \) and \( T(x) \leq 0 \leq S(x) \) if and only if \( c_1 \leq x \leq c_2 \).

**Lemma 2.** Let \( P(z) \) be in \( \mathcal{P} \) and

\[
Q(z) = a \frac{1}{0} \frac{1-t^2}{(1+zt+t^2)^2} \text{Re} \, P(t) \, dt
\]

\[
+ b \frac{1}{0} \frac{1-t^2}{(1-zt+t^2)^2} \text{Re} \, P(-t) \, dt
\]

where \( a < 0 < b, \ a + b \geq 0 \) and \(-2 < x < 2\). There exists a unique \( c, \ -2 < c < 2 \), so that \( Q(c) = 0 \).

**Proof.** It is easily checked that \( Q'(x) > 0 \) for \(-2 < x < 2\), \( \lim_{x \to -2} Q(x) = -\infty \) and \( \lim_{x \to 2} Q(x) = +\infty \). The lemma then follows.

**Lemma 3.** With the same hypotheses as in Lemma 2 and with \( a \) and \( b \) fixed we have \( c_1 \leq c \leq c_2 \) where \( c_1 \) and \( c_2 \) are given in Lemma 1. The range for \( c \) is sharp in the sense that for each \( c, \ c_1 \leq c \leq c_2 \), there exists \( P(z) \) in \( \mathcal{P} \) such that the corresponding \( Q \) given by (2.3) satisfies \( Q(c) = 0 \).
Proof. Let \( P(z) \) be in \( \mathcal{P} \) and the corresponding \( Q \) in (2.3) satisfy \( Q(c) = 0 \). Using the inequalities \((1 - |z|)/(1 + |z|) \leq \text{Re } P(z) \leq (1 + |z|)/(1 - |z|)\) for \( z \) in \( U \), we obtain

\[
\frac{(1 - t)^2}{(1 + ct + t^2)^2} \leq \frac{(1 - t^2) \text{Re } P(t)}{(1 + ct + t^2)^2} \leq \frac{(1 + t)^2}{(1 + ct + t^2)^2},
\]

\[
\frac{(1 - t)^2}{(1 - ct + t^2)^2} \leq \frac{(1 - t^2) \text{Re } P(-t)}{(1 - ct + t^2)^2} \leq \frac{(1 + t)^2}{(1 - ct + t^2)^2}.
\]

Since \( a < 0 < b \), this gives

\[
\int_0^1 \left( \frac{a(1 + t)^2}{(1 + ct + t^2)^2} + \frac{b(1 - t)^2}{(1 - ct + t^2)^2} \right) dt 
\leq Q(c) \leq \int_0^1 \left( \frac{a(1 - t)^2}{(1 + ct + t^2)^2} + \frac{b(1 + t)^2}{(1 - ct + t^2)^2} \right) dt.
\]

Thus \( T(c) \leq 0 \leq S(c) \) where \( T \) and \( S \) are given in Lemma 1. From Lemma 2 we have \( c_1 \leq c \leq c_2 \).

To see that the range of \( c \) is sharp, we note that \( Q(c_1) = 0 \) when \( P(z) = (1 - z)/(1 + z) \) and \( Q(c_2) = 0 \) when \( P(z) = (1 + z)/(1 - z) \). If \( c_1 < c < c_2 \) then \( T(c) < 0 < S(c) \). That is,

\[
\int_0^1 \left( \frac{a(1 + t)^2}{(1 + ct + t^2)^2} + \frac{b(1 - t)^2}{(1 - ct + t^2)^2} \right) dt < 0 < \int_0^1 \left( \frac{a(1 - t)^2}{(1 + ct + t^2)^2} + \frac{b(1 + t)^2}{(1 - ct + t^2)^2} \right) dt.
\]

With \( c \) fixed, let

\[
\phi(P) = a \int_0^1 \frac{(1 - t^2) \text{Re } P(t)}{(1 + ct + t^2)^2} dt + b \int_0^1 \frac{(1 - t^2) \text{Re } P(-t)}{(1 - ct + t^2)^2} dt;
\]

then \( \phi \) is a real-valued continuous functional on the convex space \( \mathcal{P} \). From (2.4) it follows that

\[
\phi \left( \frac{1 + z}{1 - z} \right) < 0 < \phi \left( \frac{1 - z}{1 + z} \right)
\]

For \( 0 \leq \lambda \leq 1 \),

\[
\phi \left( \frac{\lambda}{1 + z} + (1 - \lambda) \frac{1 + z}{1 - z} \right)
\]

is a real-valued continuous function of \( \lambda \) for \( 0 \leq \lambda \leq 1 \), with \( \phi(0) < 0 < \phi(1) \). Then there is \( \lambda_1 \) so that \( \phi(\lambda_1) = 0 \). The function \( P_1(z) = \lambda_1(1 - z)/(1 + z) + (1 - \lambda_1)(1 + z)/(1 - z) \) is a member of \( \mathcal{P} \) and the corresponding \( Q \) defined by (2.3) satisfies \( Q(c) = 0 \).
3. The class $S_H(U, \Omega(a, b))$. In the sequel the numbers $c, c_1$ and $c_2$ are those given by Lemmas 1–3.

