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On symmetry of the pluricomplex Green function for ellipsoids

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Abstract. We show that in the class of complex ellipsoids the symmetry of the pluricomplex Green function is equivalent to convexity of the ellipsoid.

For a domain $D \subset \mathbb{C}^n$ we define

$$g_D(w,z) := \sup\{u(z) : \log u \in \mathrm{PSH}(D),\ 0 \le u < 1 \text{ and there are } M, R > 0 \text{ such that } u(\zeta) \le M\|\zeta - w\| \text{ for } \|\zeta - w\| < R\},$$

$$\widetilde{k}_D^*(w,z) := \inf\{m(\lambda_1,\lambda_2) : \text{there is } \varphi \in \mathcal{O}(E,D) \text{ with } \varphi(\lambda_1) = w,\ \varphi(\lambda_2) = z\}, \quad w,z \in D,$$

where E is the open unit disk in $\mathbb{C}$,

$$m(\lambda_1,\lambda_2) := \left|\frac{\lambda_1 - \lambda_2}{1 - \overline{\lambda}_1\lambda_2}\right|, \quad \lambda_1,\lambda_2 \in E,$$

$\mathcal{O}(E,D)$ denotes the set of holomorphic functions from E to D and $\mathrm{PSH}(D)$ is the set of plurisubharmonic functions on D.

g_D (respectively, $\widetilde{k}_D^*$) is called the *pluricomplex Green function* (respectively, the *Lempert function*).

Any holomorphic mapping $\varphi : E \to D$ such that $\widetilde{k}_D^*(\varphi(\lambda_1),\varphi(\lambda_2)) = m(\lambda_1,\lambda_2)$ for some $\lambda_1 \ne \lambda_2$ is called a $\widetilde{k}_D$*-geodesic for* $(\varphi(\lambda_1),\varphi(\lambda_2))$.

Below we list some well-known properties of these functions that we shall need in the sequel (for references see [Jar-Pfl], [Lem] and [Kli]):

(1) $\log g_D(z,\cdot) \in \mathrm{PSH}(D)$ for any $z \in D$;

(2) $g_D \le \widetilde{k}_D^*$;

(3) if D is a convex domain then $g_D = \widetilde{k}_D^*$ (in particular, g_D is symmetric);

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(4) if D is a bounded pseudoconvex balanced domain, then $g_D(0,z) = \widetilde{k}_D^*(0,z) = h(z)$ for any $z \in D$, where h is the Minkowski function of D, and $\widetilde{k}_D^*(\lambda_1 b, \lambda_2 b) \le m(\lambda_1, \lambda_2)$ for any $b \in \partial D$.

Let us also define for $p = (p_1, \ldots, p_n)$, where $p_j > 0$, $j = 1, \ldots, n$, $n > 1$,

$$\mathcal{E}(p) := \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1\}$$

The domains $\mathcal{E}(p)$ are called *complex elipsoids.*

It is easy to check that a complex ellipsoid is convex iff $p_j \ge 1/2$ for $j = 1, \ldots, n$.

Our aim is the following:

THEOREM 1. *For a complex ellipsoid* $\mathcal{E}(p)$ *the following conditions are equivalent*:

(i) $k_{\mathcal{E}(p)}^*(\lambda_1 b, \lambda_2 b) = m(\lambda_1, \lambda_2)$ *for any* $b \in \partial\mathcal{E}(p)$ *and* $\lambda_1, \lambda_2 \in E$,
(ii) $g_{\mathcal{E}(p)}(\lambda b, 0) = g_{\mathcal{E}(p)}(0, \lambda b)$ *for any* $b \in \partial\mathcal{E}(p)$ *and* $\lambda \in E$,
(iii) $g_{\mathcal{E}(p)}$ *is symmetric*,
(iv) $\mathcal{E}(p)$ *is convex.*

Remark 2. Theorem 1 shows that the symmetry of the Green function is a rare phenomenon. Non-convex ellipsoids turn out to be examples of very "regular" domains failing to have the symmetry property (for other examples of such domains see e.g. [Bed-Dem], [Pol], and [Jar-Pfl]). *Moreover, our results and methods used in the proof suggest that in the class of bounded pseudoconvex complete Reinhardt domains the symmetry of the Green function is equivalent to the convexity of the domain.*

First we state the following lemma, which is part of Exercise 8.1, p. 290 of [Jar-Pfl]; for completeness we give the proof below:

LEMMA 3. *Let* D *be a domain in* $\mathbb{C}^n$. *Assume that for some* $\lambda_0, \lambda_1 \in E$, $\lambda_0 \ne \lambda_1$,

(5) $$g_D(\varphi(\lambda_0), \varphi(\lambda_1)) = m(\lambda_0, \lambda_1).$$

Then

$$g_D(\varphi(\lambda_0), \varphi(\lambda)) = m(\lambda_0, \lambda) \quad \textit{for any } \lambda \in E.$$

Proof. Define

$$a(\lambda) := \frac{\lambda_0 - \lambda}{1 - \overline{\lambda}_0 \lambda}, \quad \lambda \in E.$$

We obviously have $a \circ a = \mathrm{id}_E$. Let

$$u : E \ni \lambda \to g_D(\varphi(\lambda_0), \varphi(a(\lambda))) \in [0,1).$$

Clearly,

$$u(0) = 0, \quad \log u \in \mathrm{SH}(E).$$

Moreover,

$$u(\lambda) \le \widetilde{k}_D^*(\varphi(\lambda_0), \varphi(a(\lambda))) \le m(\lambda_0, a(\lambda)) = m(0,\lambda) = |\lambda|.$$

So

$$v(\lambda) := \log u(\lambda) - \log|\lambda| \in \mathrm{SH}(E) \quad \text{and} \quad v \le 0.$$

But, in view of (5) and the definition of u, $v(a(\lambda_1)) = 0$, so the maximum principle implies that $v \equiv 0$, and $u(\lambda) = |\lambda|$ for $\lambda \in E$. Finally,

$$g_D(\varphi(\lambda_0), \varphi(\lambda)) = g_D(\varphi(\lambda_0), \varphi(a(a(\lambda)))) = u(a(\lambda)) = |a(\lambda)| = m(\lambda_0, \lambda). \ \blacksquare$$

COROLLARY 4. *Let D be a balanced pseudoconvex bounded domain in $\mathbb{C}^n$, $b \in \partial D$, $\lambda_0 \in E$, $\lambda_0 \ne 0$. Then the following conditions are equivalent*:

(i) $g_D(\lambda_0 b, 0) = g_D(0, \lambda_0 b)$,

(ii) $g_D(\lambda_0 b, \lambda b) = \widetilde{k}_D^*(\lambda_0 b, \lambda b) = m(\lambda_0, \lambda)$ *for any* $\lambda \in E$.

Proof. This follows from $g_D(0, \lambda_0 b) = \widetilde{k}_D^*(0, \lambda_0 b) = |\lambda_0|$, the inequality $g_D \le \widetilde{k}_D^*$ and Lemma 3. ■

Before we go on to the proof of the main theorem let us collect some auxiliary results, which are similar to those in [Pfl-Zwo] (Lemmas 8 and 11) but are adapted to our situation.

LEMMA 5. *Let $\varphi : E \to \mathcal{E}(p)$ be a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $(\varphi(\lambda_1), \varphi(\lambda_2))$. Assume that*

$$\varphi_j(\lambda) = B_j(\lambda)\psi_j(\lambda), \qquad \varphi_j \not\equiv 0, \qquad j = 1, \ldots, n,$$

where ψ_j never vanishes on E and B_j is a Blaschke product (*if φ_j never vanishes*, *then $B_j :\equiv 1$*). *Let $1 \le k < n$ and $t_{k+1}, \ldots, t_n$ be positive natural numbers. Put $q_j := p_j$, $j = 1, \ldots, k$, and $q_j := t_j p_j$, $j = k+1, \ldots, n$. Define*

$$\eta(\lambda) := (\varphi_1(\lambda), \ldots, \varphi_k(\lambda), \psi_{k+1}(\lambda), \ldots, \psi_n(\lambda)),$$

$$\mu(\lambda) := (\varphi_1(\lambda), \ldots, \varphi_k(\lambda), (\psi_{k+1}(\lambda))^{1/t_{k+1}}, \ldots, (\psi_n(\lambda))^{1/t_n}), \qquad \lambda \in E.$$

Then

- *if η is not constant*, *then η is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $(\eta(\lambda_1), \eta(\lambda_2))$*,
- *if μ is not constant*, *then μ is a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $(\mu(\lambda_1), \mu(\lambda_2))$*.