Let $\mathcal{F}(a, b)$ be the class of functions which have the form

$$f(z) = A \left[ \Re \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{2(1 + cz + \zeta^2)^2} \, d\zeta + i \Im \frac{z}{(1 + cz + z^2)^2} \right]$$

where

$$A = \frac{b}{c} \int_0^1 \frac{(1 - t^2) \Re P(t)}{(1 + ct + t^2)^2} \, dt$$

with $P(z)$ in $\mathcal{P}$ and $c$ is chosen so that $c_1 \leq c \leq c_2$ and

$$b \int_0^1 \frac{(1 - t^2) \Re P(t)}{(1 + ct + t^2)^2} \, dt = a \int_0^1 \frac{(1 - t^2) \Re P(t)}{(1 + ct + t^2)^2} \, dt.$$

We note that by Lemmas 1–3, for each $P$ in $\mathcal{P}$ there is a unique $c$, $c_1 \leq c \leq c_2$, for which (3.2) is satisfied.

**Theorem 1.** If $f$ is a member of $\mathcal{F}(a, b)$, then $f$ is harmonic, sense-preserving and univalent in $U$. Moreover, $f(U)$ is convex in the direction of the real axis and $f(U) \subset \Omega(a, b)$.

**Proof.** Let $f = h + \overline{g} = \Re F + i \Re G$; then

$$F(z) = A \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{(1 + cz + \zeta^2)^2} \, d\zeta \quad \text{and} \quad G(z) = \frac{-iAz}{1 + cz + z^2}.$$

Since

$$\frac{g'(z)}{h'(z)} = \frac{F'(z) - iG'(z)}{F'(z) + iG'(z)} = \frac{P(z) - 1}{P(z) + 1},$$

it follows that $|g'(z)| < |h'(z)|$ for $z$ in $U$. Thus $f$ is locally one-to-one and sense preserving in $U$.

Also,

$$h(z) - g(z) = iG(z) = \frac{Az}{1 + cz + z^2}$$

maps $U$ onto a domain which is convex in the direction of the real axis. By a theorem of Clunie and Sheil-Small [4], $f$ is univalent and $f(U)$ is convex in the direction of the real axis. Also, $f(z)$ is real if and only if $z$ is real. Since $A > 0$ and $\Re P(z) > 0$, it follows that $f(r) = \Re F(r)$ is increasing in $[-1, 1]$ and by (3.2), $\lim_{r \to -1^+} f(r) = a$ and $\lim_{r \to -1^-} f(r) = b$. Thus $f(U)$ omits $(-\infty, a]$ and $[b, +\infty)$. Hence $f(U) \subset \Omega(a, b)$.

**Theorem 2.** $S_H(U, \Omega(a, b)) \subset \mathcal{F}(a, b)$.

**Proof.** Let $f$ be a member of $S_H(U, \Omega(a, b))$ and $f = h + \overline{g}$. Since $\Omega(a, b)$ is convex in the direction of the real axis, by a result of Clunie and
Sheil-Small [4], \( h - g = iG \) is univalent and convex in the direction of the real axis. Thus \( G \) is convex in the direction of the imaginary axis.

Let \( h(z) = a_1 z + a_2 z^2 + \ldots, \ a_1 > 0, \) and \( g(z) = b_2 z^2 + b_3 z^3 + \ldots; \) then \( G = -i(h - g) = -a_1 iz + \ldots \) Since \( f(U) = \Omega(a, b), \) it follows that \( \text{Re} \ G(z) = \text{Im} \ f(z) = 0 \) on the boundary of \( U. \) Since \( G \) is convex in the direction of the imaginary axis, it follows that \( G(U) \) is \( \mathbb{C} \) slit along one or two infinite rays along the imaginary axis. Thus \( G(z)/(-a_1 i) \) maps \( U \) into \( \mathbb{C} \) slit along one or two infinite rays along the real axis. However, \( G(z)/(-a_1 i) \) is a member of the class \( S \) of functions \( q(z) \) analytic and univalent in \( U \) and normalized by \( q(0) = q'(0) - 1 = 0. \) Making use of subordination arguments, it follows that \( G(z)/(-a_1 i) = z/(1 + cz + z^2), \ -2 \leq c \leq 2. \) Hence, \( \text{Im} \ f(r) = \text{Re} \ G(r) = 0 \) for \(-1 < r < 1. \) Since \( f \) is one-to-one and \( f_z(0) > 0, \) the function \( f(r) \) is increasing on \((-1, 1). \) Thus \( \lim_{r \to 1^+} f(r) = a \) and \( \lim_{r \to 1^-} f(r) = b. \)

Since \( |g'(z)/h'(z)| < 1, \) it follows that
\[
P(z) = (h'(z) + g'(z))/(h'(z) - g'(z))
\]
is in \( P. \) Thus, \( h'(z) + g'(z) = (h'(z) - g'(z))P(z) = iG'(z)P(z). \)

Hence,
\[
F(z) = h(z) + g(z) = \int_0^z iG''(\zeta)P(\zeta) \, d\zeta = a_1 \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{(1 + c\zeta + \zeta^2)^2} \, d\zeta.
\]
Therefore,
\[
f(z) = a_1 \left[ \text{Re} \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{(1 + c\zeta + \zeta^2)^2} \, d\zeta + i \text{Im} \frac{z}{1 + cz + z^2} \right]
\]
for some \( c, \ -2 \leq c \leq 1. \)

Since \( a = \lim_{r \to 1^+} f(r) \) and \( b = \lim_{r \to 1^-} f(r), \) we have
\[
a_1 \int_0^1 \frac{(1 - t^2) \text{Re} \ P(t)}{(1 + ct + t^2)^2} \, dt = a \quad \text{and} \quad a_1 \int_0^1 \frac{(1 - t^2) \text{Re} \ P(t)}{(1 + ct + t^2)^2} \, dt = b.
\]

Thus \( c \) must be such that
\[
a_1 \int_0^1 \frac{(1 - t^2) \text{Re} \ P(t)}{(1 + ct + t^2)^2} \, dt + b \int_0^1 \frac{(1 - t^2) \text{Re} \ P(-t)}{(1 - ct + t^2)^2} \, dt = 0.
\]

By Lemmas 2 and 3 there is a unique \( c, \ c_1 \leq c \leq c_2, \) satisfying (3.3). Thus \( f \) is a member of \( \mathcal{F}(a,b). \)

**Lemma 4.** \( \mathcal{F}(a,b) \) is closed.

**Proof.** Let \( f_n \) be a sequence in \( \mathcal{F}(a,b) \) with \( f_n \) converging to \( f \) uniformly
on compact subsets of $U$. Suppose
\[ f_n(z) = b \left( \frac{1}{0} \frac{(1-t)^2 \text{Re} P_n(t)}{1 + d_n t + t^2} dt \right)^{-1} \times \left[ \text{Re} \int_0^z \frac{(1 - \zeta^2)P_n(\zeta)}{1 + d_n \zeta + \zeta^2} d\zeta + i \text{Im} \frac{z}{1 + d_n z + z^2} \right], \]
where $P_n$ is in $\mathcal{P}$ and $d_n$ satisfies (3.2) with $c_1 \leq d_n \leq c_2$. Since $\mathcal{P}$ is normal and $c_1 \leq d_n \leq c_2$ we may assume that $P_n$ converges uniformly on compact subsets of $U$ to $P(z)$ in $\mathcal{P}$ and $d_n$ converges to some $c$. It follows that (3.2) is satisfied for this $c$ and $P(z)$ and that $f$ has the form (3.1) and hence is a member of $\mathcal{F}(a, b)$.

**Theorem 3.** $\mathcal{S}_H(U, \Omega(a, b)) = \mathcal{F}(a, b)$.

**Proof.** Let $f(z)$ have the form (3.1) where (3.2) is satisfied and let $r_n$ be a sequence with $0 < r_n < 1$ and $\lim r_n = 1$. Let $P_n(z) = P(r_n z)$ and denote by $f_n(z)$ the function obtained from (3.1) and (3.2) by replacing $P_n(z)$ with $P_n(z)$. Let $c_n$ be the value of $c$ satisfying (3.2) when $P$ is replaced by $P_n$. We claim that $f_n$ is a member of $\mathcal{S}_H(U, \Omega(a, b))$. To see this let
\[ A_n = b / \text{Re} \int_0^1 \frac{(1 - \zeta^2)P_n(\zeta)}{(1 + c_n \zeta + \zeta^2)^2} d\zeta, \quad F_n(z) = A_n \int_0^z \frac{(1 - \zeta^2)P_n(\zeta)}{(1 + c_n \zeta + \zeta^2)^2} d\zeta. \]
Let $s_n = [-c_n + i \sqrt{1 - c_n^2}] / 2$; then $(1 + c_n \zeta + \zeta^2) = (\zeta - s_n)(\zeta - \overline{s_n})$. Since $P_n$ is analytic for $|z| \leq 1$, there exists $\delta > 0$ so that for $|z - s_n| < \delta$, $P_n(z) = P_n(s_n) + P_n'(s_n)(z - s_n) + \frac{P_n''(s_n)}{2}(z - s_n)^2 + \ldots$
Thus, for $0 < |z - s_n| < \delta$,
\[ F_n'(z) = \frac{A_n(1 - z^2)P_n(z)}{(z - s_n)^2(z - s_n)} = A_n \left[ \frac{B_{-2}}{(z - s_n)^2} + \frac{B_{-1}}{(z - s_n)} + B_0 + B_1(z - s_n) + \ldots \right]. \]
Let $D = \{ z : |z - s_n| < \delta \} - \{ z : z = s_n + te^{i \arg s_n}, \ 0 \leq t \leq \delta \}$. If $z_0 = s_n + te^{i \arg s_n}$, $-\delta < t < 0$, $z_0$ fixed, then for $z \in D$,
\[ F_n(z) - F_n(z_0) = \int_{z_0}^z F_n'(\zeta) d\zeta \]
where the path of integration is in $D$. Thus for $z$ in $D$,
\[ F_n(z) = A_n \left[ \frac{d_{-1}}{z - s_n} + d \log(z - s_n) + q(z) \right] \]
where \( q(z) \) is analytic at \( z = s_n \), and
\[
d_{-1} = \frac{1 - s_n^2}{4 - c_n^2} P_n(s_n).
\]
Thus \( \text{Re} \, d_{-1} > 0 \). We take the branch of log such that for \( z \) in \( D \),
\[
\log(z - s_n) = \ln |z - s_n| + i \arg(z - s_n)
\]
where \( \arg s_n < \arg(z - s_n) < \arg s_n + 2\pi \). Thus for \( z \) in \( D \),
\[
\text{Re} \, f_n(z) = \text{Re} \, F_n(z) = A_n \left[ \text{Re} \, \frac{d_1}{z - s_n} + (\text{Re} \, d) \ln |z - s_n| \right]
\]
\[
- (\text{Im} \, d) \arg(z - s_n) + \text{Re} \, q(z) \right].
\]
We want to prove that \( f_n(z) \) cannot have a finite cluster point at \( z = s_n \).