Proof. From a result of A. Edigarian (see [Edi], Theorem 4) we know that each B_j has at most one zero and φ extends continuously to $\bar{E}$. We have clearly $\widetilde{h} \circ \eta(\lambda) \le 1$ for $\lambda \in \partial E$ ($\widetilde{h}(z) := \sum_{j=1}^n |z_j|^{2p_j}$, $z \in \mathbb{C}^n$). The maximum principle for subharmonic functions implies that $\eta(E) \subset \overline{\mathcal{E}(p)}$. But from the form of $\widetilde{k}_{\mathcal{E}(p)}$-geodesics (see [Edi], Theorem 4) we know that $\eta(E) \subset \mathcal{E}(p)$ (one may also obtain the last inclusion without the use of the results from [Edi] but applying the existence of local peak functions—cf. [Pfl-Zwo]).

If η were not a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic, then there would exist $\widetilde{\eta} \in \mathcal{O}(E, \mathcal{E}(p))$ such that $\widetilde{\eta}(E) \Subset \mathcal{E}(p)$ and $\widetilde{\eta}(\lambda_1) = \eta(\lambda_1)$, $\widetilde{\eta}(\lambda_2) = \eta(\lambda_2)$. But setting

$$\widehat{\eta} := (\widetilde{\eta}_1, \ldots, \widetilde{\eta}_k, B_{k+1}\widetilde{\eta}_{k+1}, \ldots, B_n\widetilde{\eta}_n)$$

we find that $\widehat{\eta}(E) \Subset \mathcal{E}(p)$ and $\widehat{\eta}(\lambda_1) = \varphi(\lambda_1)$ and $\widehat{\eta}(\lambda_2) = \varphi(\lambda_2)$, a contradiction with the fact that φ is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic.

For the second part of the lemma we proceed similarly. Clearly $\mu(E) \subset \mathcal{E}(q)$. If μ were not a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic, then there would exist $\widetilde{\mu} \in \mathcal{O}(E, \mathcal{E}(q))$ such that $\widetilde{\mu}(E) \Subset \mathcal{E}(q)$ and $\widetilde{\mu}(\lambda_1) = \mu(\lambda_1)$, $\widetilde{\mu}(\lambda_2) = \mu(\lambda_2)$. But setting

$$\widehat{\mu} := (\widetilde{\mu}_1, \ldots, \widetilde{\mu}_k, \ldots, (\widetilde{\mu}_{k+1})^{t_{k+1}}, \ldots, (\widetilde{\mu}_n)^{t_n}),$$

we see that $\widehat{\mu}(E) \Subset \mathcal{E}(p)$ and $\widehat{\mu}(\lambda_1) = \eta(\lambda_1)$ and $\widehat{\mu}(\lambda_2) = \eta(\lambda_2)$, a contradiction with the fact that η is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic. ■

Note that Lemma 5 may be proved without the use of the results of [Edi] (precise description of $\widetilde{k}_{\mathcal{E}(p)}$-geodesics). But in that case we have to proceed a little more delicately. For the details consult the proof of Lemma 8 in [Pfl-Zwo].

Below we present a special two-dimensional version of a result which, to some extent, is analogous to Lemma 11 of [Pfl-Zwo].

LEMMA 6. *Let $(z,0)$ and (z,w) be distinct elements of $\mathcal{E}(p) \subset \mathbb{C}^2$. Then*

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,0),(z,w)) = \frac{|w|}{(1-|z|^{2p_1})^{1/(2p_2)}}$$

and the mapping

$$E \ni \lambda \to (z, (1-|z|^{2p_1})^{1/(2p_2)}\lambda) \in \mathcal{E}(p)$$

is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $((z,0),(z,w))$.