Let \( z_j = s_n + t_j e^{i \theta_j} \) be in \( U \cap D \) with \( t_j > 0 \) and \( \lim t_j = 0 \) and such that
\[
(3.4) \lim_{j \to \infty} \text{Im} \left( \frac{z_j}{1 + c_n z_j + z_j^2} \right) = l.
\]

Straightforward computation gives
\[
\text{Im} \left[ \frac{z_j}{1 + c_n z_j + z_j^2} \right] = \frac{-2(\text{Im} \, s_n) \text{Re}(s_n e^{-i \theta_j}) + t_j T_j}{t_j |2i \text{Im} s_n + t_j e^{i \theta_j}|^2}
\]
where \( T_j \) is bounded. Because of (3.4), we must have
\[
\lim_{j \to \infty} \text{Re}(s_n e^{-i \theta_j}) = 0.
\]

We now note that
\[
d_{-1} e^{-i \theta_j} = \frac{(1 - s_n^2) e^{-i \theta_j} P_n(s_n)}{4 - c_n^2} = \frac{(1/s_n - s_n) s_n e^{-i \theta_j} P_n(s_n)}{4 - c_n^2}
\]
\[
= \frac{(\pi_n - s_n) s_n e^{-i \theta_j} P_n(s_n)}{4 - c_n^2} = \frac{-2i(\text{Im} \, s_n) s_n e^{-i \theta_j} P_n(s_n)}{4 - c_n^2}.
\]
Thus,
\[
\text{Re}(d_{-1} e^{-i \theta_j}) = \frac{2(\text{Im} \, s_n) \text{Im}(s_n e^{-i \theta_j} P_n(s_n))}{4 - c_n^2}
\]
\[
= \frac{\text{Im}(s_n e^{-i \theta_j} P_n(s_n))}{\sqrt{4 - c_n^2}}.
\]
Since \( \lim_{j \to \infty} \text{Re}(s_n e^{-i \theta_j}) = 0 \), it follows that the only possible accumulation points of \( \{s_n e^{-i \theta_j}\} \) are \( \pm i \). Thus the only possible accumulation points of \( \{s_n e^{-i \theta_j} P_n(s_n)\} \) are \( \pm i P_n(s_n) \). Moreover, \( \text{Im}(\pm i P_n(s_n)) = \pm \text{Re} P_n(s_n) \neq 0 \). Thus \( \text{Re}(d_{-1} e^{-i \theta_j}) \) is bounded away from 0.
It now follows that
\[
|\text{Re } f_n(z_j)| = |\text{Re } F_n(z_j)|
\]
\[
= A_n \left| \frac{\text{Re}(d_{-1}e^{-i\theta_j})}{t_j} + (\text{Re } d) \ln(t_j) - (\text{Im } d) \arg(t_j e^{-i\theta_j}) + \text{Re } q(z_j) \right|
\]
\[
= A_n \left| \frac{\text{Re}(d_{-1}e^{-i\theta_j}) + (\text{Re } d)t_j \ln(t_j) - t_j (\text{Im } d) \arg(t_j e^{-i\theta_j})}{t_j} + \text{Re } q(z_j) \right|
\]
approaches \(\infty\) as \(j \to \infty\). Thus \(f_n\) has no finite cluster points at \(z = s_n\).

Similarly, \(f_n\) has no finite cluster points at \(z = \tilde{s}_n\). At all other points of \(|z| = 1\), the finite cluster points of \(f_n(z)\) are real. Since \(f_n(U) \subset \Omega(a,b)\) and \(\lim_{r \to 1^-} f_n(r) = a\) and \(\lim_{r \to 1^+} f_n(r) = b\), it follows that \(f_n(U) = \Omega(a,b)\).

Thus for each \(n\), \(f_n\) is a member of \(S_H(U, \Omega(a,b))\). We know that the \(P_n\) converge to \(P\) uniformly on compact subsets of \(U\). There exists a subsequence \(c_{n_k}\) convergent to some \(c\). But then (3.2) will be satisfied with \(c\) replaced by \(s\). Since the solution to (3.2) is unique, we must have \(s = c\). Thus \(f_{n_k}\) converges to \(f\) uniformly on compact subsets of \(U\). Therefore, \(f\) is a member of \(S_H(U, \Omega(a,b))\) and \(F(a,b) \subset S_H(U, \Omega(a,b))\). Since \(F(a,b)\) is closed and \(S_H(U, \Omega(a,b)) \subset F(a,b)\), we have \(S_H(U, \Omega(a,b)) \subset F(a,b)\). Thus \(F(a,b) = S_H(U, \Omega(a,b))\).

4. The case \(a = -b\). Referring to the proof of Lemma 1, if \(a = -b\) then
\[
T(0) = \int_0^1 \frac{-4bt}{(1 + t^2)^2} \, dt < 0.
\]
Thus \(c_2 > 0\). Moreover, since \(S(-x) = -T(x)\), we have \(c_1 = -c_2\).

Since \(S_H(U, \Omega(-b,b))\) are the only classes that contain odd functions, we will be interested in \(f\) in \(F(-b,b)\) and \(f\) odd.

**Lemma 5.** Let \(f \in F(-b,b)\) and be odd. If \(f(z) = h(z) + \overline{g(z)}\), then both \(h\) and \(g\) are odd.

**Proof.** Since \(f(-z) = -f(z)\), we have \(h(z) + \overline{g(z)} = -(h(-z) + \overline{g(-z)})\). Thus \(h(z) + h(-z) = -g(z) + g(-z)\). It follows that \(h(z) + h(-z)\) and \(h(z) + h(-z)\) are both analytic in \(U\). Thus \(h(z) + h(-z)\) is constant. Since its value is 0 at \(z = 0\), we have \(h(z) = -h(-z)\). Similarly, \(g(z)\) is odd.

**Lemma 6.** If \(f \in F(-b,b)\) and \(f\) is odd then in the representation (3.1), \(P(z)\) is even and \(c = 0\).
Proof. Let \( h(z) = a_1 z + a_2 z^2 + \ldots \); then
\[
h(z) = \frac{F(z) + iG(z)}{2} = \frac{a_1}{2} \left[ \int_0^1 \frac{(1 - \zeta^2)P(\zeta) + z}{(1 + c\zeta + \zeta^2)^2} d\zeta + \frac{z}{1 + c z + z^2} \right]
\]
where \( c \) and \( P \) satisfy (3.2). Since \((1 - z^2)/(1 + c z + z^2)^2 = (z/(1 + c z + z^2))'\),
this can be written as
\[
h(z) = \frac{a_1}{2} \int_0^1 \frac{(1 - \zeta^2)(P(\zeta) + 1)}{(1 + c\zeta + \zeta^2)^2} d\zeta.
\]
By Lemma 5, \( h(z) = -h(-z) \). Thus,
\[
\int_0^r \frac{(1 - t^2)(P(t) + 1)}{(1 + c t + t^2)^2} dt = -\int_0^r \frac{(1 - t^2)(P(-t) + 1)}{(1 - c t + t^2)^2} dt.
\]
Letting \( r \to 1 \), since \(-2 < -c_2 \leq c \leq c_2 < 2\), we obtain
\[
\int_0^1 \frac{(1 - t^2)(\text{Re} P(t) + 1)}{(1 + c t + t^2)^2} dt = \int_0^1 \frac{(1 - t^2)(\text{Re} P(-t) + 1)}{(1 - c t + t^2)^2} dt.
\]
But (3.2) with \( a = -b \) gives
\[
\int_0^1 \frac{(1 - t^2)(\text{Re} P(-t))}{(1 + c t + t^2)^2} dt = \int_0^1 \frac{(1 - t^2)(\text{Re} P(t))}{(1 - c t + t^2)^2} dt.
\]
Equalities (4.1) and (4.2) imply
\[
\int_0^1 \frac{1 - t^2}{(1 + c t + t^2)^2} dt = \int_0^1 \frac{1 - t^2}{(1 - c t + t^2)^2} dt.
\]
Thus \( 1/(2 + c) = 1/(2 - c) \). Hence \( c = 0 \).
We now have
\[
h(z) = \frac{a_1}{2} \left[ \int_0^\infty \frac{(1 - \zeta^2)P(\zeta)}{(1 + \zeta^2)^2} d\zeta + \frac{z}{1 + z^2} \right]
\]
and \( h(z) \) is odd. Thus
\[
q(z) = \int_0^\infty \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2} d\zeta
\]
is odd. Hence \( q'(z) = (1-z^2)P(z)/(1+z^2)^2 \) is even and thus \( P(z) \) is even.

**Lemma 7.** Let \( f \in \mathcal{F}(-b,b) \) with representation (3.1). If \( P(z) \) is even, then \( c = 0 \) and \( f \) is odd.