Proof. Take any holomorphic mapping $\psi : E \to \mathcal{E}(p)$ such that $\psi(0) = (z,0)$ and $\psi(t) = (z,w)$, $t > 0$. Without loss of generality we may assume that ψ is continuous on $\bar{E}$. Write $\psi(\lambda) = (\psi_1(\lambda), \lambda^k\widetilde{\psi}_2(\lambda))$, where $\widetilde{\psi}_2(0) \neq 0$ and $k \geq 1$. Put $\widetilde{\psi} := (\psi_1, \widetilde{\psi}_2)$. Clearly $|\psi_1(\lambda)|^{2p_1} + |\psi_2(\lambda)|^{2p_2} \leq 1$ for $\lambda \in \partial E$, so $|\psi_1(\lambda)|^{2p_1} + |\widetilde{\psi}_2(\lambda)|^{2p_2} \leq 1$ for $\lambda \in \partial E$. The maximum principle for subharmonic functions implies that

$$|\psi_1(\lambda)|^{2p_1} + |\widetilde{\psi}_2(\lambda)|^{2p_2} \leq 1, \quad \lambda \in E.$$

In particular, putting $\lambda := t$ we have

$$|z|^{2p_1} + \frac{|w|^{2p_2}}{t^{2p_2k}} \leq 1.$$

So we obtain

$$t \geq t^k \geq \frac{|w|}{(1-|z|^{2p_1})^{1/(2p_2)}}.$$

This completes the proof. ■

In connection with the last lemma observe that for any $(z,u),(z,v)\in\mathcal{E}(p)\subset\mathbb{C}^2$,

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,u),(z,v))\le m\biggl(\frac{u}{(1-|z|^{2p_1})^{1/(2p_2)}},\frac{v}{(1-|z|^{2p_1})^{1/(2p_2)}}\biggr).$$

It turns out that the sharp inequality above has far reaching consequences.

LEMMA 7. *Let (z,u) and (z,v) be in $\mathcal{E}(p)\subset\mathbb{C}^2$. Assume that*

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,u),(z,v))< m\biggl(\frac{u}{(1-|z|^{2p_1})^{1/(2p_2)}},\frac{v}{(1-|z|^{2p_1})^{1/(2p_2)}}\biggr). \tag{6}$$

Then there are $b\in\partial\mathcal{E}(p)$ and $\lambda_1,\lambda_2\in E$ such that

$$\widetilde{k}^*_{\mathcal{E}(p)}(\lambda_1 b,\lambda_2 b)<m(\lambda_1,\lambda_2). \tag{7}$$

Proof. Define

$$b:=(b_1,b_2):=(z,(1-|z|^{2p_1})^{1/(2p_2)})\in\partial\mathcal{E}(p).$$

If we had equality in (7) for all $\lambda_1,\lambda_2\in E$, then the mapping $E\ni\lambda\to\lambda b\in\mathcal{E}(p)$ would be a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for any pair of points from the image. But due to Lemma 5, so is the mapping $(b_1,b_2\lambda)=(z,b_2\lambda)$. This, however, contradicts the assumption of the lemma. ■

Proof of Theorem 1. First notice that it is enough to prove the theorem in dimension two, because by the contractivity of $\widetilde{k}_D$ we have $\widetilde{k}^*_{\mathcal{E}(p_1,p_2)}=\widetilde{k}^*_{\mathcal{E}(p)}|_{(\mathcal{E}(p_1,p_2)\times\{0\}^{n-2})^2}$.

By (3), (4), Corollary 4, and Lemma 7 it is sufficient to find, for any non-convex ellipsoid $\mathcal{E}(p)$, points $(z,u),(z,v)\in\mathcal{E}(p)$ satisfying (6).