**Proof.** If \( P(z) \) is even, then \( Q(x) \) defined by (2.3), with \( a = -b \), satisfies
\[
Q(0) = -\frac{1}{2} (1-t^2) \text{Re} P(t) dt + \frac{1}{2} (1-t^2) \text{Re} P(-t) dt = 0.
\]
But the \( c \) given in Lemma 2 is unique. Thus \( c = 0 \). Therefore
\[
f(z) = a_1 \left[ \text{Re} \int_0^\infty \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2} d\zeta + i \text{Im} \frac{z}{1+z^2} \right],
\]
and since \( P(z) \) is even, it is easily checked that \( f(-z) = -f(z) \).

We now let
\[
G(-b,b) = \{ f \in \mathcal{F}(-b,b) : f \text{ is odd} \}.
\]
If \( f \in G(-b,b) \), then \( f \) has the representation (4.3) with \( P(z) \) in \( P \) and \( P(z) \) even. Also,
\[
a_1 = b/\int_0^1 \frac{(1-t^2) \text{Re} P(t)}{(1+t^2)^2} dt.
\]
We now easily obtain

**Theorem 4.** If \( f \in G(-b,b) \), then
\[
\frac{4b}{\pi} \leq a_1 \leq \frac{8b}{\pi}
\]
and the inequalities are sharp.

**Proof.** Since \( P \in \mathcal{P} \) and \( P \) is even, \( (1-|z|^2)/(1+|z|^2) \leq \text{Re} P(z) \leq (1+|z|^2)/(1-|z|^2) \). Thus
\[
\frac{\pi}{8} = \int_0^1 \frac{(1-t^2)^2}{(1+t^2)^3} dt \leq \int_0^1 \frac{(1-t^2) \text{Re} P(t)}{(1+t^2)^2} dt \leq \int_0^1 \frac{dt}{1+t^2} = \frac{\pi}{4}
\]
and the result follows from (4.4). Equality is attained on the right side of (4.5) when \( P(z) = (1-z^2)/(1+z^2) \) and on the left side when \( P(z) = (1+z^2)/(1-z^2) \). The corresponding extremal functions are
\[
f_1(z) = \frac{8b}{\pi} \left[ \text{Re} \left( \frac{z(1-z^2)}{2(1+z^2)^2} + \frac{1}{2} \arctan z \right) + i \text{Im} \frac{z}{1+z^2} \right].
\]
We find in Section 5 that $f_1(z)$ is actually a member of $S_H(U, \Omega(−b, b))$. Thus the right side of (4.5) is sharp for odd functions in $S_H(U, \Omega(−b, b))$.

**Theorem 5.** Let $f(z) = h(z) + g(z)$ be in $G(−b, b)$ and suppose

$$h(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}.$$

Then

$$|a_{2n+1}| \leq \frac{(n + 1)^2}{2n + 1}|a_1|, \quad n = 0, 1, 2, \ldots ,$$

$$|b_{2n+1}| \leq \frac{n^2}{2n + 1}|a_1|, \quad n = 1, 2, \ldots ,$$

and

$$|a_{2n+1} - b_{2n+1}| = |a_1|$$

and the inequalities are sharp in $S_H(U, \Omega(−b, b))$.

**Proof.** We have

$$h(z) = \frac{a_1}{2} \left[ \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{(1 + \zeta^2)^2} \, d\zeta + \frac{z}{1 + z^2} \right]$$

where $P(z)$ is in $P$ and is even. Let $P(z) = 1 + \sum_{n=1}^{\infty} p_{2n} z^{2n}$; then for $|z| < 1$,

$$\frac{1 - z^2}{(1 + z^2)^2} P(z) = 1 + \sum_{n=1}^{\infty} d_{2n} z^{2n}$$

where

$$d_{2n} = \sum_{k=0}^{n} (-1)^k (2k + 1) p_{2(n-k)} \quad \text{and} \quad p_0 = 1.$$

Then (4.11) gives

$$\frac{2a_{2n+1}}{a_1} = \frac{1}{2n + 1} \sum_{k=0}^{n} (-1)^k (2k + 1) p_{2(n-k)} + (-1)^n$$

$$= \frac{1}{2n + 1} \sum_{k=0}^{n-1} (-1)^k (2k + 1) p_{2(n-k)} + 2(-1)^n.$$
Since \(|p_n| \leq 2\) for all \(n\), we have
\[
\frac{2|a_{2n+1}|}{|a_1|} \leq \frac{2}{2n+1} \sum_{k=0}^{n-1} (2k+1) + 2 = \frac{2n^2}{2n+1} + 2 = \frac{2(n+1)^2}{2n+1},
\]
giving (4.8).

To see the sharpness, let \(P(z) = (1 - z^2)/(1 + z^2)\). With this choice of \(P\), we have \(p_{2n} = 2(-1)^n\) and from (4.12),
\[
\frac{2a_{2n+1}}{a_1} = \frac{1}{2n+1} \sum_{k=0}^{n-1} (-1)^k(2k+1)(-1)^{n-k} \cdot 2 + 2(-1)^n
\]
\[
= (-1)^n \left[ \frac{2}{2n+1} \sum_{k=0}^{n-1} (2k+1) + 2 \right] = \frac{2(-1)^n(n+1)^2}{2n+1},
\]
giving equality in (4.8). The extremal function is the \(f_1(z)\) given in (4.6).

Next we have
\[
g(z) = \frac{a_1}{2} \left[ \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{(1 + \zeta^2)^2} d\zeta - \frac{z}{1 + z^2} \right].
\]
If \(g(z) = \sum_{n=1}^{\infty} b_{2n+1}z^{2n+1}\), then
\[
(4.13) \quad \frac{2b_{2n+2}}{a_1} = \frac{1}{2n+1} \sum_{k=0}^{n-1} (-1)^k(2k+1)p_{2(n-k)}.
\]
Thus
\[
\frac{2|b_{2n+1}|}{|a_1|} \leq \frac{2}{2n+1} \sum_{k=0}^{n-1} (2k+1) = \frac{2n^2}{2n+1},
\]
giving (4.9). Equality again occurs when \(P(z) = (1 - z^2)/(1 + z^2)\) and \(f_1(z)\) is given in (4.6).

Finally, from (4.10) and (4.11),
\[
|a_{2n+1} - b_{2n+1}| = |(-1)^n a_1| = |a_1|.
\]

We remark that the inequalities involved are actually sharp for odd functions in \(S_H(U, \Omega(-b, b))\) since \(f_1 \in S_H(U, \Omega(-b, b))\).

**Theorem 6.** Let \(f(z) = h(z) + g(z)\) be a member of \(G(-b, b)\). Then for \(|z| = r < 1\),
\[
(4.14) \quad \frac{|a_1|(1 - r^2)}{(1 + r^2)^3} \leq |f_z(z)| \leq \frac{|a_1|(1 + r^2)}{(1 - r^2)^3}
\]
and the inequalities are sharp.
Proof. We have
\[ h(z) = \frac{a_1}{2} \left[ \int_0^z \frac{(1 - \zeta^2)P(\zeta)}{(1 + \zeta^2)^2} \, d\zeta + \frac{z}{1 + z^2} \right]. \]
Thus,
\[
(4.15) \quad f_z = h'(z) = \frac{a_1(1 - z^2)}{2(1 + z^2)^2} (P(z) + 1).
\]
Since \( P(z) \) is in \( \mathcal{P} \) and is even, we can write \( P(z) = (1 - w(z))/(1 + w(z)) \) where \( w(z) = d_2 z^2 + \ldots \) is analytic in \( U \) and \( |w(z)| \leq |z|^2 \) for \( z \) in \( U \). Thus \( P(z) + 1 = 2/(1 + w(z)) \). Hence
\[
(4.16) \quad \frac{2}{1 + r^2} \leq \frac{2}{1 + |w(z)|} \leq |P(z) + 1| \leq \frac{2}{1 - |w(z)|} \leq \frac{2}{1 - r^2}.
\]
Using (4.10) and (4.15) we obtain the inequalities (4.14). Equality on the right side of (4.14) is attained by \( f_1(z) \) at \( z = \pm iv \) and equality on the left side of (4.14) is attained by \( f_1(z) \) when \( z = \pm r \).

5. The extremal functions. We now verify that the extremal function \( f_1(z) \) given by (4.6) is actually a member of \( S_H(U, \Omega(-b, b)) \), while the function \( f_2(z) \) given by (4.7) maps \( U \) into the strip \( \{ z : -b < \Re z < b \} \) and hence is a member of \( G(-b, b) - S_H(U, \Omega(-b, b)) \).

To see this we first prove that \( f_1(z) \) has no non-real finite cluster points at \( z = i \). Let \( z_j = i + t_j e^{i\theta_j} \) be such that \( 0 < t_j, \pi < \theta_j < 2\pi, \ |z_j| < 1, \) and \( \lim_{j \to \infty} \Im(z_j/(1 + z_j^2)) = l \neq 0. \) Necessarily \( l > 0. \) A brief computation gives
\[
A_j = \Im \left( \frac{z_j}{1 + z_j^2} \right) = \frac{-(t_j + 2 \sin \theta_j)(1 + t_j \sin \theta_j)}{t_j |z_j + i|^2}.
\]
Thus \( -(t_j + 2 \sin \theta_j)(1 + t_j \sin \theta_j) = t_j |z_j + i|^2 A_j = t_j B_j \) where \( \lim B_j = 4l > 0. \) Hence
\[
-2 \sin \theta_j [1 + t_j \sin \theta_j] = t_j B_j + t_j [1 + t_j \sin \theta_j] = t_j c_j,
\]
where \( \lim c_j = 4l + 1. \) Therefore
\[
(5.1) \quad \sin \theta_j = \frac{t_j c_j}{-2(1 + t_j \sin \theta_j)} = t_j D_j
\]
where \( \lim D_j = -(4l + 1)/2. \) In particular, \( \lim \sin \theta_j = 0, \) so \( \lim \cos \theta_j = 1. \) Let
\[
T(z) = \frac{z(1 - z^2)}{(z - i)^2(z + i)^2};
\]
then in a neighborhood of \( z = i, \)
\[
T(z) = \frac{-i}{2(z - i)^2} - \frac{1}{2(z - i)} + q(z)
\]
where \( q(z) \) is analytic at \( z = i \). Further,

\[
T(z_j) = \frac{-ie^{-i2\theta_j}}{2t_j^2} - \frac{e^{-i\theta_j}}{2t_j} + q(z_j).
\]