We consider two cases:

Case I: $p_1,p_2<1/2$. For $t_1,t_2\in(0,1)$ define, on E,

$$\varphi(\lambda):=\biggl(\Bigl(\frac{t_2}{(t_2+t_1)(1+t_1t_2)}\Bigr)^{1/(2p_1)}(1-t_1\lambda)^{1/p_1},\ \Bigl(\frac{t_1}{(t_2+t_1)(1+t_1t_2)}\Bigr)^{1/(2p_2)}(1+t_2\lambda)^{1/p_2}\biggr).$$

Notice that φ is exactly of one of the forms from [Jar-Pfl-Zei] and [Edi] (with

$$a_j=\Bigl(\frac{t_{3-j}}{(t_2+t_1)(1+t_1t_2)}\Bigr)^{1/(2p_j)},\qquad j=1,2,$$

$\alpha_1=t_1$, $\alpha_2=-t_2$, $\alpha_0=0$). One may easily verify that $\varphi(E)\subset\mathcal{E}(p)$.

The numbers t_1 and t_2 and consequently φ will be fixed later. Our aim is to find φ (or equivalently t_1,t_2), $\lambda_1=x+iy\in E$, $\lambda_2=\overline{\lambda}_1$ (with $x,y>0$)

such that

$$\varphi_1(\lambda_1) = \varphi_1(\lambda_2) =: z, \tag{8}$$

$$u := \varphi_2(\lambda_1) = \overline{\varphi_2(\lambda_2)} =: \overline{v}, \quad \operatorname{Arg}(\varphi_2(\lambda_1)) = \operatorname{Arg}(\lambda_1) \in (0, \pi/2), \tag{9}$$

$$\frac{|u|}{(1 - |z|^{2p_1})^{1/(2p_2)}} > |\lambda_1|. \tag{10}$$

In fact, assuming that the conditions (8)–(10) are satisfied, by elementary properties of m and the definition of $\widetilde{k}^*$ we have (remember the equality $\lambda_1 = \overline{\lambda}_2$)

$$m\Big(\frac{u}{(1 - |z|^{2p_1})^{1/(2p_2)}}, \frac{v}{(1 - |z|^{2p_1})^{1/(2p_2)}}\Big)$$
$$> m(\lambda_1, \lambda_2) \geq \widetilde{k}^*_{\mathcal{E}(p)}(\varphi(\lambda_1), \varphi(\lambda_2)) = \widetilde{k}^*_{\mathcal{E}(p)}((z, u), (z, v)),$$

which gives (6) and finishes the proof (in Case (I)).

To get properties (8) and (9) it is enough to have

$$\frac{1}{p_1} \arctan \frac{t_1 y}{1 - t_1 x} = \pi, \tag{11}$$

$$\arctan \frac{y}{x} = \frac{1}{p_2} \arctan \frac{t_2 y}{1 + t_2 x} \quad (=: \alpha \in (0, \pi/2)), \tag{12}$$

which gives

$$y = x \tan \alpha =: a_3 x, \tag{13}$$

$$t_2 = \frac{\tan(p_2\alpha)}{y - x\tan(p_2\alpha)} = \frac{\tan(p_2\alpha)}{x(\tan\alpha - \tan(p_2\alpha))} =: \frac{a_2}{x}, \tag{14}$$

$$t_1 = \frac{\tan(p_1\pi)}{x(\tan\alpha + \tan(p_1\pi))} =: \frac{a_1}{x}. \tag{15}$$

Let us recall the restrictions imposed on the numbers inolved:

$$x + iy \in E, \quad x, y > 0, \quad t_1, t_2 \in (0, 1), \quad \alpha \in (0, \pi/2).$$

Therefore, we have, in particular, $x < 1/\sqrt{1 + \tan^2\alpha}$.

We impose on t_2 the condition $t_2 < 1$. Substituting $x = 1/\sqrt{1 + \tan^2\alpha}$ in (14) we have

$$t_2 = \frac{\tan(p_2\alpha)\sqrt{1 + \tan^2\alpha}}{\tan\alpha - \tan(p_2\alpha)} < \frac{\tan(\alpha/2)\sqrt{1 + \tan^2\alpha}}{\tan\alpha - \tan(\alpha/2)} = 1$$

since $p_2 < 1/2$. This implies that for $x < 1/\sqrt{1 + \tan^2\alpha}$ close enough, t_2 given by (14) is smaller than 1.