Using (5.1), we can write

\[
\text{Re} T(z_j) = \frac{\sin \theta_j \cos \theta_j}{t_j^2} - \frac{\cos \theta_j}{2t_j} + \text{Re} q(z_j)
\]

\[
= -D_j \cos \theta_j \frac{\cos \theta_j}{t_j} + \text{Re} q(z_j) = -\frac{\cos \theta_j(2D_j + 1)}{2t_j} + \text{Re} q(z_j).
\]

Since \( \lim(2D_j + 1) = -4l \neq 0 \) and \( \lim |\cos \theta_j| = 1 \) it follows that \( \lim |\text{Re} T(z_j)| = \infty \) and hence \( \lim |\text{Re} f_1(z_j)| = \infty \). Thus \( f_1 \) has only real cluster points at \( z = i \). Since \( f_1(z) \) is odd, it has only real cluster points at \( z = -i \) as well.

If \( z_0 \neq \pm i \) and \( |z_0| = 1 \), then \( \lim_{z \to z_0} f_1(z) = \pm b \). Since \( f_1(U) \subset \Omega(-b, b) \) and since the interval \((-b, b)\) is covered by \( f_1(U) \), it follows that \( f_1(U) = \Omega(-b, b) \). Thus \( f_1 \) is a member of \( S_{\mathbb{H}}(U, \Omega(-b, b)) \).

We now prove that \( f_2(U) = \{ z : -b < \text{Re} z < b \} \) where \( f_2(z) \) is given by (4.7). We have

\[
\text{Re} f_2(z) = \frac{4b}{\pi} \text{Re}(\text{arctan} z) = \frac{4b}{\pi} \text{Re} \left( i \frac{1}{2} \log \frac{1 - iz}{1 + iz} \right) = -\frac{2b}{\pi} \text{arg} \left( \frac{1 - iz}{1 + iz} \right).
\]

Since \( \text{Re}[(1 - iz)/(1 + iz)] > 0 \), it follows that

\[
|\text{Re} f_2(z)| = \frac{2b}{\pi} \text{arg} \frac{1 - iz}{1 + iz} < \frac{2b}{\pi} \cdot \frac{\pi}{2} = b.
\]

We claim that the cluster points of \( f_1(z) \) at \( z = \pm i \) form the two lines \( \text{Re} z = \pm b \). To see this, let \( l > 0 \). We can choose a sequence \( z_j = i + t_j e^{-i\theta_j} \) with \( \pi < \theta_j < 2\pi \), \( t_j > 0 \) and \( \lim t_j = 0 \), such that

\[
\lim_{j \to \infty} \text{Im} \frac{z_j}{1 + z_j^2} = l.
\]

As in the previous example, \( \lim \sin \theta_j = 0 \) and \( \lim |\cos \theta_j| = 1 \). We have

\[
\text{Re} f_2(z_j) = -\frac{2b}{\pi} \text{arg} \left( \frac{1 - iz_j}{1 + iz_j} \right).
\]

Moreover,

\[
\tan \left[ \text{arg} \left( \frac{1 - iz_j}{1 + iz_j} \right) \right] = -\frac{2 \text{Re} z_j}{1 - |z_j|^2} = -\frac{-2t_j \cos \theta_j}{-2t_j \sin \theta_j - t_j^2} = \frac{2 \cos \theta_j}{2 \sin \theta_j + t_j}.
\]

Making use of computations from the last example, we get

\[
\tan \left[ \text{arg} \left( \frac{1 - iz_j}{1 + iz_j} \right) \right] = \frac{2 \cos \theta_j}{2t_j D_j + t_j} = \frac{2 \cos \theta_j}{t_j(2D_j + 1)}
\]

where \( \lim 2D_j + 1 = -4l < 0 \).
If \( \theta_j \) is chosen so that \( \lim \theta_j = \pi \) then \( \tan(\arg((1 - iz_j)/(1 + iz_j))) \) tends to \( \infty \) and \( \arg((1 - iz_j)/(1 + iz_j)) \) tends to \( \pi/2 \), and thus \( \Re f_2(z_j) \) tends to \(-b\). Hence \(-b + il, l > 0\), is a cluster point. If \( \theta_j \) is chosen so that \( \lim \theta_j = 2\pi \), then we see that \( b + il, l > 0\), is a cluster point. Since \( f_2 \) is odd, it follows that \( \pm b + il, l < 0\), are cluster points at \( z = -i \). It now follows that \( f_2(U) = \{z : -b < \Re(z) < b\} \).

References


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