But we also want $t_1 < 1$. Utilizing formula (15), after substituting as previously $x = 1/\sqrt{1 + \tan^2\alpha}$ we have

$$\tan^2(p_1\pi)\tan\alpha < \tan\alpha + 2\tan(p_1\pi)$$

for $\alpha > 0$ small enough, so as before $t_1 < 1$ for $x < 1/\sqrt{1+\tan^2\alpha}$ close enough with α small.

We have proved so far the existence of x, y, t_1, t_2 such that (11) and (12) are satisfied (with $\alpha > 0$ small enough). In other words, to complete that case it is sufficient to prove that (10) holds for $\alpha > 0$ small enough, and $x < 1/\sqrt{1+\tan^2\alpha}$ close enough. More precisely, we want to show that (see (8)–(10))

$$\frac{\frac{t_1}{(t_1+t_2)(1+t_1t_2)}((1+t_2x)^2+t_2^2y^2)}{\left(1-\frac{t_2}{(t_1+t_2)(1+t_1t_2)}((1-t_1x)^2+t_1^2y^2)\right)} > (x^2+y^2)^{p_2},$$

which is equivalent to (use (13)–(15))

$$\begin{aligned} &a_1((1+a_2)^2+a_2^2a_3^2) \\ &\quad > x^{2p_2}(1+a_3^2)^{p_2}\left((a_1+a_2)\left(1+\frac{a_1a_2}{x^2}\right)-a_2((1-a_1)^2+a_1^2a_3^2)\right). \end{aligned}$$

Equivalently,

$$\begin{aligned} 0 &> x^{2p_2}(1+a_3^2)^{p_2}(a_1+2a_1a_2-a_1^2a_2-a_1^2a_2a_3^2) \\ &\quad + x^{2p_2-2}(1+a_3^2)^{p_2}a_1a_2(a_1+a_2)-a_1((1+a_2)^2+a_2^2a_3^2) =: \psi(x). \end{aligned}$$

Our aim is to prove that if α is sufficiently small then for $x < 1/\sqrt{1+a_3^2}$ close enough, the above inequality holds.

One may easily verify that $\psi(1/\sqrt{1+a_3^2}\,) = 0$. To get the desired inequality it is sufficient to show that

$$\psi'(1/\sqrt{1+a_3^2}\,) > 0$$

if α is small enough. But the last inequality is equivalent to

$$p_2(a_1+2a_1a_2-a_1^2a_2-a_1^2a_2a_3^2)+(p_2-1)a_1a_2(a_1+a_2)(1+a_3^2) > 0,$$

or

$$p_2((1+a_2)^2+a_2^2a_3^2) > a_2(a_1+a_2)(1+a_3^2).$$

Substituting the formulas (13)–(15) we get

$$\begin{aligned} &p_2\left(\frac{\tan^2\alpha}{(\tan\alpha-\tan(p_2\alpha))^2}+\frac{\tan^2\alpha\tan^2(p_2\alpha)}{(\tan\alpha-\tan(p_2\alpha))^2}\right) \\ &\quad > \frac{\tan(p_2\alpha)}{\tan\alpha-\tan(p_2\alpha)}\frac{\tan\alpha(\tan(p_2\alpha)+\tan(p_1\pi))}{(\tan\alpha+\tan(p_1\pi))(\tan\alpha-\tan(p_2\alpha))}(1+\tan^2\alpha) \end{aligned}$$

or equivalently

$$p_2\frac{\tan\alpha}{1+\tan^2\alpha}\frac{1+\tan^2(p_2\alpha)}{\tan(p_2\alpha)} > \frac{\tan(p_2\alpha)+\tan(p_1\pi)}{\tan\alpha+\tan(p_1\pi)}$$

and, finally,

$$\beta(\alpha) := p_2 \sin(2\alpha)(\tan\alpha + \tan(p_1\pi)) - \sin(2p_2\alpha)(\tan(p_2\alpha) + \tan(p_1\pi)) > 0.$$

Note that (remember that $0 < p_2 < 1/2 < 1$)

$$\beta(0) = \beta'(0) = 0, \quad \beta''(0) = 4p_2(1 - p_2) > 0,$$

which implies that $\beta(\alpha) > 0$ for $\alpha > 0$ small enough. This completes the proof.

Case (II): $p_1 < 1/2 \le p_2$. There is an $n \in \mathbb{N}$ ($n \ge 2$) such that $q_2 := \frac{1}{n} p_2 < \frac{1}{2}$ ($q_1 := p_1$). Then by the proof of Case (I), there are $(z, u), (z, v) \in \mathcal{E}(q)$ such that (see (7))

$$\text{(16)} \qquad \widetilde{k}^*_{\mathcal{E}(q)}((z,u),(z,v)) < m\left(\frac{u}{(1-|z|^{2q_1})^{1/(2q_2)}}, \frac{v}{(1-|z|^{2q_1})^{1/(2q_2)}}\right).$$

Let φ be a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $((z,u),(z,v))$ with $\varphi(\lambda_1) = (z,u)$ and $\varphi(\lambda_2) = (z,v)$ and let B_2 be the Blaschke product associated with φ_2. We have clearly $\varphi_1 \not\equiv z$ (a consequence of the Schwarz–Pick Lemma). By Lemma 5,

$$\mu(\lambda) := \left(\varphi_1(\lambda), \left(\frac{\varphi_2(\lambda)}{B_2(\lambda)}\right)^{1/n}\right), \quad \lambda \in E,$$

is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $(\mu(\lambda_1), \mu(\lambda_2)) := ((z, \widetilde{u}), (z, \widetilde{v}))$. It is enough to show that

$$\text{(17)} \qquad \widetilde{k}^*_{\mathcal{E}(p)}((z,\widetilde{u}),(z,\widetilde{v})) < m\left(\frac{\widetilde{u}}{(1-|z|^{2p_1})^{1/(2p_2)}}, \frac{\widetilde{v}}{(1-|z|^{2p_1})^{1/(2p_2)}}\right).$$

Consider the mapping

$$\psi : E \ni \lambda \to (z, \lambda(1-|z|^{2p_1})^{1/(2p_2)}) \in \mathcal{E}(p).$$

If (17) did not hold, then we would have equality there. Then ψ is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $((z,\widetilde{u}),(z,\widetilde{v})) =: (\psi(\lambda_3), \psi(\lambda_4))$ with some $\lambda_3, \lambda_4 \in E$. Consequently, the mapping $\widetilde{\psi}(\lambda) := (z, (\psi_2(\lambda))^n B_2(\lambda))$ is a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $((z,u),(z,v))$ (because $\widetilde{\psi}(\lambda_3) = \varphi(\lambda_1) = (z,u)$, $\widetilde{\psi}(\lambda_4) = \varphi(\lambda_2) = (z,v)$, $m(\lambda_1,\lambda_2) = m(\lambda_3,\lambda_4)$ and φ is a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $((z,u),(z,v))$). This, however, contradicts the fact that no such geodesic has constant first component (remember (17) and apply the Schwarz–Pick Lemma); one may alternatively exclude that case using the description of geodesics from [Edi]: namely, no geodesic has a component with more than one zero (counted with multiplicities), which happens here. This finishes Case (II) and the proof of Theorem 1. ∎

References

[Bed-Dem] E. Bedford and J.-P. Demailly, *Two counterexamples concerning the pluri-complex Green function in* $\mathbb{C}^{\ltimes}$, Indiana Univ. Math. J. 37 (1988), 865–867.

[Edi] A. Edigarian, *On extremal mappings in complex ellipsoids*, Ann. Polon. Math. 62 (1995), 83–96.

[Jar-Pfl] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter 1993.

[Jar-Pfl-Zei] M. Jarnicki, P. Pflug and R. Zeinstra, *Geodesics for convex complex ellipsoids*, Ann. Scuola Norm. Sup. Pisa 20 (1993), 535–543.

[Kli] M. Klimek, *Pluripotential Theory*, Oxford University Press, 1991.

[Lem1] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–474.

[Pfl-Zwo] P. Pflug and W. Zwonek, *The Kobayashi metric for non-convex complex ellipsoids*, Complex Variables Theory Appl. 29 (1996), 59–71.

[Pol] E. A. Poletsky, *Holomorphic currents*, Indiana Univ. Math. J. 42 (1993), 85–144.

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